Memorandum on Dimension Formulas for Spaces of Jacobi Forms

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1   Introduction

Denote by $S_{k,m}(\Gamma)$ the space of Jacobi cusp forms of weight k, index m on a subgroup Γ of finite index in $\Gamma(1) = \mathrm{SL}(2,\mathbb{Z})$. In [S-Z1] one finds an explicit trace formula for Jacobi forms. One of the first applications of such a trace formula is to calculate the dimensions of the spaces of Jacobi forms. Although the cited trace formula is a "ready to compute" formula, it can still be considerably simplified if one is merely interested in dimensions, i.e. the trace of the identity operator. That this can be done, what one has to do and what the outcoming formula is looking like, at least qualitatively, is without doubt known to specialists. Nevertheless, there is no place in the literature where this has been put down in sufficient generality. The purpose of the present note is to fill this gap. The resulting dimension formulas are summarized in Theorems 1 to 4.

2   A first computation

We start with the trace formula as given in [S-Z1, Theorem 1]. According to this theorem one has

$$\dim S_{k,m}(\Gamma) = \sum_A I(A)g(A) + \sum_B I(B)g(B).$$

Here the notation is as follows: The symbol Γ denotes an arbitrary subgroup of (finite index in) $\Gamma(1)$. In the first sum A runs through a complete set of representatives for the Γ-conjugacy classes of all non-parabolic elements of Γ, and in the second sum B runs through a complete set of representatives

of all parabolic elements of Γ modulo the equivalence $\sim$, where $B_1 \sim B_2$ if and only if GB_1 is Γ-conjugate to B_2 for some $G \in \mathcal{C}_{\Gamma \cap \Gamma(4m)}(B_1)$. Here, for any given matrix B, parabolic or not, and any given subgroup Γ of $\Gamma(1)$, the symbol $\mathcal{C}_\Gamma(B)$ stands for the centralizer of B in Γ. Moreover,

$$I(1) = [\Gamma(1) : \Gamma] \cdot \frac{2k - 3}{48},$$

and for parabolic B with positive trace

$$I(B) = -\frac{1}{2} \left[\mathcal{C}_\Gamma(B) : \mathcal{C}_{\Gamma \cap \Gamma(4m)}(B) \right]^{-1} \cdot \left(1 - i\, C \left(\frac{r}{s} \right) \right),$$

where r, s stand for those uniquely determined positive integers such that B and $\mathcal{C}_{\Gamma \cap \Gamma(4m)}(B)$ are $\Gamma(1)$-conjugate to[1] $(1, r; 0, 1)$ and $\langle (1, s; 0, 1) \rangle$, respectively, and where $C(z) = \cot(\pi z)$ for $z \notin \mathbb{Z}$, and $C(z) = 0$ for $z \in \mathbb{Z}$. For all other A, the expression $I(A)$ is somehow defined and will be recalled later; the only important point for the moment is that $I(A) = 0$ for non-split hyperbolic A (i.e. for those A with trace t satisfying $t^2 - 4 \neq$ square in $\mathbb{Q}^*$). Finally, $g(1) = 2m$, and for a parabolic B which is $\Gamma(1)$-conjugate to $(1, r; 0, 1)$ for some r one has

$$g(A) = \sum_{\lambda \bmod 2m} e^{2\pi i \left(\frac{r\lambda^2}{4m} \right)}.$$

If the reader wishes to compare the above formula for the dimensions with the formula given in [S-Z1, Theorem 1] he should note that (i) $\dim S_{k,m}(\Gamma) = \operatorname{tr}(H_{k,m,\Gamma}(\Gamma \ltimes \mathbb{Z}^2), S_{k,m}(\Gamma))$ in the notation of [S-Z1]; (ii) we have dropt here various subscripts and parameters: in the notations of [S-Z1] we have $I(A) = I_{k,m,\Gamma}(A)$, and $g(A) = g_m(\Gamma \ltimes \mathbb{Z}^2, A)$; (iii) $g_m(\Gamma \ltimes \mathbb{Z}^2, A) = G_m(A)$, where the latter expression is given by [S-Z1, Theorem 2] (when applying this theorem, note that the quadratic forms Q_A and Q'_A occurring in the statement of the theorem are equivalent modulo $\Gamma(1)$ if A and A' are $\Gamma(1)$-conjugate; this implies that $G_m(A)$ depends only on the $\Gamma(1)$-conjugacy class of A; in particular $g(B) = g((1, r; 0, 1))$ for the parabolic B as above, and $Q_{(1,r;0,1)}(\lambda, \mu) = r\lambda^2$).

To be correct it must be added that the quoted dimension formula holds strictly true only for $k \geq 3$. The given formula becomes true for arbitrary k

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[1] We use $(a, b; c, d)$ to denote matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

if one subtracts on the left hand side a certain correction term, which for $k = 1, 2$, however, turns out to be non-trivial (cf. [S-Z1, formulas (9), (10) of §3]). In fact, it can be shown that this correction term equals $J^+_{3-k,m}(\Gamma)$, where $J^+_{3-k,m}(\Gamma)$ denotes the space of skew-holomorphic Jacobi forms of weight $3-k$, index m on Γ (as defined e.g. in [S2]). Thus the given formula holds true for arbitrary k if one replaces the left hand side by $\dim S_{k,m}(\Gamma) - \dim J^+_{3-k,m}(\Gamma)$, keeping in mind that $\dim J^+_{3-k,m}(\Gamma) = 0$ for $k \geq 3$. The dimensions of $J^+_{1,k}(\Gamma)$ and $J^+_{2,k}(\Gamma)$ can be explicitly calculated for congruence subgroups Γ using the Theorem of Serre and Starck on modular forms of weight $1/2$ (cf. [S1], [I-S] or [S3] for details of the method which has to be applied). Thus, in principle, it would be possible to give an effective formula for $\dim S_{k,m}(\Gamma)$ for arbitrary k. However, for simplicity we concentrate here on the case $k \geq 3$ and leave the correction terms undetermined for $k \leq 2$.

Furthermore, we assume first of all that Γ contains nor elliptic matrices neither the matrices with trace -2, i.e. that Γ is torsion-free and that the cusps of Γ are all regular, i.e. that any parabolic subgroup of Γ is $\Gamma(1)$-conjugate to $\langle (1, b; 0, 1) \rangle$ for a suitable b. Note that all these assumptions hold for the principal congruence subgroups $\Gamma(N)$ with $N \geq 3$ (as it follows easily from the fact that any elliptic matrix in $\Gamma(1)$ is conjugate to one of the matrices $\pm(0, -1; 1, 0)$, $\pm(0, -1; 1, 1)$ ore $\pm(-1, -1; 1, 0)$). Under these assumptions only $A = 1$ and parabolic B with trace$= 2$ contribute to the given formula for $\dim S_{k,m}(\Gamma)$ (Here one has also to use that $\Gamma(1)$ contains no split hyperbolic matrices, i.e. matrices with trace$^2 - 4 =$ square in $\mathbb{Q}^*$).

Concerning the parabolic contribution one easily verifies the following:

(i) $C_\Gamma(A) = \Gamma_p$ (=stabilizer of p in Γ) for all parabolic $A \in \Gamma$ with fixed point $p \in \mathbb{P}_1(\mathbb{Q})$;

(ii) for any two parabolic A and A' there exists a matrix $G \in C_\Gamma(A)$ such that GA and A' are $\Gamma(1)$-conjugate if and only if the fixed points p and p' of A and A' are equivalent modulo Γ;

(iii) for any two parabolic A and A' having the same fixed point one has $A \sim A'$ if and only if A and A' lie in the same coset modulo $C_{\Gamma \cap \Gamma(4m)}(A)$.

Taking into account these facts the parabolic contribution can now be written as $\sum_{p \in \mathbb{P}_1(\mathbb{Q})} t_p$ where

$$t_p = \sum_{A \in \Gamma_p / (\Gamma_p \cap \Gamma(4m))} I(A) g(A).$$

To simplify the t_p fix a cusp p. Then there exist uniquely positive inte-

gers b, f such that Γ_p and $\Gamma_p \cap \Gamma(4m)$ are $\Gamma(1)$-conjugate to $\langle(1, b; 0, 1)\rangle$ and $\langle(1, bf; 0, 1)\rangle$, respectively. Thus

$$t_p = \sum_{0 < \nu \leq f} I(R(1, b\nu; 0, 1)R^{-1}) g(R(1, b\nu; 0, 1)R^{-1})$$

with a suitable $R \in \Gamma(1)$. Inserting the quoted values for the functions I and g one obtains

$$t_p = -\frac{1}{2f} \sum_{0 < \nu \leq f} \left(1 - iC\left(\frac{\nu}{f}\right)\right) \sum_{\lambda \bmod 2m} e^{2\pi i \left(\frac{b\nu\lambda^2}{4m}\right)}.$$

Now $f = \frac{4m}{(4m, b)}$ (note that $\Gamma_p \cap \Gamma(4m)$ is $\Gamma(1)$-conjugate to $\langle(1, [b, 4m]; 0, 1)\rangle$ on the one hand, and to $\langle(1, bf; 0, 1)\rangle$, by the definition of f, on the other hand; thus $bf = [b, 4m]$, whence $f = \frac{4m}{(4m, b)}$). Using this we can write

$$t_p = -\frac{1}{2}\#\{\lambda \bmod 2m | b\lambda^2 \equiv 0 \bmod 4m\}$$

$$+ \frac{m}{f^2} i \sum_{\nu \bmod f} C\left(\frac{\nu}{f}\right) \sum_{\lambda \bmod f} e^{2\pi i \left(\frac{b}{(4m, b)}\nu\lambda^2\right)}.$$

The first term equals $-\frac{m}{f}Q(f)$ where $Q(n)$, for any positive integer n, denotes the greatest integer whose square divides n.

To simplify the second term we apply the following Lemma, which is Proposition A.2 in [S-Z2]; for the proof the reader is referred to loc. cit..

Lemma 1. *Let a and f be positive integers. Then*

$$\frac{i}{f} \sum_{\nu \bmod f} C\left(\frac{\nu}{f}\right) \sum_{\lambda \bmod f} e^{2\pi i \left(\frac{a\nu\lambda^2}{f}\right)} = -2(a, f) \sum_{\Delta} \left(\frac{\Delta}{a/(a, f)}\right) H(\Delta).$$

Here the sum on the right side is over all $\Delta < 0$ dividing $\frac{f}{(a,f)}$ such that $\frac{f}{(a,f)\Delta}$ is square-free, and $H(\Delta)$ denotes the Hurwitz class number of Δ.

Recall that $H(\Delta)$ equals the number of $\Gamma(1)$-equivalence classes of all integral, positive definite binary quadratic forms of discriminant Δ, counting forms $\Gamma(1)$-equivalent to a multiple of $x^2 + y^2$ (resp. $x^2 + xy + y^2$) with multiplicity $\frac{1}{2}$ (resp. $\frac{1}{3}$). In particular, $H(\Delta) = 0$ if $\Delta \not\equiv 0, 1 \bmod 4$.

Accordingly to this Lemma the second term of the last formula for t_p can be written as $-\frac{2m}{f} \sum \left(\frac{\Delta}{b/(4m, b)}\right) H(\Delta)$ with Δ running through all negative integers dividing f such that f/Δ is square-free.

3 A special case

Summing up the result of the calculations of the foregoing section, we have proved

Theorem 1. *Let k and m be integers, $m \geq 1$. Let Γ be a torsion-free subgroup of finite index in $\Gamma(1)$ which contains no matrices with trace equal to -2. Then the dimension of the space of Jacobi cusp forms of weight k, index m on Γ is given by*

$$\dim S_{k,m}(\Gamma) - \dim J^+_{3-k,m}(\Gamma) = m \cdot [\Gamma(1) : \Gamma] \frac{2k - 3}{24}$$

$$- \sum_p \frac{m}{f_p} Q(f_p) - \sum_p \frac{2m}{f_p} \sum_{\substack{\Delta | f_p,\ \Delta < 0 \\ f_p/\Delta \ squarefree}} \left(\frac{\Delta}{b_p/(4m, b_p)} \right) H(\Delta).$$

Here p runs through a set of representatives for $\Gamma \backslash \mathbb{P}_1(\mathbb{Q})$, and for each such p we use $b_p = \frac{1}{2}[\Gamma(1)_p : \Gamma_p]$, $f_p = 4m/(4m, b_p)$. Moreover, $H(\Delta)$ denotes the Hurwitz class number (as explained in the last but not least paragraph of section 2), and $Q(n)$, for any positive integer n, denotes the greatest integer whose square divides n.

Note that this dimensions formula becomes even simpler if Γ is normal in $\Gamma(1)$ since then the numbers b_p, f_p do not depend on p, and are all equal to, say, $b := \frac{1}{2}[\Gamma(1)_\infty : \Gamma_\infty]$, $f := 4m/(4m, b)$. The sums over p in the theorem can then simply be replaced by $\sharp\Gamma \backslash \mathbb{P}_1(\mathbb{Q})$, which equals $[\Gamma(1) : \Gamma]/2b$. In particular, for the group $\Gamma(N)$, where $b = N$, we find

Corollary 1. *Let N, k, m be positive integers, $N, k \geq 3$. Then*

$$\dim S_{k,m}(\Gamma(N)) =$$

$$\varphi(N)\psi(N) \left(mN\frac{2k - 3}{24} - \frac{d}{8}Q\left(\frac{4m}{d}\right) - \frac{d}{4} \sum_\Delta \left(\frac{\Delta}{N/d} \right) H(\Delta) \right).$$

Here $d = (4m, N)$, and Δ runs through all negative integers dividing $4m/d$ such that $4m/d\Delta$ is square-free. Moreover, $\varphi(N)$ denotes the Euler phi-function, and $\psi(N) = \#\mathbb{P}_1(\mathbb{Z}/N\mathbb{Z}) = N \prod_{p|N} \left(1 + \frac{1}{p} \right)$.

The simplest instance of this formula occurs for $4m|N$ since then the sum containing the Hurwitz class numbers vanishes. Here, for $k \geq 3$, we obtain

$$\dim S_{k,m}\left(\Gamma(N)\right) = m\varphi(N)\psi(N)\left(N\frac{2k-3}{24} - \frac{1}{2}\right).$$

This formula was also proved in [K] by considering Jacobi forms as holomorphic sections of certain line bundles, to which the Hirzebruch-Riemann-Roch theorem could be explicitly applied if $4m|N$.

4   The general case

In this section we compute the dimension formulas for arbitrary subgroups Γ of $\Gamma(1)$. The computations are essentially the same as in the section 2. However, in view of the various contributions and cases to consider in the general case, a straightforward calculation would lead to rather complicated formulas. The main goal of this section is to state these formulas in a more concise and possibly meaningful way.

To begin with we rewrite the formula of Theorem 1. To this end we introduce first of all some notation. As in [S-Z2] we define a function $H_n(\Delta)$ for integers $n \geq 1$ and $\Delta \leq 0$. The function $H_1(\Delta)$ equals the Hurwitz class number $H(\Delta)$, i.e. $H(0) = -\frac{1}{12}$ and $H(\Delta)$, for $\Delta \neq 0$ as recalled in the last but not least paragraph of section 2. For general $n \geq 1$ write $(n, \Delta) = a^2 b$ with square-free b and set

$$H_n(\Delta) = \begin{cases} a^2 b \left(\frac{\Delta/a^2 b^2}{n/a^2 b}\right) H_1(\Delta/a^2 b^2) & \text{if } a^2 b^2|\Delta, \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, for integers $k \geq 2$, we define the polynomial[2] $p_k(s)$ as the coefficient of x^{k-2} in the power series development of $(1 - sx + x^2)^{-1}$. Note that $p_{2k-2}(2) = (2k - 3)$ and $p_{2k-2}(0) = (-1)^k$.

Finally, for an exact divisor[3] n of m with codivisor $n' = m/n$ and integers

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[2]These are, up to a scaling of the argument and a shift in the indices, the classical Gegenbauer polynomials.

[3]i.e. n and m/n are relatively prime

$k \geq 2$, $b \geq 1$ and $t = 0, \pm 1$ we set[4]

$$s_{k,m;b}^{\text{top}}(n) = -p_{2k-2}(2)H_{bn'}(0) - \frac{1}{2}Q\left(n'(4n', bn')\right),$$

$$s_{k,m;b}^{\text{par.}}(n) = -\frac{1}{2}(4n, bn')\, p_{2k-2}(0) \sum_{\substack{\Delta \mid 4n/(4n,bn'),\ \Delta < 0 \\ 4n/(4n,bn')\Delta \text{ square-free}}} H_{bn'/(4n,bn')}(\Delta)$$

$$s_{k,m;t}^{\text{ell.}}(n) = -\delta\left((t+2)|n\right) p_{2k-2}\left(\sqrt{t+2}\right) H_{n'}(t^2 - 4).$$

Here, in the definition of $s_{k,m;b}^{\text{par.}}(n)$, the sum is over all negative integers Δ dividing $4n/(4n, bn')$ such that $4n/(4n, bn')\Delta$ is square-free. Moreover, $\delta(a|n)$ equals 1 or 0 accordingly as a divides n or not. Recall from the previous section that $Q(n)$, for any positive integer n, denotes the greatest integer whose square divides n.

We can now reformulate Theorem 1 as follows:

Theorem 2. *Let k and m be positive integers, $k \geq 2$, and Γ be a subgroup of finite index in $\Gamma(1)$. Denote by r the number of cusps of Γ and by $b_1, \ldots, b_r$ the cusp widths of a complete set of representatives for the cusps $\Gamma \backslash \mathbb{P}_1(\mathbb{Q})$.*

If Γ is torsion-free and contains no matrices with trace equal to -2, then the dimension of the space of Jacobi cusp forms of weight k and index m on Γ is given by

$$\dim S_{k,m}(\Gamma) - \dim J_{3-k,m}^{+}(\Gamma) = \sum_{j=1}^{r} \left(s_{k,m;b_j}^{\text{top}}(1) + (-1)^k\, s_{k,m;b_j}^{\text{par.}}(m) \right).$$

Recall that the cusp width b_p of a cusp p is by definition equal to $b_p = [\Gamma(1)_p : \{\pm 1\} \cdot \Gamma_p]$. To deduce Theorem 2 from Theorem 1 one merely needs to recall that for any subgroup Γ of $\Gamma(1)$, one has $\sum_{j=1}^{r} b_r = [\Gamma(1) : \{\pm 1\} \cdot \Gamma]$.

The formulation of the dimension formula as in Theorem 2 has so far no advantage over the one given in the preceding section. However, the usage of the auxiliary functions $s_{k,m} \ldots$ will allow us to rewrite more systematically the dimension formulas for not necessarily torsion-free groups Γ, which we shall discuss now. More precisely, we shall prove the following formula.

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[4]The sum $s_{k,m,1}^{\text{top}}(n) + s_{k,m,1}^{\text{par.}}(n) + s_{k,m,-1}^{\text{ell.}}(n) + s_{k,m,0}^{\text{ell.}}(n) + s_{k,m,1}^{\text{ell.}}(n)$ equals the function $s_{k,m}(1, n)$ introduced in [S-Z2, Theorem 1], which describes the trace of the Atkin-Lehner operator W_n on the *certain space* of modular forms of level m and weight $2k - 2$.

Theorem 3. *Let k and m be positive integers, $k \geq 2$, and Γ be a subgroup of finite index in $\Gamma(1)$. Denote by r the number of cusps of Γ and by $b_1, \ldots, b_r$ the cusp widths of a complete set of representatives for the cusps $\Gamma \backslash \mathbb{P}_1(\mathbb{Q})$, and let $e(0)$ and $e(-1) = e(+1)$ be the number of Γ-orbits of the elliptic fixed points of Γ which are $\Gamma(1)$-equivalent to i and $e^{2\pi i/3}$, respectively.*

If Γ contains the matrix -1, then the dimension of the space of Jacobi cusp forms of weight k and index m on Γ is given by

$$\dim S_{k,m}(\Gamma) - \dim J^+_{3-k,m}(\Gamma) = \sum_{j=1}^{r} \frac{1}{2} \left(s^{top}_{k,m;b_j}(1) + (-1)^k s^{top.}_{k,m;b_j}(m) \right)$$

$$+ \sum_{j=1}^{r} \frac{1}{2} \left(s^{par.}_{k,m;b_j}(1) + (-1)^k s^{par.}_{k,m;b_j}(m) \right)$$

$$+ \sum_{t=-1}^{+1} \frac{e(t)}{2} \left(s^{ell.}_{k,m;t}(1) + (-1)^k s^{ell.}_{k,m;t}(m) \right).$$

Note that the dimension formula for $S_{k,m}(\Gamma)$, for varying Γ, depends only on the "branching scheme" $b_1, \ldots, b_r$, $e(0)$, $e(1)$ of Γ.

Proof of Theorem 3. In addition to the computation of § 2, we have first of all to take into account in our general trace formula the term $I(-1)g(-1)$. By [S-Z1, Theorem 1, Theorem 2] this equals $[\Gamma(1) : \Gamma] (-1)^k \frac{2k-3}{24}$. In the notation introduced in the beginning of this section this equals the contribution of $H_b(0)$ in $\frac{1}{2}(-1)^k \sum_j s^{top}_{k,m;b_j}(m)$. Similarly, the term $I(1)g(1) = m[\Gamma(1) : \Gamma](2k-2)/24$ equals the contribution of $H_{bm}(0)$ in $\frac{1}{2} \sum_j s^{top}_{k,m;b_j}(1)$.

Next, let p be a cusp. Then there is a positive integer b such that the groups Γ_p and $\Gamma_p \cap \Gamma(4m)$ are $\Gamma(1)$-conjugate to $\langle \pm 1 \rangle \times \langle (1,b;0,1) \rangle$ and $\langle (1,bf;0,1) \rangle$ with $f = 4m/(4m,b)$, respectively. Accordingly, we find $t_p = t_p^+ + t_p^-$, where

$$t_p^\varepsilon = \sum_{0 < \nu \leq f} I\left(\varepsilon(1,b\nu;0,1) \right) g\left(\varepsilon(1,b\nu;0,1) \right).$$

Here t_p^+ equals one half of the t_p of section § 2 (note that in the case considered here $[\Gamma_p : \Gamma_p \cap \Gamma(4m)] = 2f$ due to the presence of -1 in Γ). Accordingly, t_p^+ equals $\frac{1}{2}(-1)^k \sum_j s^{par.}_{k,m;n_j}(m)$ plus the contribution of the Q-terms in $\frac{1}{2} \sum_j s^{top}_{k,m;n_j}(1)$.

For the calculation of t_p^-, we use

$$I\left(-(1, b\nu; 0, 1)\right) = -\frac{1}{4f} i^{1-2k} \left(1 - i\, C(\nu/f)\right),$$

$$g\left(-(1, b\nu; 0, 1)\right) = -i \sum_{\lambda \bmod 2} e^{2\pi i \left(\frac{mb\nu\lambda^2}{4}\right)}$$

(cf. [S-Z1, Theorem 1, 2]). By a similar calculation as in section § 2 we find

$$t_p^- = -\frac{(-1)^k}{4} \left(Q((4, bm)) + \left(\frac{-4}{bm}\right) H(-4)\right).$$

But this equals the Q-term in $\frac{1}{2}(-1)^k s_{k,m;b}^{\mathrm{top}}(m)$ plus $\frac{1}{2} s_{k,m;b}^{\mathrm{par.}}(1)$.

If $A = (a, b; c, d)$ is an elliptic matrix in Γ with trace t, then by [S-Z1, Theorem 1, 2], we have

$$I(A) = \frac{1}{|\Gamma_e|} \operatorname{sign}(c) \frac{\rho^{3/2-k}}{\rho - \overline{\rho}}, \quad g(A) = -i|t - 2|^{-3/2} \sum_{\lambda,\mu \bmod t-2} e^{2\pi i \left(\frac{m}{t-2} Q_A(\lambda,\mu)\right)}.$$

Here ρ and $\overline{\rho}$ are the roots of $x^2 - tx + 1 = 0$ such that the imaginary part of ρ and c have the same sign, Γ_e is the stabilizer in Γ of the elliptic fixed point e of A in the upper half plane, and $Q_A(\lambda, \mu) = b\lambda^2 + (d - a)\lambda\mu - c\mu^2$. Note that $I(A) = -I(A^{-1})$ and that the same identity holds true for $g(A)$. Thus, A and A^{-1} add the contribution

$$t_A = 2\operatorname{Re}(I(A))\operatorname{Re}(g(A)) - 2\operatorname{Im}(I(A))\operatorname{Im}(g(A))$$

to our general trace formula. One easily verifies

$$\operatorname{Re}(I(A)) = -\frac{p_{2k-2}\left(\sqrt{2+t}\right)}{2|\Gamma_e|\sqrt{2+t}}, \quad \operatorname{Im}(I(A)) = -(-1)^k \frac{p_{2k-2}\left(\sqrt{2-t}\right)}{2|\Gamma_e|\sqrt{2-t}}$$

(using $\rho = \left(\frac{\sqrt{t+2} + \sqrt{t-2}}{2}\right)^2$) and

$$\operatorname{Re}(g(A)) = \frac{1}{2}\delta(2 + t = 1) |\Gamma_e| \sqrt{2+t}\, H_m(t^2 - 4),$$

$$\operatorname{Im}(g(A)) = -\frac{1}{2}\delta((2 - t)|m) |\Gamma_e| \sqrt{2-t}\, H_1(t^2 - 4)$$

(by a case by case inspection; note that $I(A)$ and $g(A)$ depend only on the $\Gamma(1)$-conjugacy class of A, thus it suffices to verify the latter two formulas for $A = (0, -1; 1, 0)$, $A = (0, -1; 1, 1)$ and $A = (-1, -1; 1, 0)$, respectively). Hence

$$t_A = \frac{1}{2} \left(s_{k,m;t}^{\text{ell.}}(1) + (-1)^k s_{k,m;-t}^{\text{ell.}}(m) \right).$$

It is now clear that the contributions of the elliptic matrices add up to the term as stated in the theorem. $\square$

We leave it to the reader to verify the last theorem, which describes the remaining case, i.e. the case of a Γ which does not contain the matrix -1 but possibly elliptic fixed points and irregular cusps (i.e. cusps p such that Γ_p is generated by an element with negative trace). Here the corresponding dimension formulas run as follows:

Theorem 4. *Let the notations be as in Theorem 3. Suppose that Γ does not contain the matrix -1, and let $b_1, \ldots, b_{r_1}$ the cups widths of the regular cusps and $b_{r_1+1}, \ldots, b_r$ the cups widths of the irregular ones. Then one has*

$$\dim S_{k,m}(\Gamma) - \dim J_{3-k,m}^+(\Gamma) = \sum_{j=1}^{r_1} \left(s_{k,m;b_j}^{top}(1) + (-1)^k s_{k,m;b_j}^{par.}(m) \right)$$

$$+ \sum_{j=r_1+1}^{r_2} \frac{1}{2} \left(s_{k,m;2b_j}^{top}(1) + (-1)^k s_{k,m;2b_j}^{par.}(m) \right)$$

$$+ \sum_{j=r_1+1}^{r_2} \left(s_{k,m;b_j}^{par}(1) + (-1)^k s_{k,m;b_j}^{top}(m) \right)$$

$$- \sum_{j=r_1+1}^{r_2} \frac{1}{2} \left(s_{k,m;2b_j}^{par}(1) + (-1)^k s_{k,m;2b_j}^{top}(m) \right)$$

$$+ e(-1) \left(s_{k,m;-1}^{ell.}(1) + (-1)^k s_{k,m;+1}^{ell.}(m) \right).$$

5   Concluding remarks

Theorem 1 to Theorem 4 summarize the dimension formulas for holomorphic Jacobi cusp forms of arbitrary integral weight $k \geq 2$ and integral index $m \geq 1$ on arbitrary subgroups Γ of $\Gamma(1)$. However, for the important case $k = 2$, we would still have to compute the term $\dim J_{3-k,m}^+(\Gamma)$ to obtain an explicit

formula. In principle this computation could be done, however, this seems to be a rather cumbersome task. In essence, this computation would reduce to an analysis of the action of $\Gamma(1)$ on the space of modular form of weight $\frac{1}{2}$. For an example of this kind of computation the reader is referred to [I-S], where we proved vanishing results for spaces of (holomorphic) Jacobi forms of weight 1 on groups $\Gamma_0(l)$.

The general trace formula of [S-Z1] admits also to derive explicit dimension formulas for spaces of Jacobi forms with characters, like e.g. for the spaces $S_{k,m}(\Gamma_0(l), \chi)$, where χ is a Dirichlet character modulo l. It also admits the derivation of explicit formulas for the traces of Atkin-Lehner operators W_n (as considered in [S-Z2] for Jacobi forms on $\Gamma(1)$) on spaces of Jacobi forms on general Γ. It it the very likely that the function $s^*_{k,m;b}(n)$ for nontrivial divisors n of m are related to these trace formulas.

It might be interesting to ask for the geometric interpretation of the decomposition of the dimension formulas into the $s_{k,m}\ldots$-parts. A clue to this would be the article [K].

Finally, it might be interesting to compare the dimension formulas for Jacobi forms to the dimension formulas for ordinary elliptic modular forms. For example, the dimension of the space $S_{2k-2}(m)$ of modular cups forms of weight $2k-2$ on $\Gamma_0(m)$ is given by

$$\dim S_{2k-2}(m) = \sum_{\substack{m'|m \\ \frac{m}{m'} \text{ square-free}}} \left(s^{\text{top}}_{k,m',1}(1) + s^{\text{par}}_{k,m',1}(1) + \sum_{t=-1}^{+1} s^{\text{ell}}_{k,m',t}(1) \right)$$

(cf. [S-Z2]). This reflects the existence of a certain natural subspace of $S_{2k-2}(m)$, whose dimension equals the term corresponding to m, and which, in the cited article, was proved to be Hecke-equivariantly isomorphic to $S_{k,m}(\Gamma(1))$. Similar lifting maps exist also for Jacobi forms on proper subgroups of $\Gamma(1)$[5], and a comparison of dimension formulas might give a first clue towards an explicit description of the images of such liftings. These liftings suggest Hecke-equivariant relations e.g. between Jacobi forms of index 1 on $\Gamma_0(l)$ and Jacobi forms of index l and on $\Gamma(1)$. Again, our dimension formulas may help to pinpoint what exactly one should expect. From our formulas we find e.g., for primes $p \equiv 1 \mod 12$ and even $k \geq 4$, that the dimension of $\dim S_{k,1}(\Gamma_0(p))$ equals the dimension of $S_{k,1}(\Gamma(1)) \oplus S_{k,p}(\Gamma(1)) \oplus S^+_{k,p}(\Gamma(1))$

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[5]However, to our knowledge this has never been worked out in detail for groups different from $\Gamma(1)$.

(assuming the so far unproved fact[6] that the dimension of the space of skew-holomorphic cusp forms $S^+_{k,p}(\Gamma(1))$ is given by the same formula as for $S_{k,p}(\Gamma(1))$, but with the $(-1)^k$ replaced by $-(-1)^k$.).

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[6]We hope to prove this eventually in another article.