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A Generalization of Morley's Omitting Types Theorem

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概要

T を可算言語 L で表現された理論とする．タイプの排除定理は孤立的でないタイプを排除する T のモデルの存在を保証する．この定理は証明方法を変えることなく，次のように拡張される：可算個の非孤立的タイプの集合を排除するモデルが存在する．

Shelah は完全タイプの場合はタイプの集合の濃度が連続濃度未満という条件でも同じ定理が成立することを示している．一方 Morley によるタイプの排除定理は，十分に大きな濃度を持つモデルの上でタイプが排除されれば，そのタイプを排除する任意に大きなモデルの存在を主張する．タイプの排除定理と同様に Morley の定理も，可算個のタイプの集合に対して容易に拡張することができる．本研究においては，完全タイプの集合の場合に，濃度が連続未満の場合への Morley の上記定理の拡張を行う．

Theorem 1 *Let T be a countable complete theory and R a set of complete types with $|R| < 2^\omega$. Suppose that for each $i < \omega_1$, there is a model $M_i \models T$ with the following properties:*

1. $|M_i| \geq \beth_i(\omega)$,
2. *M_i omits each member of R.*

Then for each κ there is a model M omitting R and with $|M| \geq \kappa$.

Proof: We expand T by adding Skolem functions. Let $\{t_n\}_{n\in\omega}$ be an enumeration of all the Skolem functions with variables among $\{x_i\}_{i\in\omega}$. We may assume that the variables appearing in t_n are among $\bar{x}_n = x_0 ... x_{n-1}$. By adding dummy variables, we assume $t_n = t_n(\bar{x}_n)$.

First we introduce some terminologies. Let $\{I_i : i \in X\}$ be a set of n-indiscernible sequence with respect to the expanded language, where X is an uncountable set. We will say that $\{I_i : i \in X\}$ is t_n-uniform if the followin condition holds:

(*) If $i_0, i_1 \in X$ and $\bar{a}_j$ is an n-tuple (ordered increasingly) from I_{i_j} $(j = 0, 1)$, then $\mathrm{tp}(t_n(\bar{a}_0)) = \mathrm{tp}(t_n(\bar{a}_1))$.

If there is an uncoutable subset Y of X such that $\{I_i : i \in Y\}$ is t_n-uniform, then we will say that $\{I_i : i \in X\}$ is essentially t_n-uniform.

Claim A *Let $\{I_i : i \in X\}$ be a set of n-indiscernible sequences with $|X| = \omega_1$. Then one of the following cases holds:*

1. *$\{I_i : i \in X\}$ is essentially t_n-uniform.*
2. *There is a formula $\varphi(x)$ such that both $X^{\varphi,t_n} = \{i \in X : \varphi(x) \in \mathrm{tp}(t_n(a_{i,0}, ..., a_{i,n-1}))\}$ and $X^{\neg\varphi,t_n} = \{i \in X : \neg\varphi(x) \in \mathrm{tp}(t_n(a_{i,0}, ..., a_{i,n-1}))\}$ are uncountable, where $a_{i,0}, ..., a_{i,n-1}$ is the beginning part of I_i.*

Proof: Suppose that 2 is not the case. Then for each φ, X^{φ,t_n} or $X^{\neg\varphi,t_n}$ is countable. So, for each φ, cutting off X^{φ,t_n} or $X^{\neg\varphi,t_n}$ from X, we obtain an uncountable set $X_0 \subset X$ such that $\{I_i : i \in X_0\}$ is t_n-uniform. So 1 holds. (End of proof of Claim)

Under the same notation as in claim A, we have the following.

Claim B *Let $\{I_i : i \in X\}$ be a set of n-indiscernible sequences with $|X| = \omega_1$. Suppose that $\{I_i : i \in X\}$ is not essentially t_k-uniform, where $k \le n$. Then for any uncoutable subsets $X_i \subset X$ $(i = 0, 1)$, we can find uncountable sets $X'_i \subset X_i$ $(i = 0, 1)$ and $\varphi(x)$ such that $X'_0 \subset X^{\varphi,t_k}$ and $X'_1 \subset X^{\neg\varphi,t_k}$.*

Proof: Since $\{I_i : i \in X\}$ is not essentially t_k-uniform, neither is $\{I_i : i \in X_0\}$. By lemma A, there is a formula $\psi(x)$ such that both $(X_0)^{\psi,t_k}$ and $(X_0)^{\neg\psi,t_k}$ are uncountable. If $\psi(x)$ also divides X_1 into two uncountable sets, we are done. So, by cutting off a countable set from X_1, we can assume that $(X_1)^{\psi,t_k} = X_1$. Then we choose $\theta(x)$ such that both $(X_1)^{\theta,t_k}$ and $(X_1)^{\neg\theta,t_k}$ are uncountable. Again we can assume that $(X_0)^{\theta,t_k} = X_0$. Let $\varphi(x)$ be the formula $(\psi(x) \leftrightarrow \theta(x))$. Let $X'_0 = (X_0)^{\varphi,t_k}$ and $X'_1 = (X_1)^{\neg\varphi,t_k}$. It is easy to check that $\varphi(x)$, X'_0 and X'_1 satisfy our requirements. (End of Proof of Claim)

We put $X_\emptyset = \omega_1$, and for each $i \in X_\emptyset$ we fix a sequence $I_\emptyset(i)$ enumerating the universe M_i. Using Erdös-Rado theorem and claim B, for $\eta \in 2^{<\omega}$, we can inductively choose $X_\eta \subset \omega_1$ and $\{I_\eta(i) : i \in X_\eta\}$ with the following properties:

- If $\eta < \nu$ then
  - X_ν is an uncoutable subset of X_η;
  - $I_\nu(i)$ is a subsequence of $I_\eta(i)$ for each $i \in X_\nu$;

- $|I_\eta(i)| < |I_\eta(j)|$ for $i, j \in X_\eta$ with $i < j$, and $\sup\{|I_\eta(i)| : i \in X_\eta\} = \beth_{\omega_1}(\omega)$;
- If $\eta \in 2^n$ then
  - each $I_\eta(i)$ is an infinite n-indiscernible sequence (with respect to the expanded language);
  - $\{I_\eta(i) : i \in X_\eta\}$ is essentially t_n-uniform $\Rightarrow$ it is t_n-uniform;
- If $\eta \in 2^n$ and $k \leq n$ then
  - $\{I_i : i \in X_\eta\}$ is not t_k-uniform $\Rightarrow$ $X_{\eta\frown 0} \subset (X_\eta)^{(\varphi_{\eta,k}),t_k}$ and $X_{\eta\frown 1} \subset (X_\eta)^{(\neg\varphi_{\eta,k}),t_k}$, for some formula $\varphi_{\eta,k}(x)$.

For $\eta \in 2^n$ such that $\{I_i : i \in X_\eta\}$ is t_n-uniform, let

$$p_\eta(x) = \mathrm{tp}(t_n(a_{i_0}, ..., a_{i_{n-1}})),$$

where $a_{i_1}, ..., a_{i_{n-1}}$ is the beginning part of I_i. By the t_n-uniformity, this definition is well-defined. p_η is a type realized in M_i, so p_η does not belong to R.

Although it is not used in our proof, we remark the following: Suppose that there is an infinite path $\nu \in 2^\omega$ such that each $\{I_{\nu|n}(i) : i \in X_{\nu|n}\}$ is t_n-uniform. Then we can easily find an infinite indiscernible sequence I whose Skolem closure only realizes types in $\{p_{\nu|n} : n \in \omega\}$. This can be shown as below. Let $\Gamma(\{x_i\}_{i\in\omega})$ be the following set of formulas.

$$\{\{x_i\}_{i\in\omega} \text{ is indiscernible}\} \cup \bigcup_{n\in\omega} p_{\nu|n}(t_n(\bar{x}_n))$$

Clearly it is consistent. Choose a realization I of Γ. Let M be the Skolem closure of I. Then each element of M has the form $t_n(\bar{a})$, where $\bar{a}$ is an n-tuple from I. So, by the indiscernibility, $t_n(\bar{a})$ realizes $p_{\eta|n}$, which is not a member of R. By a compactness argument, the cardinality of I can be chosen arbitrarily large.

For $\nu \in 2^\omega$, we define the following:

- K_ν is the set of all $k \in \omega$ such that $\{I_{\nu|k}(i) : i \in X_{\nu|k}\}$ is not t_k-uniform.
- For $n \in K_\nu$, let $\Gamma^n_\nu(x)$ be the set

  $$\bigcup_{n\leq m\in\omega} \{\varphi_{\nu|m,n}(x) : \nu(m) = 0\} \cup \bigcup_{n\leq m\in\omega} \{\neg\varphi_{\nu|m,n}(x) : \nu(n) = 1\}$$

  (Recall that $\varphi_{\eta,n}(x)$ was a formula dividing X_η into two uncountable sets.)
- Finally let $\Delta_\nu(\{x_i\}_{i<\kappa})$ be the set

  $$\{\{x_i\}_{i<\kappa} \text{ is indiscernible}\} \cup \bigcup_{n\in K_\nu} \Gamma^n_\nu(t_n(\bar{x}_n)) \cup \bigcup_{n\notin K_\nu} p_{\nu|n}(t_n(\bar{x}_n)).$$

Claim C $\Delta_\nu(\{x_i\}_{i<\kappa})$ *is finitely satisfiable.*

Let $n^* \in \omega$. Choose $i \in X_{\nu|n^*}$ arbitrarily, and let $I = I_{\nu|n^*}(i)$. Clearly I is an n^*-indiscernible sequence. Let $n < n^*$. First assume $n \notin K_\nu$. Then the first n-elements of I clearly realize the type $p_{\nu|n}(t_n(\bar{x}))$. Then assume $n \in K_\nu$ and let $n \leq m < n^*$. Suppose $\nu(m) = 0$. Then $X_{\nu|n^*} \subset X_{\nu|m^\frown 0} \subset (X_{\nu|m})^{\varphi_{\nu|m}, t_m}$. So if $\bar{a}$ is the first n-element of I, then it satisfies $\varphi_{\nu|m,n}(t_n(\bar{x}_n))$. For the same reason, if $\nu(m) = 1$, $\bar{a}$ satisfies $\neg\varphi_{\nu|m,n}(t_n(\bar{x}_n))$. This shows that $\Delta_\nu(\{x_i\}_{i<\kappa})$ is finitely satisfiable.

Claim D *If $\eta \neq \eta'$ are infinite paths with $\eta|n = \eta'|n$ for some $n \in K_\nu \cap K_{\nu'}$. Then $\Gamma^k_\eta(x)$ and $\Gamma^k_{\eta'}(x)$ are contradictory.*

Choose the largest $m \geq n$ with $\eta|m = \eta'|m$. We can assume $\eta(m) = 0$ and $\eta'(m) = 1$. Then $\Gamma^n_\eta(x)$ contains $\psi_{\eta|m,n}(x)$, while $\Gamma^n_{\eta'}(x)$ contains $\neg\psi_{\eta|m,n}(x)$.

Claim E *There is a model N such that*

- *N omits each member of R;*
- *the domain of N is the Skolem hull of an infinite indiscernible sequence.*

For each $\nu \in 2^\omega$, choose $J_\nu = (a_{\nu,i})_{i<\kappa}$ realizing Δ_η. Let $\mathcal{K} = \{\nu|n : \nu \in 2^\omega, n \in K_\nu\}$. For $\eta \in \mathcal{K}$, let S_η be the set

$$\{\nu \in 2^\omega : \eta < \nu,\ \mathrm{tp}(t_n(a_{\nu,0}, \ldots, a_{\nu,n-1})) \in R\},$$

where $n = \mathrm{len}(\eta)$. By claim B and our assumption that $|R| < 2^\omega$, we have $|S_\eta| < 2^\omega$. So we can choose $\nu \in 2^\omega \setminus \bigcup_{\eta\in\mathcal{K}} S_\eta$.

J_ν is an infinite indiscernible sequence. By our choice of ν, if $\nu|n \in \mathcal{K}$, then $t_n(a_{\nu|n,0}, \ldots, a_{\nu|n,n-1})$ does not realize R. If $\nu \notin \mathcal{K}$, then $t_n(a_{\nu|n,0}, \ldots, a_{\nu|n,n-1})$ realizes $p_{\nu|n}$, which is not a member of R. Let N be the Skolem hull of J_ν. By the indiscernibility (and the fact that t_k's enumerate all the Skolem terms), N does not realize a member of R.

参考文献

[1] Chen Chung ChangH. Jerome Keisler, Model Theory (Studies in Logic and the Foundations of Mathematics) , 1990.

[2] Saharon Shelah, Classification Theory and the Number of Non-Isomorphic Models (Studies in Logic and the Foundations of Mathematics), 1990.

[3] Newelski, Omitting types and the real line, Journal of Symbolic Logic, vol. 62 (1987), no. 4, pp. 1020-1026.