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On the consistency between Schwinger-Dyson Equation and Bethe-Salpeter Equation

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Abstract

This is a brief comment on the consistency between Schwinger-Dyson (SD) equation and Bethe-Salpeter (BS) equation. Some of the pathological phenomena in the approximate BS equations may be avoided if the propagators used there are the solutions of the SD equation which is given in a consistent order of approximation with that for the Bethe-Salpeter equation. A systematic method is given which gives consistent approximations to SD and BS equations such that the solution to them satisfies the Ward-Takahashi identities.

§1. Introduction

It is well-known that the approximate Bethe-Salpeter (BS) equations have several pathological boundstate solutions, such as negative metric states, tachyons and so on, in particular when the coupling constant becomes strong. [1] However, such pathology occurs, for instance, for the simple BS equation in ladder approximation in which the free propagators are used independently of the coupling strength. Here I would like to emphasize the importance of mutual consistency of the approximate BS equation and Schwinger-Dyson (SD) equation. The BS equation is the equation determining the boundstates and contains the propagators of the constituents. The constituent propagators are determined by the SD equation. If the coupling constant becomes larger, not only the BS equation but also the SD equation changes. Therefore the simple ladder BS equation with free constituent propagators is clearly inconsistent. The propagators used in the approximate BS equation should be those determined by the approximate SD equation. The approximations of BS and SD equations should be consistent with each other. I suspect that many pathological phenomena may be avoided if this consistency between the BS and SD equations are satisfied.

Well-known example is the ladder approximation *both* for the BS and SD equations in QED. Maskawa and Nakajima [2] proved that those two equations in the ladder approximation is consistent with the chiral symmetry. Namely the vertex function determined by the ladder BS equation satisfies the chiral Ward identity with the propagator determined by the ladder SD equation. Even if the coupling constant is stronger than the critical coupling, the BS equation has no tachyon states and the lowest boundstate is the massless pion. They suspected that there might be no approximations beyond the ladder one that satisfy the mutual consistency between BS and SD equations concerning the chiral symmetry.

I will discuss this kind of consistency between the BS and SD equations very systematically in the QCD-like theory. Despite the suspicion of Maskawa and Nakajima, there exist many (actually, infinite number of) other approximations beyond ladder one satisfying such mutual consistency concerning the flavor symmetries. I will show some explicit examples.

This is a talk dedicated to Prof. Noboru Nakanishi who made many important contributions to the researches of the Bethe-Salpeter equations. This talk is based on my old work in collaboration with Masako Bando and Masayasu Harada. [3]

§2. Formulation of SD and BS equations

I will now present a systematic way obtaining such a consistent pair of approximate SD and BS equations.

2.1. *Effective action*

Let us consider QCD system in the presence of external background gauge fields A_μ coupling to the flavor charges which are orthogonal to the color degrees of freedom:

$$\mathcal{L}_{\mathrm{QCD}}(A) = -\frac{1}{4}F^{\alpha}_{\mu\nu}F^{\alpha\,\mu\nu} + \bar{\psi}i\gamma^{\mu}\left(\partial_{\mu} - ig_s T^{\alpha}G^{\alpha}_{\mu} - iA_{\mu}\right)\psi, \tag{2·1}$$

where we have assumed vector coupling $\mathcal{L}_{\mathrm{int}} = \bar{\psi}\gamma^{\mu}A_{\mu}\psi$ for the external flavor gauge field

$$A_{\mu}(x) \equiv A^{a}_{\mu}(x)\lambda^{a}\ , \tag{2·2}$$

just for notational simplicity. The axial vector case can be obtained simply by replacement $\gamma^{\mu} \longrightarrow \gamma^{\mu}\gamma_5$. Note that the external gauge fields are added purely as a convenient devise to derive consistent equations and will be eventually set equal to zero.

The generating functional W of the Green functions with bi-local source function $J(x,y)$ is defined by

$$\exp iW[J,A] \equiv \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\mathcal{D}G_{\mu}\exp i\left[\int d^4x\,\mathcal{L}_{\mathrm{QCD}}(A) + \int d^4x d^4y\, J(x,y)\psi(x)\bar{\psi}(y)\right]. \tag{2·3}$$

The effective action $\Gamma[S_{\mathrm{F}},A]$ for the quark propagator $S_{\mathrm{F}}(x,y)$ in the presence of the external flavor gauge field A_μ is defined by performing the Legendre transformation of $W[J,A]$ as

$$\Gamma[S_{\mathrm{F}},A] = W[J,A] - \int d^4x d^4y\, S_{\mathrm{F}}(x,y)J(x,y) \tag{2·4}$$

where

$$S_{\mathrm{F}}(x,y) \equiv \frac{\delta W[J,A]}{\delta J(x,y)} = \langle \mathrm{T}\psi(x)\bar{\psi}(y)\rangle\ . \tag{2·5}$$

As usual in the Legendre transformation, the dual relation to this equation holds:

$$-\,J(x,y) = \frac{\delta\Gamma[S_{\mathrm{F}},A]}{\delta S_{\mathrm{F}}(x,y)}\ . \tag{2·6}$$

In particular, whcxen $J=0$, this gives the SD equation determining the fermion propagator $S_{\mathrm{F}}(x,y)$ in the presence of the external flavor gauge field A_μ.

$$\frac{\delta\Gamma[S_{\mathrm{F}},A]}{\delta S_{\mathrm{F}}(x,y)} = 0\ . \tag{2·7}$$

$\Gamma[S_{\mathrm{F}},A]$ defined here is the effective action introduced by Dominicis and Martin [4] and Cornwall, Jackiw and Tomboulis. [5] They have given the formula

$$\Gamma[S_{\mathrm{F}},A] = i\mathrm{Tr}\,\mathrm{Ln}S_{\mathrm{F}} - \mathrm{Tr}\,(i\not{D}S_{\mathrm{F}}) + i^{-1}\mathcal{K}_{2\mathrm{PI}}[S_{\mathrm{F}}], \tag{2·8}$$

where the external gauge field A_μ is present only at the covariant derivative $D_\mu = \partial_\mu - A_\mu$. $\mathcal{K}_{\rm 2PI}$ stands for the two particle irreducible (w.r.t. fermion-line) diagram contributions: in the present QCD-like theory, we can expand the $\mathcal{K}_{\rm 2PI}$ into power series of the gauge coupling $\alpha_s = g_s^2/4\pi$,

$$\mathcal{K}_{\rm 2PI} = \mathcal{K}_{\rm 2PI}^{(1)} + \mathcal{K}_{\rm 2PI}^{(2)} + \cdots$$

and $\mathcal{K}_{\rm 2PI}^{(1)}$ and $\mathcal{K}_{\rm 2PI}^{(2)}$ are diagrammatically given by Fig. 1. More explicitly the first term $\mathcal{K}_{\rm 2PI}^{(1)}$

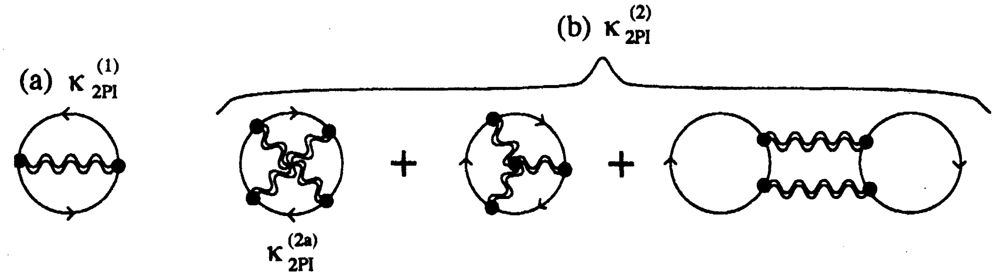

Fig. 1. Two particle irreducible (w.r.t. fermion-line) diagrams contributing to $\mathcal{K}_{\rm 2PI}^{(1)}$ and $\mathcal{K}_{\rm 2PI}^{(2)}$ in QCD. The double wavy line represents the gluon propagator $D_{\mu\nu}$ and the solid line represents the fermion propagator $S_{\rm F}$.

is given by

$$\mathcal{K}_{\rm 2PI}^{(1)} = -\frac{g_s^2}{2}\int d^4x d^4y \ {\rm tr}\left(S_{\rm F}(x,y) i\gamma_\mu T^\alpha S_{\rm F}(y,x) i\gamma_\nu T^\alpha\right) D^{\mu\nu}(x-y) \ ,$$

where T^α ($\alpha = 1, \cdots, N_c^2 - 1$) are color matrices in the quark representation and $D_{\mu\nu}$ is tree level gluon propagator given by

$$D^{\mu\nu}(x) = \int \frac{d^4p}{i(2\pi)^4} e^{-ipx}\left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right)\left(\frac{1}{p^2} - \frac{1}{p^2 - \Lambda^2}\right) \qquad (\Lambda \to \infty) \ ,$$

where we have included an ultraviolet cutoff Λ for definiteness. If we use the running coupling constant as was done in the improved ladder approximation by Higashijima[6] and Miransky,[7] the coupling constant g_s should be replaced by the running coupling function $g_s(p^2)$ with gluon momentum p_μ in the argument.[8]

2.2. *SD equation*

The SD equation (2·7) gives

$$iS_{\rm F}^{-1} = i\not{\partial} - i^{-1}\frac{\delta \mathcal{K}_{\rm 2PI}}{\delta S_{\rm F}} \ . \tag{2·9}$$

If we take only the lowest order term in $\mathcal{K}_{\rm 2PI}$, $\mathcal{K}_{\rm 2PI}^{(1)}$, then this SD equation reduces to

$$iS_{\rm F}^{-1} = i\not{\partial} + \not{A} + i^{-1}K * S_{\rm F} \ , \tag{2·10}$$

with $K * S_{\rm F}$ defined by

$$K * S_{\rm F} \equiv g_s^2(i\gamma_\mu T^\alpha)S_{\rm F}(y,x)(i\gamma_\nu T^\alpha)D^{\mu\nu}(x-y)\ . \tag{2·11}$$

diagrammatically, this reads

$$iS_{\rm F}^{-1} = i\partial\!\!\!/ + A\!\!\!/ - i$$

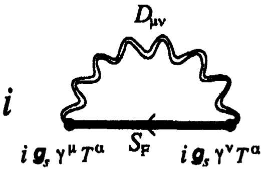

Fig. 2. Schwinger Dyson equation derived from the effective action Γ using $\mathcal{K}_{\rm 2PI} = \mathcal{K}_{\rm 2PI}^{(1)}$.

Equation (2·9) is the SD equation determining a solution $S_{\rm F} = S_{\rm F}[A]$ for the fermion propagator, on an *arbitrary* external background gauge field A_μ. The solution $S_{\rm F}[A]$ is expanded into a power series in the external gauge field A_μ:

$$S_{\rm F}[A] = S_{\rm F} + iA_\mu^a G_3^{a\mu} + \frac{i^2}{2}A_\mu^a A_\nu^b G_4^{a\mu,b\nu} + \frac{i^3}{3}A_\mu^a A_\nu^b A_\rho^c G_5^{a\mu,b\nu,c\rho} + \cdots\ , \tag{2·12}$$

where a, b and c denote the flavor indices. Here and henceforth the space-time coordinates and the integrations are suppressed, i.e., $A_\mu^a G_3^{a\mu} \equiv \int d^4z A_\mu^a(z) G_3^{a\mu}(x,y;z)$, etc.. The function $G_{n+2}^{a_1\mu_1,\cdots,a_n\mu_n}(x,y;z_1,\cdots,z_n)$ defines a fermion 2-point function with n vector vertices inserted (see Fig. 3):

$$\begin{aligned} G_3^{a\mu}(x,y;z) &\equiv \frac{1}{i}\frac{\delta S_{\rm F}(x,y;A)}{\delta A_\mu^a(z)}\bigg|_{A=0} = \langle 0|\,{\rm T}j^{a\mu}(z)\psi(x)\bar\psi(y)\,|0\rangle\ , \\ G_4^{a\mu,b\nu}(x,y;z,w) &\equiv \frac{1}{i^2}\frac{\delta S_{\rm F}(x,y;A)}{\delta A_\mu^a(z)\delta A_\nu^b(w)}\bigg|_{A=0} = \langle 0|\,{\rm T}j^{a\mu}(z)j^{b\nu}(w)\psi(x)\bar\psi(y)\,|0\rangle\ , \end{aligned} \tag{2·13}$$

and so on. This is because $\delta/\delta A_\mu^a$ yields an insertion of the vector current operator $j^{a\mu} = \bar\psi\gamma^\mu\lambda^a\psi$ to which the external gauge boson A_μ^a couples. Hereafter in this section, we suppress the flavor indices to denote $G_{n+2}^{a_1\mu_1,\cdots,a_n\mu_n}$ simply as $G_{n+2}^{\mu_1\cdots\mu_n}$, and write only γ^μ in place of $\gamma^\mu\lambda^a$ as vertex factors in the figures, accordingly.

2.3. *BS equations for the vertices*

Therefore the SD equation (2·9) for $S_{\rm F}[A]$ in fact gives not only the SD equation for the propagator $S_{\rm F} = S_{\rm F}[A{=}0]$ but also the Bethe-Salpeter (BS) equations for the $n+2$-point Green functions $G_{n+2}^{\mu_1\cdots\mu_n}$. That is, the functional differentiation w.r.t. A_μ (and then setting $A=0$) of the SD eq. (2·9) successively generates the BS equations for the $G_{n+2}^{\mu_1\cdots\mu_n}$ functions. It is convenient to define the following vertex function by amputating the fermion legs:

$$\Gamma_{n+2}^{\mu_1\cdots\mu_n} \equiv S_{\rm F}^{-1}G_{n+2}^{\mu_1\cdots\mu_n}S_{\rm F}^{-1}.$$

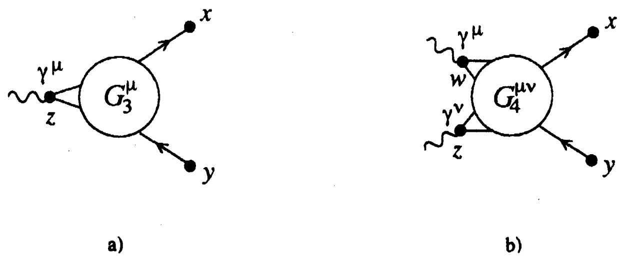

Fig. 3. Graphical representations of a) G_3^μ and b) $G_4^{\mu\nu}$ defined in Eq. (2·13) where wavy line represents the external gauge field.

To show what is going on as explicitly as possible, from here on in this section, we confine ourselves to the simplest case using the lowest order kernel ($\mathcal{K}_{\rm 2PI} = \mathcal{K}_{\rm 2PI}^{(1)}$).

First differentiation $\delta/\delta A_\mu|_{A=0}$ of Eq. (2·10) gives (see Fig. 4)

$$\Gamma_3^\mu = \gamma^\mu + \tilde{K} * \Gamma_3^\mu \ , \tag{2·14}$$

where $\tilde{K} * \Gamma_3^\mu \equiv K * (S_{\rm F}\Gamma_3^\mu S_{\rm F}) = K * G_3^\mu$ is defined in the same way as in Eq.(2·11).

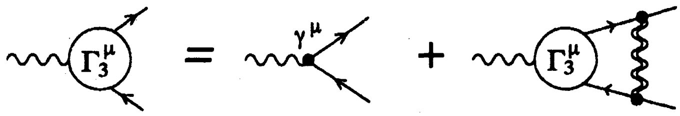

Fig. 4. BS equation for Γ_3.

Second differentiation $\delta^2/\delta A_\mu A_\nu|_{A=0}$ of Eq. (2·10) gives (see Fig. 5)

$$\Gamma_4^{\mu\nu} - \Gamma_3^\nu S_{\rm F}\Gamma_3^\mu - \Gamma_3^\mu S_{\rm F}\Gamma_3^\nu = \tilde{K} * \Gamma_4^{\mu\nu} \ . \tag{2·15}$$

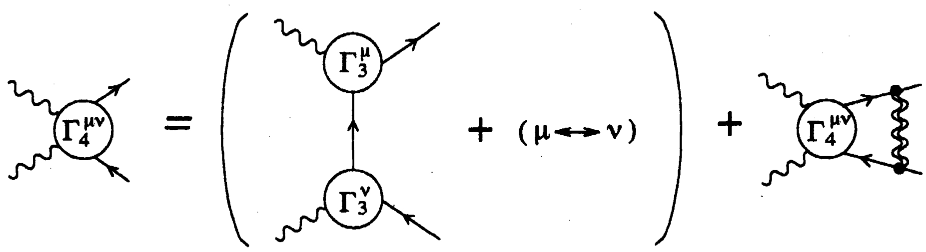

Fig. 5. BS equation for Γ_4.

§3. External Gauge Invariance

By our assumption that the flavor freedom is orthogonal to the color, the flavor matrices λ^a commute with the color matrices T^α. Then we have the following lemma.

Lemma: *For any approximation for $\mathcal{K}_{2\mathrm{PI}}$ by an arbitrary subset of diagrams contributing to $\mathcal{K}_{2\mathrm{PI}}$, the effective action Eq.(2·8) is (external) gauge invariant:*

$$\Gamma[S_{\mathrm{F}}, A] = \Gamma[S_{\mathrm{F}}^{U}, A^{U}] \ , \tag{3·1}$$

where the gauge transformation with $U(x) = \exp\left(i\theta^a(x)\lambda^a\right)$ is given explicitly by

$$\begin{aligned} A_\mu \to A_\mu^U &= U A_\mu U^{-1} + \frac{1}{i}\partial_\mu U \cdot U^{-1} \ , \\ S_{\mathrm{F}}(x,y) \to S_{\mathrm{F}}^U(x,y) &= U(x) S_{\mathrm{F}}(x,y) U^{-1}(y) \ . \end{aligned} \tag{3·2}$$

The proof is easy as written in Ref.3) in detail, and each term in Eq. (2·8) is separately gauge-invariant. In particular, each diagram contributing to $\mathcal{K}_{2\mathrm{PI}}$ is also separately invariant. Indeed, in any diagram, all the fermion lines are connected. Although they are separated by the interaction vertex factor $g_s\gamma^\mu T^\alpha$ at each vertex, the gauge transformation matrices $U(x)$ and $U^{-1}(x)$ which appear from the two propagators of both sides of the vertex point x cancel each other since U acts only in the the flavor space and is commutative with the color matrix T^α at the vertex. This also explains the reason why the argument of the running coupling function must be the momentum of the gluon, since otherwise the vertex becomes non-local for the fermion lines. [8]

Thus the gauge invariance of $\Gamma[S_{\mathrm{F}}, A]$ holds at any order of approximation for $\mathcal{K}_{2\mathrm{PI}}$.

§4. Ward-Takahashi identity

We now show that the external gauge invariance of the effective action implies that the vertex functions determined by those BS equations satisfy the Ward-Takahashi identities.

It immediately follows from the external gauge invariance relation (3·1) that the solution of the SD equation (2·7) on the gauge transformed background A_μ^U is given by the gauge transformation $U S_{\mathrm{F}}[A] U^{-1}$ of the solution $S_{\mathrm{F}}[A]$ on the original background A_μ: that is,

$$S_{\mathrm{F}}[A^U] = U S_{\mathrm{F}}[A] U^{-1} \ . \tag{4·1}$$

Substituting the expansion (2·12) into both sides of Eq. (4·1), we have

$$\begin{aligned} \mathrm{LHS} &= S_{\mathrm{F}} + iA_\mu^U G_3^\mu + \frac{i^2}{2} A_\mu^U A_\nu^U G_4^{\mu\nu} + \cdots , \\ \mathrm{RHS} &= U S_{\mathrm{F}} U^{-1} + iA_\mu U G_3^\mu U^{-1} + \frac{i^2}{2} A_\mu A_\nu U G_4^{\mu\nu} U^{-1} + \cdots \ . \end{aligned} \tag{4·2}$$

Considering, in particular, an infinitesimal gauge transformation $U = 1 + i\theta$ ($\theta = \theta^a\lambda^a$) and $A_\mu^U = A_\mu + D_\mu\theta$, and equating the same power terms in A_μ on both sides, we find

$$-i\partial_\mu^z G_3^{a\mu}(x,y;z) = i\delta^4(z-x)\lambda^a S_{\mathrm{F}}(x-y) - i\delta^4(z-y) S_{\mathrm{F}}(x-y)\lambda^a, \tag{4·3}$$

and so on. These are just the Ward-Takahashi identities required by the external gauge invariance. Thus this proves that the fermion propagator S_F and the vertices $\Gamma_{n+2}^{\mu_1\cdots\mu_n}$ determined by our SD and BS equations satisfy the Ward-Takahashi identities giving relations among them; namely, *our approximations for the SD and BS equations are mutually consistent and gauge invariant.*

We emphasize again that the WT identities are satisfied if we use SD and BS equations in the same order of approximation, that is, if they are both derived from the same effective action $\mathcal{K}_{2PI}$, irrespectively of the order of the approximation for it.

If we take the the lowest order approximation with $\mathcal{K}_{2PI} = \mathcal{K}_{2PI}^{(1)}$, we have the ladder SD equation in Fig. 2 with $A_\mu = 0$ and the ladder BS equation for the 3-point vertex Γ_3^μ in Fig. 4. This gauge invariance for $\Gamma_3^{a\mu}$ in the simplest ladder approximation has been known for a long time to Maskawa and Nakajima.[2] *)

If we take $\mathcal{K}_{2PI} = \mathcal{K}_{2PI}^{(1)} + \mathcal{K}_{2PI}^{(2a)}$, then the SD equation for S_F is changed into the form given in Fig. 6 (with $A_\mu = 0$) and the BS equation for $\Gamma_3^{a\mu}$ into the form shown in Fig. 7. Note that the Fig. 7 can be obtained by acting the differentiation $\delta/\delta A_\mu|_{A=0}$ on the Fig. 6 with $A_\mu \neq 0$. These equations are much more complicated than the simple ladder ones, nevertheless they satisfy the gauge invariance. Important is the mutual consistency of the approximations between the SD equation and BS equations.

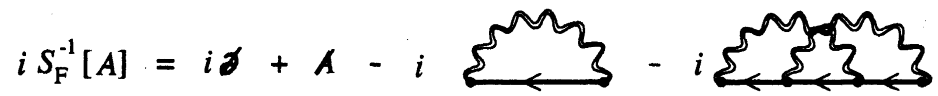

Fig. 6. SD equation using $\mathcal{K}_{2PI} = \mathcal{K}_{2PI}^{(1)} + \mathcal{K}_{2PI}^{(2a)}$.

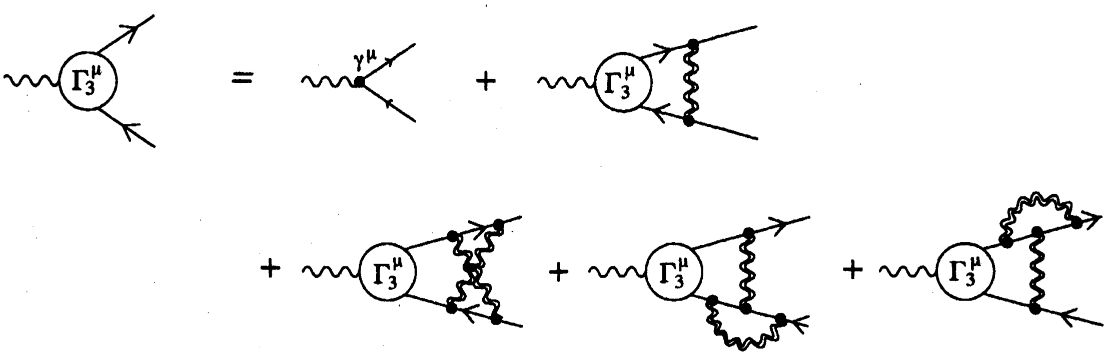

Fig. 7. BS equation for Γ_3^μ using $\mathcal{K}_{2PI} = \mathcal{K}_{2PI}^{(1)} + \mathcal{K}_{2PI}^{(2a)}$.

Although we have shown only the inhomogeneous BS equations for the vertices, the homogeneous BS equations are of course obtained by picking up the pole part from them so

*) A refinement of the proof and the generalization to the running coupling case was given by Kugo and Mitchard.[8]

that the homogeneous BS equations are given by simply dropping the inhomogeneous term from the inhomogeneous BS equations.

Acknowledgments

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