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INTERSECTION NUMBERS OF HECKE CYCLES ON HILBERT MODULAR VARIETIES

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ABSTRACT. Let $\mathcal{O}$ be the ring of integers of a totally real number field E and set $\mathbf{G} := \mathrm{Res}_{E/\mathbb{Q}}(\mathrm{GL}_2)$. Fix an ideal $\mathfrak{c} \subset \mathcal{O}$. For each ideal $\mathfrak{m} \subset \mathcal{O}$ let $T(\mathfrak{m})$ denote the $\mathfrak{m}$th Hecke operator associated to the standard compact open subgroup $U_0(\mathfrak{c})$ of $\mathbf{G}(\mathbb{A}^f)$. Setting

$$X_0(\mathfrak{c}) := \mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A})/K_\infty U_0(\mathfrak{c}),$$

where K_∞ is a certain subgroup of $\mathbf{G}(\mathbb{R})$, we use $T(\mathfrak{m})$ to define a Hecke cycle

$$Z(\mathfrak{m}) \in IH_{2[E:\mathbb{Q}]}(X_0(\mathfrak{c}) \times X_0(\mathfrak{c})).$$

Here $IH_\bullet$ denotes intersection homology. We use Zucker's conjecture (proven by Looijenga and independently by Saper and Stern) to obtain a formula relating the intersection numbers $Z(\mathfrak{m}) \cdot Z(\mathfrak{n})$ to the trace of ${}^*T(\mathfrak{m}) \circ T(\mathfrak{n})$ considered as an endomorphism of the space of Hilbert cusp forms on $U_0(\mathfrak{c})$.

1. SYNOPSIS

This note is a synopsis of the main results of the forthcoming paper [Get] as described during a talk at RIMS. We begin by recalling the main theorem of Hirzebruch and Zagier's famous paper [HZ76]. In their paper they examine a set of pairs of CM points, namely the intersection points of certain "Hirzebruch-Zagier cycles" Z_m which are the the closures of images of modular "embeddings" of the usual (affine) modular curve $Y_0(p)$ and certain compact Shimura curves into the Hilbert modular surface $\mathrm{SL}_2(\mathcal{O}_{\mathbb{Q}(\sqrt{p})})\backslash\mathfrak{H}^2$ in a toroidal compactification. Here $\mathfrak{H}$ is the usual complex upper half plane, $p \equiv 1 \pmod 4$ is a prime and $\mathcal{O}_{\mathbb{Q}(\sqrt{p})}$ is the ring of integers of $\mathbb{Q}(\sqrt{p})$. In particular, if $Z_m \cdot Z_n$ denotes the "number of intersections" of Z_m and Z_n, then the bulk of [HZ76] is devoted to proving that for each $m \in \mathbb{Z}_{>0}$ the generating series

$$\sum_{n=0}^{\infty}(Z_m \cdot Z_n)q^n \tag{1.1}$$

is a weight 2 modular form for $\Gamma_0(p)$ with character $(\frac{p}{\cdot})$. Here $Z_m \cdot Z_0$ is essentially the volume of Z_m and $q := e^{2\pi iz}$ for $z \in \mathfrak{H}$.

In this paper we provide a kind of generalization of Hirzebruch-Zagier. Since their seminal work, a number of cohomology theories useful for studying topological intersection theory on Hermitian locally symmetric spaces such as Hilbert modular varieties have developed, including intersection homology, L^2-cohomology, and Harder's theory of Eisenstein cohomology. These theories have intrinsic interest, but our treatment of them is utilitarian; they provide a natural framework for studying the interplay between intersection theory on modular varieties and the coefficients of modular forms. Our goal is to revisit Hirzebruch-Zagier armed with these theories with the idea of proving comparison theorems for pairings that arise naturally in the context of intersection homology on the one hand, and Hecke algebras on the other.

Concretely, we consider the graphs of Hecke correspondences on the product of two Hilbert modular varieties associated to totally real fields of arbitrary dimension. In order to make this precise, we must develop some notation: Let E be a totally real number field with degree $[E : \mathbb{Q}] = n$, ring of integers $\mathcal{O}$, and narrow class number h^+; thus h^+ is the order of the ray class group modulo β_∞, the product of the infinite places. To every ideal $\mathfrak{c} \subset \mathcal{O}$ we associate the Hilbert modular variety

$$Y_0(\mathfrak{c}) := \bigcup_{j=1}^{h^+} \Gamma_j\backslash\mathfrak{H}^n$$

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via a well-known construction (see [vdG88, §I.7]). This variety has h^+ components, each of complex dimension n (the Γ_j are discrete subgroups of $(\mathrm{GL}_2(\mathbb{R})^+)^n$). For convenience, denote its Baily-Borel compactification by $X_0(\mathfrak{c})$. We then define, for every ideal $\mathfrak{m} \subset \mathcal{O}$, a Hecke cycle

$$Z(\mathfrak{m}) \in IH_{2n}(X_0(\mathfrak{c}) \times X_0(\mathfrak{c})).$$

Here $IH_\bullet$ denotes Goresky and MacPherson's intersection homology theory (with middle perversity). The class $Z(\mathfrak{m})$ is determined by the Hecke operator $T(\mathfrak{m})$ for $X_0(\mathfrak{c})$.

Recall the standard compact open subgroups $U_0(\mathfrak{c}) \le (\mathrm{Res}_{E/\mathbb{Q}}\mathrm{GL}_2)(\mathbb{A}^f)$ of Hecke type, where $\mathfrak{c} \subset \mathcal{O}$ is an ideal as above and $\mathbb{A}^f$ denotes the finite adèles of $\mathbb{Q}$. Let $\mathbb{T}_\mathfrak{c}$ denote the Hecke algebra associated to $U_0(\mathfrak{c})$; thus $\mathbb{T}_\mathfrak{c}$ acts on the space $M(U_0(\mathfrak{c}))$ of weight $(2,\dots,2)$ Hilbert modular forms on $U_0(\mathfrak{c})$. Moreover, this action preserves the subspace $S(U_0(\mathfrak{c}))$ of Hilbert cusp forms. There are two natural bilinear pairings associated to $\mathbb{T}_\mathfrak{c}$. The first is just

$$\begin{array}{rcl} \mathbb{T}_\mathfrak{c} \times \mathbb{T}_\mathfrak{c} & \longrightarrow & \mathbb{C} \\ (T_1, T_2) & \longmapsto & \mathrm{Tr}({}^*T_1 \circ T_2), \end{array} \tag{1.2}$$

where we view T_1 and T_2 as endomorphisms of $S(U_0(\mathfrak{c}))$ and the adjoint is taken with respect to the Petersson inner product. Secondly, attached to each element $\sum c_\mathfrak{m} T(\mathfrak{m})$ of the subspace of $\mathbb{T}_\mathfrak{c}$ spanned by the $T(\mathfrak{m})$, we have the cycle $\sum c_\mathfrak{m} Z(\mathfrak{m})$, and thus can consider the pairing on these cycles induced by the generalized Poincaré pairing

$$IH_{2n}(X_0(\mathfrak{c}) \times X_0(\mathfrak{c})) \times IH_{2n}(X_0(\mathfrak{c}) \times X_0(\mathfrak{c})) \longrightarrow H_0(X_0(\mathfrak{c}) \times X_0(\mathfrak{c})) \longrightarrow \mathbb{C}.$$

Theorem 1.1 below, the main result of this work, is essentially a comparison of these two pairings. To state it, we set notation for the finite sum

$$\sigma'_\mathfrak{c}(\mathfrak{a}) := \sum_{\substack{\mathfrak{n} \supset \mathfrak{a} \\ \mathfrak{n}+\mathfrak{c}=\mathcal{O}}} \mathrm{N}(\mathfrak{n}^{-1}\mathfrak{a}), \tag{1.3}$$

where $\mathrm{N}(\mathfrak{n}) := \mathrm{Norm}_{E/\mathbb{Q}}(\mathfrak{n})$. The reader may recognize $\sigma'_\mathfrak{c}(\mathfrak{a})$ as a generalized divisor function that appears in the Fourier coefficients of a certain Eisenstein series. One can also relate $\sigma'_\mathfrak{c}(\mathfrak{m})$ in a simple manner to the volume of $Z(\mathfrak{m})$.

We are now ready to state Theorem 1.1. In our setting, in which we consider the product of two copies of a Hilbert modular variety $X_0(\mathfrak{c})$ associated to a totally real number field E of arbitrary degree n and narrow class number h^+, this theorem gives a formula for $Z(\mathfrak{m}) \cdot Z(\mathfrak{n})$ in terms of traces of Hecke operators and coefficients of Eisenstein series:

Theorem 1.1. *Let $\mathfrak{m}, \mathfrak{n} \subset \mathcal{O}$ be ideals. If $\mathfrak{m}, \mathfrak{n}$ are ideals that are equivalent in the ray class group modulo β_∞, then we have the formula*

$$Z(\mathfrak{m}) \cdot Z(\mathfrak{n}) = 2^n \left((-1)^n \mathrm{Tr}({}^*T(\mathfrak{m}) \circ T(\mathfrak{n})) + h^+ \sigma'_\mathfrak{c}(\mathfrak{m}) \sigma'_\mathfrak{c}(\mathfrak{n}) \right).$$

Otherwise, $Z(\mathfrak{m}) \cdot Z(\mathfrak{n}) = 0$.

Remark. We remark that the method of proof of Theorem 1.1, which is different from the approach of Hirzebruch and Zagier, boils down to a formal argument using a Lefschetz coincidence formula in intersection homology and Zucker's conjecture (now a theorem). Both of these hold for arbitrary Hermitian locally symmetric varieties and cohomology groups with coefficients in a representation, which indicates that some analogue of the formula in Theorem 1.1 should hold in great generality.

From the shape of the formulae in Theorem 1.1, one would expect that some analogue of the generating series (1.1) would be modular. This is indeed the case. More precisely, we have the following:

Theorem 1.2. *If $S(U_0(\mathfrak{c})) = S^{new}(U_0(\mathfrak{c}))$ and either $[E : \mathbb{Q}] > 1$ or $\mathfrak{c} \neq \mathcal{O}$, then for each (integral) ideal $\mathfrak{m} \subset \mathcal{O}$ the formal Fourier expansion*

$$\Phi_{\mathfrak{m},\mathfrak{c}}\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\right) := \delta(\mathfrak{c})(Z(\mathfrak{m}) \cdot Z(0))(y)|y| + \sum_{0 \ll \xi \in E} Z(\mathfrak{m}) \cdot Z(\xi y)|y| e(\mathrm{tr}(\xi i y_\infty))\chi_E(\xi x)$$

defines an element of $M(U_0(\mathfrak{c}))$. Here we set $Z(\mathfrak{a}) = 0$ if the ideal associated to $\mathfrak{a}$ is not integral, and $0 \ll \xi$ means that ξ is totally positive.

Here

$$\delta(\mathfrak{c}) := \begin{cases} 1 \text{ if } \mathfrak{c} = \mathcal{O}_E \\ 0 \text{ otherwise} \end{cases}$$

and $S^{\mathrm{new}}(U_0(\mathfrak{c}))$ is the new subspace (also known as the primitive subspace, see [Shi78, p. 652]). The constant term $(Z(\mathfrak{m}) \cdot Z(0))(y)$ is essentially the volume of $Z(\mathfrak{m})$ in one component and zero elsewhere.

Remark. One should compare Theorem 1.2 with the formula for the generating series (1.1) obtained by Zagier in [Zag76], which we now recall. Let $(,)$ denote the Petersson inner product (we won't specify the normalization), and denote by $\widehat{f}$ the Doi-Naganuma lift of an elliptic modular form f for $\Gamma_0(p)$ with nebentypus $(\frac{p}{\cdot})$ (see [Zag76]). From Zagier's results, one can reformulate the main identity of [HZ76] as

$$\sum_{n=0}^{\infty}(Z_m \cdot Z_n)q^n = \pi_+ \left(t(m)E_{2,p}(z) - \frac{4}{p^2}\sum_{n=1}^{\infty}\left({\sum}' \frac{(\widehat{f},\widehat{f})}{(f,f)^2}a_f(m)a_f(n)\right) q^n \right). \tag{1.4}$$

Here the prime indicates the summation over the basis of normalized weight two elliptic newforms (i.e. eigenforms for all the Hecke operators)

$$f(z) := \sum_{n=1}^{\infty} a_f(n)q^n$$

for $\Gamma_0(p)$ with character $(\frac{p}{\cdot})$, π_+ is the canonical projection to the plus space, $t(m)$ is a rational number depending only on m, and $E_{2,p}$ is a weight two Eisenstein series for $\Gamma_0(p)$ with character $(\frac{p}{\cdot})$ (see [Zag76, (98-99)]). In the course of the proof of Theorem 1.2, we shall see that $\Phi_{\mathfrak{m},\mathfrak{c}}$ can be similarly decomposed as the sum of an Eisenstein series and a sum of weighted newforms.

We can rephrase Theorem 1.2 in the language of modular forms as follows: let $M(\Gamma_j)$ be the space of classical Hilbert modular forms of weight $(2,\ldots,2)$ for Γ_j. Moreover, let $s_1,\ldots,s_{h^+}$ be idèles of E with $(s_j)_\nu = 1$ for each infinite place ν such that the the fractional ideals associated to $s_1,\ldots,s_{h^+}$ form a set of representatives for the classes in the ray class group modulo β_∞. Using this notation we can phrase Corollary 1.2 in terms of classical Hilbert modular forms simply by inverting the formulae for converting classical Fourier expansions to adelic Fourier expansions:

Corollary 1.3. *Suppose that* $\mathfrak{m}$ *is in the ray class of* s_j^{-1} *modulo* β_∞. *If* $S(U_0(\mathfrak{c})) = S^{\mathrm{new}}(U_0(\mathfrak{c}))$ *and either* $[E:\mathbb{Q}] > 1$ *or* $\mathfrak{c} \neq \mathcal{O}$, *then*

$$\delta(\mathfrak{c})\mathrm{N}(s_j)(Z(\mathfrak{m})\cdot Z(0))(s_j^{-1}) + \sum_{\substack{\xi\in s_j\\ 0\ll\xi}} Z(\mathfrak{m})\cdot Z(\xi s_j)\mathrm{N}(s_j)e(\mathrm{tr}(\xi z)) \in M(\Gamma_j(\mathfrak{c})).$$

Remark. The case $n=1$, $E=\mathbb{Q}$, $\mathfrak{c}=\mathbb{Z}$ is not covered by Theorem 1.2 or Corollary 1.3. However, using Theorem 1.1 in this case and an easy analogue of the argument given in [Get, §8], one can prove that

$$-\frac{1}{24}E_2(z) := -\frac{1}{24} + \sum_{n\geq 1}\left(\sum_{d|n} d\right) e(nz) = \mathrm{N}(s_j)(Z(\mathfrak{m})\cdot Z(0))(s_1^{-1}) + \sum_{\substack{\xi\in s_1\\ 0\ll\xi}} Z(\mathfrak{m})\cdot Z(\xi s_1^{-1})\mathrm{N}(s_j)e(\mathrm{tr}(\xi z)).$$

Here we choose $s_1 = \mathbb{Z}$. The function $E_2(z)$ is not a modular form (the space $M(\mathrm{SL}_2(\mathbb{Z})) = M_2(\mathrm{SL}_2(\mathbb{Z}))$ is zero dimensional). However, this "quasi-modular" function is often useful in the classical theory of elliptic modular forms. Furthermore, it is in some sense the first nontrivial example of a p-adic modular form (see [Ser73]).

Under the assumptions of Theorem 1.2, one can use the automorphic forms $\Phi_{\mathfrak{m},\mathfrak{c}}$ to define a $\mathbb{C}$-linear map Ψ by

$$\begin{array}{rccccc} \Psi: & HC(\mathfrak{c}) & \longrightarrow & M(U_0(\mathfrak{c})) & \longrightarrow & S(U_0(\mathfrak{c})) \\ & \sum \alpha_{\mathfrak{m}} Z(\mathfrak{m}) & \longmapsto & \sum \alpha_{\mathfrak{m}}\Phi_{\mathfrak{m},\mathfrak{c}} & & \end{array} \tag{1.5}$$

where $HC(\mathfrak{c})$ is the subspace of $IH_{2n}(X_0(\mathfrak{c}) \times X_0(\mathfrak{c}))$ spanned by the Hecke cycles $Z(\mathfrak{m})$ as $\mathfrak{m}$ varies over the ideals of $\mathcal{O}$ and the second map is the canonical projection. In [Get, §8] we prove that this map has full image:

Theorem 1.4. *If $S^{\mathrm{new}}(U_0(\mathfrak{c})) = S(U_0(\mathfrak{c}))$ and either $\mathfrak{c} \neq \mathcal{O}$ or $\mathfrak{c} = \mathcal{O}$ and $n \neq 1$, then the map*

$$\Psi : HC(\mathfrak{c}) \longrightarrow S(U_0(\mathfrak{c}))$$

of (1.5) is surjective.

Here $S^{\mathrm{new}}(U_0(\mathfrak{c}))$ denotes the new subspace (see [Shi78]).

Now that we have stated the main theorems of this paper, we pause to point the reader to some related results in the literature:

Remark.

(1) Kudla and Millson prove theorems similar to Theorem 1.2 in their study of special cycles on orthogonal and unitary groups (see [KM90]).
(2) Arithmetic analogues of (1.1) and Theorem 1.2 have appeared in ([BK], [KK]) and the program of Kudla, Rapoport, and Yang (see [Kud04], [RY04]).
(3) Lefschetz fixed point numbers of Hecke correspondences on certain Hermitian locally symmetric varieties in weighted cohomology and L^2-cohomology are computed in [KM97] and [Ste89], respectively.

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References

[BK] J.H. Bruinier, J.I. Burgos Gil, and U. Kühn. Borcherds products and arithmetic intersection theory on Hilbert modular surfaces. *to appear*. http://www.arxiv.org/abs/math.NT/0310201.

[Get] J. R. Getz. Intersection numbers of Hecke cycles on Hilbert modular varieties. *Recommended for publication in the American Journal of Mathematics.*

[HZ76] F. Hirzebruch and D. Zagier. Intersection numbers of curves on Hilbert modular surfaces and modular forms of Nebentypus. *Invent. Math.*, 36:57–114, 1976.

[KK] J.I. Burgos Gil, J. Kramer and U. Kühn. Cohomological arithmetic chow groups. *to appear in Journal of the Institut of Math. Jussieu.* http://www.arxiv.org/abs/math.AG/0404122.

[KM90] S. Kudla and J. Millson. Intersection numbers of cycles on locally symmetric spaces and Fourier coefficients of holomorphic modular forms in several complex variables. *Publ. math. de l'I.H.É.S.*, 71:121–172, 1990.

[KM97] M. Goresky, R. Kottwitz, and R. MacPherson. Discrete series characters and the Lefschetz formula for Hecke operators. *Duke Math. J.*, 89:477–554, 1997.

[Kud04] S. Kudla. Special cycles and derivatives of Eisenstein series. In H. Darmon and S. Zhang, editors, *Heegner Points and Rankin L-series*, volume 49 of *MSRI Publications*, pages 243–270, Cambridge, 2004. MSRI, Cambridge University Press.

[RY04] S. Kudla, M. Rapoport, and T. Yang. Derivatives of Eisenstein series and Faltings heights. *Compos. Math.*, 140(4):887–951, 2004.

[Ser73] J-P. Serre. Formes modulaires et fonctions zêta p-adiques. In *Modular Functions of One Variable III*, volume 350 of *Lecture Notes in Mathematics*, pages 191–268, Antwerp, 1973. Springer.

[Shi78] G. Shimura. The special values of zeta functions associated with Hilbert modular forms. *Duke. Math. J.*, 45:637–679, 1978.

[Ste89] M. Stern. L^2 index theorems on locally symmetric spaces. *Invent. Math.*, 96:231–282, 1989.

[vdG88] G. van der Geer. *Hilbert Modular Surfaces.* Number 3. Folge, Band 16 in Egrebnisse der Math. und ihrer Grenzgebiete. Springer-Verlag, 1988.

[Zag76] D. Zagier. Modular forms whose Fourier coefficients involve zeta functions of quadratic fields. In *Modular Functions in One Variable VI*, volume 627 of *Lecture Notes in Mathematics*, pages 105–169. Springer-Verlag, 1976.

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