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On the zeta function for the space of binary cubic forms and distributions of discriminants of cubic ring extensions

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1 Introduction

The aim of this note is to give a brief introduction on applications of Sato-Shintani's zeta functions (so called the zeta functions of *prehomogeneous vector spaces*) to algebraic number theory along the line with the author's preprints [T06a, T06b], which is a generalization of Shintani's papers [Sh72, Sh75]. For simplicity we mainly consider the situation of [T06a]. We state some of the main results of [T06b] in Section 3.

We start with the main results of this note. Let k be a number field and $\mathcal{O}$ the ring of integers of k. Let $\mathfrak{M}_{\mathbb{R}}$ and $\mathfrak{M}_{\mathbb{C}}$ respectively the set of real places and complex places of k. Further let $\mathfrak{M}_\infty = \mathfrak{M}_{\mathbb{R}} \amalg \mathfrak{M}_{\mathbb{C}}$. We put $r_1 = \#\mathfrak{M}_{\mathbb{R}}$, $r_2 = \#\mathfrak{M}_{\mathbb{C}}$ and $n = [k : \mathbb{Q}]$. We denote by Δ_k, h_k and $\zeta_k(s)$ the absolute discriminant, the class number and the Dedekind zeta function of k, respectively.

To classify cubic extensions of k via the splitting type at places of $\mathfrak{M}_\infty$, we introduce the following notation. Let $k_\infty = k \otimes_{\mathbb{Q}} \mathbb{R}$. We fix a separable cubic k_∞-algebra $L_\infty = \prod_{v\in\mathfrak{M}_\infty} L_v$, where $L_v \in \{\mathbb{R}^3, \mathbb{R}\times\mathbb{C}\}$ if $v \in \mathfrak{M}_{\mathbb{R}}$ and $L_v = \mathbb{C}^3$ if $v \in \mathfrak{M}_{\mathbb{C}}$. Let

$$h(L_\infty, n) := \#\left\{(R,F) \,\middle|\, \begin{array}{l} F \text{ is a cubic extension of } k,\ F\otimes_{\mathbb{Q}}\mathbb{R} \cong L_\infty, \\ R \text{ is an order of } F \text{ containing } \mathcal{O}, \text{ and } N(\Delta_{R/\mathcal{O}}) = n. \end{array}\right\}.$$

Here $\Delta_{R/\mathcal{O}}$ is the relative discriminant of $R/\mathcal{O}$ (which is an integral ideal of $\mathcal{O}$) and $N(\Delta_{R/\mathcal{O}})$ is its ideal norm. We count pairs (R, F) up to isomorphism. We put $i(L_\infty) = \#\{v \in \mathfrak{M}_{\mathbb{R}} \mid L_v = \mathbb{R}^3\}$. The following is a main result of [T06a].

Theorem 1.1 *For any* $\varepsilon > 0$,

$$\sum_{n<X} h(L_\infty, n) = \frac{\mathfrak{A}_k}{3^{i(L_\infty)+r_2}} X + \frac{\mathfrak{B}_k}{3^{i(L_\infty)/2}} X^{5/6} + O(X^{\frac{5n-1}{5n+1}+\varepsilon}) \qquad (X\to\infty),$$

where we put

$$\mathfrak{A}_k := (\mathrm{Res}_{s=1}\,\zeta_k(s))\cdot\frac{\zeta_k(2)}{2^{r_1+r_2+1}}, \quad \mathfrak{B}_k := (\mathrm{Res}_{s=1}\,\zeta_k(s))\cdot\frac{3^{r_1+r_2/2}\zeta_k(1/3)}{5\cdot 2^{r_1+r_2}\Delta_k^{1/2}}\left(\frac{\Gamma(1/3)^3}{2\pi}\right)^n.$$

Remark 1.2 *The case $k = \mathbb{Q}$ is essentially known by Shintani [Sh75]. In the formula above, $X^{5/6}$-term is relevant when $n = 1, 2$.*

We explain one more theorem we consider in this note. We call a finite $\mathcal{O}$-algebra a *cubic algebra* if it is projective of rank 3 as an $\mathcal{O}$-module. We denote by $\mathcal{C}(\mathcal{O})$ the set of isomorphism classes of cubic algebras of $\mathcal{O}$. For a fractional ideal $\mathfrak{a}$ of k, we put $\mathcal{C}(\mathcal{O}, \mathfrak{a}) = \{R \in \mathcal{C}(\mathcal{O}) \mid \bigwedge^3 R \cong \mathfrak{a}\}$. It is known that $\mathcal{C}(\mathcal{O}, \mathfrak{a})$ depends only on the ideal class of $\mathfrak{a}$ and that $\mathcal{C}(\mathcal{O}) = \coprod_{\mathfrak{a} \in \mathrm{Cl}(k)} \mathcal{C}(\mathcal{O}, \mathfrak{a})$ (we use the same symbol $\mathfrak{a}$ to denote its ideal class.) In general for a projective $\mathcal{O}$-module M of rank m, the class of the ideal isomorphic to $\bigwedge^m M$ is called the *Steinitz class* of M. It is known that finite generated projective modules over a Dedekind domain are classified by the rank and the Steinitz class. For this fact, see Milnor's book [M71].

We count the number of $\mathcal{C}(\mathcal{O}, \mathfrak{a})$ for each $\mathfrak{a}$. More precisely, for each L_∞ we count the set $\mathcal{C}(\mathcal{O}, \mathfrak{a}, L_\infty) = \{R \in \mathcal{C}(\mathcal{O}, \mathfrak{a}) \mid R \otimes_{\mathbb{Z}} \mathbb{R} \cong L_\infty\}$. An interesting phenomenon we prove in the case k is a quadratic field is that, the Steinitz class is *not uniformly distributed* in the $X^{5/6}$-term if $\mathrm{Cl}(k)$ contains a non-trivial 3-torsion element. Let $\#(\mathrm{Aut}\,(R))$ be the cardinality of the automorphisms of R as an $\mathcal{O}$-algebra and $h_k^{(3)}$ the number of 3-torsions of $\mathrm{Cl}(k)$ (which is a power of 3.)

Theorem 1.3 *For any $\varepsilon > 0$,*

$$\sum_{\substack{R \in \mathcal{C}(\mathcal{O}, \mathfrak{a}, L_\infty) \\ N(\Delta_{R/\mathcal{O}}) \le X}} \frac{1}{\#(\mathrm{Aut}\,(R))} = \left(1 + \frac{1}{3^{i(L_\infty)+r_2}}\right) \frac{\mathfrak{A}_k}{h_k} X + \tau(\mathfrak{a}) \frac{\mathfrak{B}_k h_k^{(3)}}{3^{i(L_\infty)/2} h_k} X^{5/6} + O(X^{\frac{4n-1}{4n+1}+\varepsilon})$$

as $X \to \infty$. Here for $\mathfrak{a} \in \mathrm{Cl}(k)$, we put $\tau(\mathfrak{a}) = 1$ if there exists $\mathfrak{b} \in \mathrm{Cl}(k)$ such that $\mathfrak{a} = \mathfrak{b}^3$ and $\tau(\mathfrak{a}) = 0$ otherwise.

Our approach to prove the theorems above are the use of the zeta function theory of prehomogeneous vector spaces founded by Sato-Shintani [SS74]. In proving density theorems, this is an alternative approach to using reduction theory. These two approaches are both useful and have different strength. One advantage of zeta function theory is that we can obtain a sharp error term estimate because our zeta function satisfies the functional equation. For the reduction theory approach, see [DH71] or [B05], for example.

2 The space of binary cubic forms and the zeta function of Sato-Shintani

We first prove Theorem 1.3 and after that Theorem 1.1. We first sketch the proof of Theorem 1.3 and next of Theorem 1.1. Theorem 1.3 is proved by studying the space of binary cubic forms $(\mathrm{GL}_2, \mathrm{Sym}^3\mathrm{Aff}^2)$ both algebraically and analytically. The ideal class group $\mathrm{Cl}(k)$ naturally arises from both parts.

Let G be the general linear group of rank 2 and V the space of binary cubic forms;

$$G := \mathrm{GL}_2,$$
$$V := \{x(u, v) = au^3 + bu^2v + cuv^2 + dv^3 \mid a, b, c, d \in \mathrm{Aff}\}.$$

We define the action of G on V by

$$(g \cdot x)(u,v) = \frac{1}{\det(g)} x((u,v)g).$$

The twist by $\det(g)^{-1}$ is to make the representation faithful. For $x = x(u,v) = au^3 + bu^2v + cuv^2 + dv^3 \in V$, let $P(x)$ be the discriminant;

$$P(x) := b^2c^2 - 4ac^3 - 4b^3d + 18abcd - 27a^2d^2.$$

Then we have $P(g \cdot x) = (\det g)^2 P(x)$.

2.1 Parameterizations of cubic algebras (algebraic part)

We consider a group theoretical parameterization of $\mathcal{C}(\mathcal{O}, \mathfrak{a})$, which is a natural generalization of Delone-Faddeev's correspondence [DF64] over $\mathbb{Z}$.

Definition 2.1 *We put*

$$\begin{aligned} V(k) \supset V_{\mathfrak{a}} &:= \mathfrak{a} \oplus \mathcal{O} \oplus \mathfrak{a}^{-1} \oplus \mathfrak{a}^{-2} \\ &:= \{au^3 + bu^2v + cuv^2 + dv^3 \mid a \in \mathfrak{a}, b \in \mathcal{O}, c \in \mathfrak{a}^{-1}, d \in \mathfrak{a}^{-2}\}, \\ G(k) \supset G_{\mathfrak{a}} &:= \begin{pmatrix} \mathcal{O} & \mathfrak{a} \\ \mathfrak{a}^{-1} & \mathcal{O} \end{pmatrix}^{\times} := \left\{ \begin{pmatrix} p & q \\ r & s \end{pmatrix} \mid p \in \mathcal{O}, q \in \mathfrak{a}, r \in \mathfrak{a}^{-1}, s \in \mathcal{O}, ps - qr \in \mathcal{O}^{\times} \right\}. \end{aligned}$$

Then $G_{\mathfrak{a}} \cdot V_{\mathfrak{a}} \subset V_{\mathfrak{a}}$.

Remark 2.2 *We can regard* $V_{\mathfrak{a}}$ *as the space of cubic maps from* $\mathcal{O} \oplus \mathfrak{a}$ *to* $\mathfrak{a} \cong \bigwedge^2(\mathcal{O} \oplus \mathfrak{a})$.

Proposition 2.3 (1) *There exists the canonical bijection between* $\mathcal{C}(\mathcal{O}, \mathfrak{a})$ *and* $G_{\mathfrak{a}} \backslash V_{\mathfrak{a}}$ *making the following diagram commutative.*

$$\begin{tikzcd} G_{\mathfrak{a}} \backslash V_{\mathfrak{a}} \arrow[r] \arrow[d, "P"] & \mathcal{C}(\mathcal{O}, \mathfrak{a}) \arrow[d, "\text{discriminant}"] \\ (\mathcal{O}^{\times})^2 \backslash \mathfrak{a}^{-2} \arrow[r, "\times \mathfrak{a}^2"] & \{\text{integral ideals of } \mathcal{O}\}. \end{tikzcd}$$

Here, the right vertical arrow is to take the discriminant, and the low horizontal arrow is given by multiplying $\mathfrak{a}^2$. *Moreover, this diagram is functorial with respect to the ring homomorphism of Dedekind domains.*

(2) *For each* $R \in \mathcal{C}(\mathcal{O}, \mathfrak{a})$, *we denote by* x_R *the corresponding element in* $G_{\mathfrak{a}} \backslash V_{\mathfrak{a}}$. *Then* $\mathrm{Aut}\,(R) \cong \mathrm{Stab}(G_{\mathfrak{a}}; x_R) := \{\gamma \in G_{\mathfrak{a}} \mid \gamma \cdot x_R = x_R\}$.

Construction of the map For each $R \in \mathcal{C}(\mathcal{O}, \mathfrak{a})$, the binary cubic form

$$x_R \colon R/\mathcal{O} \longrightarrow \wedge^2(R/\mathcal{O}), \qquad \xi \longmapsto \xi \wedge \xi^2$$

can be regarded an element of $G_{\mathfrak{a}} \backslash V_{\mathfrak{a}}$, since $R \in \mathcal{C}(\mathcal{O}, \mathfrak{a})$ implies $R/\mathcal{O} \cong \mathcal{O} \oplus \mathfrak{a}$. This map $R \mapsto x_R$ gives the desired bijection. For the proof see [T06a, Proposition 3.6].

2.2 Zeta function (analytic part)

The representation (G, V) is an example of what is called the *prehomogeneous vector space* and for such a representation, M. Sato and Shintani [SS74] associated a zeta function. This zeta function is a Dirichlet series satisfying certain functional equation. We recall the adelic version of the zeta function for (G, V). Let $V' = \{x \in V \mid P(x) \neq 0\}$. Let $\mathbb{A}$ be the adele ring of k. We denote by $\mathscr{S}(V(\mathbb{A}))$ the space of Schwartz-Bruhat functions on $V(\mathbb{A})$. Let $\mathrm{Cl}(k)^*$ be the set of characters of $\mathrm{Cl}(k)$. Via the canonical surjection $\mathbb{A}^\times/k^\times \to \mathbb{A}^\times/k_\infty^\times k^\times \widehat{\mathcal{O}}^\times \cong \mathrm{Cl}(k)$, we regard elements of $\mathrm{Cl}(k)^*$ as characters on $\mathbb{A}^\times/k^\times$.

Definition 2.4 *For $\Phi \in \mathscr{S}(V_{\mathbb{A}})$, $s \in \mathbb{C}$, $\omega \in \mathrm{Cl}(k)$, we define*

$$Z(\Phi, s, \omega) := \int_{G(\mathbb{A})/G(k)} \omega(\det g) |\det g|_{\mathbb{A}}^{2s} \sum_{x \in V'(k)} \Phi(g \cdot x) dg$$

and call it the global zeta function.

We consider the meaning of this function. As usual, let $\widehat{\mathcal{O}} = \widehat{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathcal{O}$ where $\widehat{\mathbb{Z}} = \prod_p \mathbb{Z}_p$ and $\mathbb{A}_{\mathrm{f}} = \widehat{\mathcal{O}} \otimes_{\mathcal{O}} k$. Recall that we put $k_\infty = k \otimes_{\mathbb{Q}} \mathbb{R}$. For our purpose, we assume the following.

Assumption 2.5 *We assume $\Phi \in \mathscr{S}(V(\mathbb{A}))$ to be of the form $\Phi = \Phi_\infty \otimes \Phi_f$, where Φ_f is the characteristic function on $V(\widehat{\mathcal{O}}) \subset V(\mathbb{A}_{\mathrm{f}})$, and Φ_∞ is an arbitrary Schwartz-Bruhat function on $V(k_\infty)$.*

For a fractional ideal $\mathfrak{a}$, we use the same symbol $\mathfrak{a}$ to denote the corresponding finite idele, which is well defined up to $\widehat{\mathcal{O}}^\times$-multiple. That is, $\mathfrak{a} \in \mathbb{A}_{\mathrm{f}}^\times (\subset \mathbb{A}^\times)$ is characterized by $\mathfrak{a} = k \cap \mathfrak{a}\widehat{\mathcal{O}}$. It is known that the double coset space $G(k_\infty)G(\widehat{\mathcal{O}})\backslash G(\mathbb{A})/G(k)$ is represented by $\mathrm{Cl}(k)$. More precisely, we have

$$G(\mathbb{A}) = \coprod_{\mathfrak{a} \in \mathrm{Cl}(k)} G(k_\infty)G(\widehat{\mathcal{O}}) \begin{pmatrix} 1 & 0 \\ 0 & \mathfrak{a} \end{pmatrix} G(k).$$

According to this decomposition, we define the partial zeta integral by

$$Z_{\mathfrak{a}}(\Phi, s) := \int_{G(k_\infty)G(\widehat{\mathcal{O}})\left(\begin{smallmatrix} 1 & 0 \\ 0 & \mathfrak{a} \end{smallmatrix}\right)G(k)/G(k)} |\det g|_{\mathbb{A}}^{2s} \sum_{x \in V'(k)} \Phi(g \cdot x) dg.$$

Then since $\omega(\det(G(k_\infty)G(\widehat{\mathcal{O}})G(k))) = \omega(k_\infty^\times \widehat{\mathcal{O}}^\times k^\times) = 1$, we have

$$Z(\Phi, s, \omega) = \sum_{\mathfrak{a} \in \mathrm{Cl}(k)} \omega(\mathfrak{a}) Z_{\mathfrak{a}}(\Phi, s).$$

Definition 2.6 (1) *Let $\mathcal{T}_\infty$ be the set of all possible separable cubic algebras L_∞ of the form $\prod_{v \in \mathfrak{M}_\infty} L_v$. Then set of orbits $G(k_\infty)\backslash V'(k_\infty)$ corresponds bijectively to $\mathcal{T}_\infty$. We denote by $V_{L_\infty} \subset V'(k_\infty)$ the $G(k_\infty)$-orbit corresponding to L_∞. (This* should not be confused *to the set of L_∞ rational points of V.) We define the local zeta function at $\mathfrak{M}_\infty$ by*

$$\mathcal{Z}_{L_\infty}(\Phi_\infty, s) = \int_{G(k_\infty)} |P(g_\infty x)|_\infty^s \Phi_\infty(g_\infty \cdot x) dg_\infty$$

where x is an arbitrary element of V_{L_∞}. Here the invariant measure dg_∞ on $G(k_\infty)$ is chosen so that $dg = dg_{\mathrm{f}} dg_\infty$ where dg_{f} is the invariant measure on $G(\mathbb{A}_{\mathrm{f}})$ giving the volume of $G(\widehat{\mathcal{O}})$ one.

(2) *We define*

$$\xi(L_\infty, \mathfrak{a}; s) = \sum_{R \in \mathcal{C}(\mathcal{O}, \mathfrak{a}, L_\infty)} \frac{(\#\mathrm{Aut}\,(R))^{-1}}{N(\Delta_{R/\mathcal{O}})^s}.$$

Proposition 2.7 *We have*

$$Z_{\mathfrak{a}}(\Phi, s) = \sum_{L_\infty \in \mathcal{T}_\infty} \mathcal{Z}_{L_\infty}(\Phi_\infty, s) \xi(L_\infty, \mathfrak{a}; s).$$

Let $G(\widehat{\mathcal{O}})_{\mathfrak{a}} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & \mathfrak{a} \end{smallmatrix}\right)^{-1} G(\widehat{\mathcal{O}}) \left(\begin{smallmatrix} 1 & 0 \\ 0 & \mathfrak{a} \end{smallmatrix}\right)$ and $\Phi_{\mathfrak{a}}(x) = \Phi(\left(\begin{smallmatrix} 1 & 0 \\ 0 & \mathfrak{a} \end{smallmatrix}\right) x)$. Then since $\Phi_{\mathfrak{a}}$ is $G(\widehat{\mathcal{O}})_{\mathfrak{a}}$-invariant, $|\det(G(\widehat{\mathcal{O}})_{\mathfrak{a}} G(k))|_{\mathbb{A}} = 1$ and $|\mathfrak{a}|_{\mathbb{A}} = N(\mathfrak{a})^{-1}$, we have

$$Z_{\mathfrak{a}}(\Phi, s) = N(\mathfrak{a})^{-2s} \int_{G(k_\infty)G(\widehat{\mathcal{O}}_{\mathfrak{a}})/G(k) \cap G(k_\infty)G(\widehat{\mathcal{O}})_{\mathfrak{a}}} |\det g_\infty|_\infty^{2s} \sum_{x \in V'(k)} \Phi_{\mathfrak{a}}(g_\infty \cdot x) dg_\infty dg_{\mathrm{f}}.$$

We can easily see that, as a subset of $V(k_\infty)$ or $G(k_\infty)$,

$$V(k) \cap \left(\begin{smallmatrix} 1 & 0 \\ 0 & \mathfrak{a} \end{smallmatrix}\right)^{-1} V(\widehat{\mathcal{O}}) = V_{\mathfrak{a}},$$
$$G(k) \cap \left(\begin{smallmatrix} 1 & 0 \\ 0 & \mathfrak{a} \end{smallmatrix}\right)^{-1} G(\widehat{\mathcal{O}}) \left(\begin{smallmatrix} 1 & 0 \\ 0 & \mathfrak{a} \end{smallmatrix}\right) = G_{\mathfrak{a}}.$$

Hence

$$Z_{\mathfrak{a}}(\Phi, s) = N(\mathfrak{a})^{-2s} \int_{G(k_\infty)/G_{\mathfrak{a}}} |\det g_\infty|_\infty^{2s} \sum_{x \in V_{\mathfrak{a}} \cap V'(k)} \Phi_\infty(g_\infty \cdot x) dg_\infty \times \int_{G(\widehat{\mathcal{O}})_{\mathfrak{a}}} dg_{\mathrm{f}}.$$

Since $G(\mathbb{A}_{\mathrm{f}})$ is unimodular, $\int_{G(\widehat{\mathcal{O}})_{\mathfrak{a}}} dg_{\mathrm{f}} = \int_{G(\widehat{\mathcal{O}})} dg_{\mathrm{f}} = 1$. Now by the usual unfolding method we have

$$Z_{\mathfrak{a}}(\Phi, s) = \sum_{L_\infty \in \mathcal{T}_\infty} \mathcal{Z}_{L_\infty}(\Phi_\infty, s) \left(\sum_{x \in G_{\mathfrak{a}} \backslash (V_{\mathfrak{a}} \cap V_{L_\infty})} \frac{(\#\mathrm{Stab}(G_{\mathfrak{a}}; x))^{-1}}{N(\mathfrak{a})^{2s} |P(x)|_\infty^s} \right).$$

Now the proposition follows from Proposition 2.3.

2.3 Analytic properties of the zeta function and Tauberian theorem

Let $\{a_n\}$ be a positive sequence. We put

$$A(X) = \sum_{n \le X} a_n, \qquad a(s) = \sum_{n \ge 1} a_n n^{-s}.$$

Then Tauberian theorem states that, from analytic properties of $a(s)$ as a complex function, we can obtain some informations on the asymptotic behavior of $A(X)$ as $X \to \infty$. If $a(s)$ is the Dirichlet series $\xi(L_\infty, a; s)$ in Definition 2.6 then $A(X)$ is nothing but the left hand side of Theorem 1.3. Hence we can reduce the proof of Theorem 1.3 to the analysis of $Z(\Phi, s, \omega)$. Since V is a vector space, we can use the Fourier analysis to study the zeta functin. The analytic properties of $Z(\Phi, s, \omega)$ was studied extensively by Shintani [Sh72] when $k = \mathbb{Q}$ with the trivial character and later generalized by Wright [Wr85] using adelic language. For $\omega \in \mathrm{Cl}(k)^*$, let $\delta(\omega) = 1$ if ω is trivial and $\delta(\omega) = 0$ otherwise.

Theorem 2.8 (Shintani [Sh72], Wright [Wr85]) *The zeta function $Z(\Phi, s, \omega)$ can be continued holomorphically to the entire $\mathbb{C}$ except for possible simple poles at $s = 0, 1/6, 5/6, 1$. We have*

$$\mathrm{Res}_{s=1} Z(\Phi, s, \omega) = \delta(\omega)\Sigma_a(\Phi), \qquad \mathrm{Res}_{s=5/6} Z(\Phi, s, \omega) = \delta(\omega^3)\Sigma_b(\Phi)$$

for appropriate invariant distributions Σ_a, Σ_b. Also it satisfies the functional equation

$$Z(\Phi, s, \omega) = Z(\widehat{\Phi}, 1-s, \omega^{-1})$$

where $\widehat{\Phi}$ is an appropriate Fourier transform of Φ.

For the definitions of Σ_a, Σ_b and $\widehat{\Phi}$, see [Wr85]. From this theorem, combined with archimedean local theory, we know the functional equation and residue formula of $\xi(L_\infty, \mathfrak{a}; s)$.

Corollary 2.9 (Datskovsky-Wright [DW86]) *The Dirichlet series $\xi(L_\infty, \mathfrak{a}; s)$ can be continued holomorphically to the whole complex plane except for a simple pole at $s = 1$ and a possible simple pole at $s = 5/6$. The residues are $(1 + 3^{-r(L_\infty) - r_2})\mathfrak{A}_k/h_k$ and $\tau(\mathfrak{a})(5/6)\mathfrak{B}_k 3^{-r(L_\infty)/2}(h_k^{(3)}/h_k)$, respectively. Also it satisfies functional equation of the form*

$$\xi(L_\infty, \mathfrak{a}; 1-s) = \left(\Gamma(s)^2\Gamma(s+\frac{1}{6})\Gamma(s-\frac{1}{6})\right)^n \sum_{\lambda \in \Lambda} P_\lambda(e^{\pi\sqrt{-1}s}, e^{-\pi\sqrt{-1}s})\hat{\xi}_\lambda(s),$$

where Λ is a finite set, $P_\lambda(x, y)$ are polynomials in x, y with degrees do not exceed $2n$, and $\hat{\xi}_\lambda(s)$ are certain Dirichlet series with the absolute convergence domains $\mathrm{Re}(s) > 1$.

Now Theorem 1.3 follows from Sato-Shintani's Tauberian theorem [SS74] which is a modification of Landau's Tauberian theorem [L15].

2.4 Contributions from "reducible" algebras

The step from Theorem 1.3 to Theorem 1.1 is to separate the "reducible" algebras i.e., $R \in \mathcal{O}$ with $R \otimes k$ not fields. Let us define

$$G \supset B := \left\{ \begin{pmatrix} * & 0 \\ * & * \end{pmatrix} \right\}, \quad V \supset W := \{v(bu^2 + cuv + dv^2) \mid b, c, d \in \mathrm{Aff}\}.$$

Then (B, W) is also a prehomogeneous vector space. Shintani [Sh75] showed in the case $k = \mathbb{Q}$ that the representation (B, W) parameterizes the reducible algebras. We see in [T06a, Section 3] that it is true for a general number field. We briefly recall the argument. Let $\mathfrak{a}$, $\mathfrak{c}$ be non-zero fractional ideals of k.

Definition 2.10 *We put*

$$B(k) \supset B_{\mathfrak{a},\mathfrak{c}} = \left\{ \begin{pmatrix} t & 0 \\ u & p \end{pmatrix} \middle| \ t, p \in \mathcal{O}^{\times}, u \in \mathfrak{a}^{-1}\mathfrak{c}^{-2} \right\},$$
$$W(k) \supset W_{\mathfrak{a},\mathfrak{c}} = \{y \mid y_1 \in \mathfrak{c}, y_2 \in \mathfrak{a}^{-1}\mathfrak{c}^{-1}, y_3 \in \mathfrak{a}^{-2}\mathfrak{c}^{-3}\}.$$

Then $W_{\mathfrak{a},\mathfrak{c}}$ is $B_{\mathfrak{a},\mathfrak{c}}$-invariant.

Let $V_{\mathfrak{a}}^{\mathrm{red}} = \{x \in V_{\mathfrak{a}} \mid R_x \otimes k \text{ is not a field}\}$ where we denote by $R_x \in \mathcal{C}(\mathcal{O}, \mathfrak{a})$ the element corresponding to $x \in V_{\mathfrak{a}}$. We fix $\mathfrak{a}$.

Proposition 2.11 *For each $\mathfrak{c}$, there exists the canonical map $\psi_{\mathfrak{a},\mathfrak{c}}\colon B_{\mathfrak{a},\mathfrak{c}} \backslash W_{\mathfrak{a},\mathfrak{c}} \to G_{\mathfrak{a}} \backslash V_{\mathfrak{a}}^{\mathrm{red}}$ which preserve the value of P up to $(\mathcal{O}^{\times})^2$-multiple. Moreover,*

$$\coprod_{\mathfrak{c} \in \mathrm{Cl}(k)} B_{\mathfrak{a},\mathfrak{c}} \backslash W_{\mathfrak{a},\mathfrak{c}} \longrightarrow G_{\mathfrak{a}} \backslash V_{\mathfrak{a}}^{\mathrm{red}}$$

is "almost bijective".

For the precise meaning of "almost bijective", see [T06a, Proposition 3.12]. We give the construction of $\psi_{\mathfrak{a},\mathfrak{c}}$. We fix $q, s \in k$ such that $q\mathfrak{a}^{-1} + s\mathcal{O} = \mathfrak{c}$. Then, $q \in \mathfrak{a}\mathfrak{c}$, $s \in \mathfrak{c}$, and also there exist $p \in \mathfrak{c}^{-1}, r \in \mathfrak{a}^{-1}\mathfrak{c}^{-1}$ such that $ps - qr \in \mathcal{O}^{\times}$. We can choose such elements because $\mathcal{O}$ is a Dedekind domain. We put $g_{\mathfrak{a},\mathfrak{c}} = \left(\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right) \in G(k)$. We define

$$\widetilde{\psi}_{\mathfrak{a},\mathfrak{c}}\colon W_{\mathfrak{a},\mathfrak{c}} \longrightarrow V_{\mathfrak{a},\mathfrak{c}}^{\mathrm{red}}, \qquad y \longmapsto g_{\mathfrak{a},\mathfrak{c}} y.$$

We see by computation that $g_{\mathfrak{a},\mathfrak{c}}^{-1} G_{\mathfrak{a}} g_{\mathfrak{a},\mathfrak{c}} \cap B(k) = B_{\mathfrak{a},\mathfrak{c}}$. This shows that the map $\widetilde{\psi}_{\mathfrak{a},\mathfrak{c}}$ induces a well defined map $\psi_{\mathfrak{a},\mathfrak{c}}\colon B_{\mathfrak{a},\mathfrak{c}} \backslash W_{\mathfrak{a},\mathfrak{c}} \to G_{\mathfrak{a}} \backslash V_{\mathfrak{a},\mathfrak{c}}^{\mathrm{red}}$. It turns out that this map does not depend on the choice of $g_{\mathfrak{a},\mathfrak{c}}$.

Hence the same analytic process yields the asymptotic formula of reducible algebras. The global theory for (B, W) was done by Shintani [Sh75] and the author [T06a, Section 4] gave an adelic version of Shintani's treatment. By subtracting this from the formula of Theorem 1.3, we have Theorem 1.1. For details, see [T06a].

3 Splitting conditions at non-archimedean places

In Theorems 1.1, 1.3 we classify $\mathcal{C}(\mathcal{O})$ via the splitting type at infinite places. Recently the author [T06b] consider the same problem under imposing finite number of splitting conditions at non-archimedean places. We state some of its main results here.

Let k be a general number field. We put $n = [k : \mathbb{Q}]$. For a place v of k let k_v be the completion of k at v. Let T be a finite set of places. Take a separable cubic algebra L_v of k_v for each $v \in T$ and let $L_T = (L_v)_{v\in T}$ the T-tuple. We let

$$\mathcal{C}(\mathcal{O}, L_T)^{\mathrm{ird}} := \left\{ R \in \mathcal{C}(\mathcal{O}) \;\middle|\; \begin{array}{l} F = R \otimes_{\mathcal{O}} k \text{ is a cubic field extension of } k, \text{ and} \\ F \otimes_k k_v \cong L_v \text{ for all } v \in T. \end{array} \right\}.$$

We define

$$\vartheta_{L_T}^{\mathrm{ird}}(s) := \sum_{R\in\mathcal{C}(\mathcal{O},L_T)^{\mathrm{ird}}} \frac{\#(\mathrm{Aut}\,(R))^{-1}}{N(\Delta_{R/\mathcal{O}})^s},$$
$$h_{L_T}(X) := \#\{R \in \mathcal{C}(\mathcal{O}, L_T)^{\mathrm{ird}} \mid N(\Delta_{R/\mathcal{O}}) < X\}.$$

Theorem 3.1 *There exist constants $\mathfrak{A}_{L_T}$ and $\mathfrak{B}_{L_T}$ described explicitly such that;*

(1) *$\vartheta_{L_T}^{\mathrm{ird}}(s)$ has meromorphic continuation to the whole complex plane which is holomorphic for $\mathrm{Re}(s) > 1/2$ except for simple poles at $s = 1$ and $5/6$ with residues $\mathfrak{A}_{L_T}$ and $\mathfrak{B}_{L_T}$, respectively, and*

(2) *for any $\varepsilon > 0$,*

$$h_{L_T}(X) = \mathfrak{A}_{L_T}X + (5/6)^{-1}\mathfrak{B}_{L_T}X^{5/6} + O(X^{\frac{5n-1}{5n+1}+\varepsilon}) \qquad (X \to \infty).$$

Note that the $X^{5/6}$-term in the formula is relevant only when $n = 1, 2$. We give the formulae of $\mathfrak{A}_{L_T}$ and $\mathfrak{B}_{L_T}$. We denote by $\mathfrak{M}_{\mathrm{f}}$ the set of all finite places. For $v \in \mathfrak{M}_{\mathrm{f}}$, let q_v be the order of the residue field of k_v. We put $\theta_{L_v} = \#(\mathrm{Aut}_{k_v\text{-algebra}}(L_v))$. For a non-archimedean local field K with the order of residue field q, we define its local zeta function by $\zeta_K(s) = (1 - q^{-s})^{-1}$. The cubic algebra L_v is in general a product of local fields. We define $\zeta_{L_v}(s)$ as the product of the zeta functions of those fields. The relative discriminant Δ_{L_v/k_v} is also defined as the product of relative discriminants of those local fields. We denote by Δ_{L_v} its norm. We put $i_\infty(L_T) = \#\{v \in \mathfrak{M}_{\mathbb{R}} \mid L_v = \mathbb{R}^3\}$. We give the value in case of $T \supset \mathfrak{M}_\infty$. The general case is easily obtained from this by taking a suitable summation.

Theorem 3.2 *When $T \supset \mathfrak{M}_\infty$, the constants $\mathfrak{A}_{L_T}$ and $\mathfrak{B}_{L_T}$ are given by*

$$\mathfrak{A}_{L_T} = \frac{\mathrm{Res}_{s=1}\,\zeta_k(s)\cdot\zeta_k(2)}{2^{r_1+r_2+1}3^{i_\infty(L_T)+r_2}} \prod_{v\in T\cap\mathfrak{M}_{\mathrm{f}}} \alpha_v(L_v),$$
$$\mathfrak{B}_{L_T} = \frac{\mathrm{Res}_{s=1}\,\zeta_k(s)\cdot\zeta_k(1/3)}{6\Delta_k^{1/2}(\sqrt{3})^{r_2-i_\infty(L_T)}} \left(\frac{3\Gamma(1/3)^3}{2\pi}\right)^n \prod_{v\in T\cap\mathfrak{M}_{\mathrm{f}}} \beta_v(L_v),$$

where

$$\alpha_v(L_v) = \frac{(1-q_v^{-1})(1-q_v^{-2})}{(1-q_v^{-4})(1-q_v^{-5})} \cdot \theta_{L_v}^{-1}\Delta_{L_v}^{-1} \cdot \frac{\zeta_{L_v}(2)}{\zeta_{L_v}(4)},$$

$$\beta_v(L_v) = \frac{(1-q_v^{-1/3})(1-q_v^{-1})}{(1-q_v^{-10/3})(1-q_v^{-4})} \cdot \theta_{L_v}^{-1}\Delta_{L_v}^{-1} \cdot \frac{\zeta_{L_v}(1/3)\zeta_{L_v}(5/3)}{\zeta_{L_v}(2/3)\zeta_{L_v}(10/3)}.$$

Let $v \in \mathfrak{M}_{\mathrm{f}}$. We see by computation that $\sum_{L_v} \alpha_v(L_v) = \sum_{L_v} \beta_v(L_v) = 1$ where L_v runs through all the separable cubic algebras of k_v. Hence $\alpha_v(L_v)$ and $\beta_v(L_v)$ give the proportion of the contributions of cubic algebras with local splitting type L_v. The computation of $\alpha_v(L_v)$ is reduced to the determination of certain orbital volume in a v-adic vector space. The meaning of $\beta_v(L_v)$ is more subtle and the computation requires a careful local theory.

4 Quartic case

Let $V = (\mathrm{Sym}^2\mathrm{Aff}^3)^* \otimes \mathrm{Aff}^2$ be the space of pairs of ternary quadratic forms. The group $G = \mathrm{GL}_3 \times \mathrm{GL}_2$ naturally acts on V as a linear representation. It is known that there exist a non-zero relative invariant polynomial P in V and $V' := \{x \in V \mid P(x) \neq 0\}$ is a single orbit over algebraically closed fields. For any field k, Wright and Yukie [WY92] showed that the set of non-degenerate rational orbits $G(k)\backslash V'(k)$ corresponds bijectively to the set of separable quartic algebras of k. Hence we can regard this representation as the quartic analogy of the space of binary cubic forms, thus it is naturally to carry out the similar program to find the density theorems of distributions of discriminants of quartic algebras or analytic properties of the Dirichlet series counting quartic algebras.

As for the algebraic part, Bhargava [B04] recently discovered that the set of integral orbits $G(\mathbb{Z})\backslash V(\mathbb{Z})$ corresponds bijectively to the set $\{(R, S)\}$ where R is a quartic ring and S is a cubic resolvent ring of R over $\mathbb{Z}$. (For the notion "resolvent ring" due to Bhargava, see [B04].) This correspondence has a direct generalization to over a Dedekind domain as we did for the cubic case in Proposition 2.3. The proof will be appear in elsewhere. Also there are three pairs (P_i, W_i) parameterizing the "reducible" algebras, where P_i are parabolic subgroups and W_i their invariant subspaces.

For the analytic part, the global theory of the quartic case was achieved by Yukie [Y93] with large amount of technical computations. The remaining important problem is the global theory for those (P_i, W_i). We hope this to carry out in the future.

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