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COMPLETIONS OF GBL-ALGEBRAS AND ACYCLIC MODAL ALGEBRAS: NEGATIVE RESULTS

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ABSTRACT. We prove that several well-known varieties of modal and GBL-algebras are not closed under completions.

1. INTRODUCTION

In recent years, a lot of attention has been devoted to completions in algebraic logic — in particular, to varieties of algebras closed under canonical extensions and Dedekind-McNeille completions. The most general notion of closure under completions is clearly the following: we say that a variety V is *closed under completions* if for every $\mathfrak{A} \in V$ there a lattice complete $\mathfrak{B} \in V$ and an embedding $f : \mathfrak{A} \mapsto \mathfrak{B}$. Observe that we *do not* require f to be a complete homomorphisms, that is, to preserve arbitrary joins. Now, if we can prove that a variety is not closed under completions, it follows that it cannot be canonical. Notice also, that strictly speaking we do not need to assume that algebras in our variety V have lattice reducts: being partially ordered would be sufficient. In the paper however, we will deal exclusively with varieties that do have lattice reducts. We show that a number of varieties corresponding to well-known modal and substructural logics fail to be closed under completions.

Closure under completions should not be confused with being *completely generated*, that is, with the existence of a class X of lattice-complete algebras such that $V = HSP(X)$. Being completely generated is a much weaker property then closure under completions. In particular, every variety generated by its finite members is completely generated; most varieties discussed in the present paper have the finite model property and hence are completely generated.

Failure of closure under completions seems harder to show than failure of canonicity, at least in the sense that there are non-canonical varieties closed under completions. Good examples can be found in the modal realm, for instance, Wolter [17] shows that the variety corresponding to tense logics of the reals is not canonical, but every algebra in this variety can be embedded in a dual algebra of a Kripke frame satisfying the axioms of this logics. This property is known as *complexity*[1] and is properly contained between canonicity and closure under completions. As follows

[1]This is perhaps not the most fortunate name, it is however justified by the fact that the dual algebra of a Kripke frame is a particular case of a *complex* algebra.

from results of Surendonk [15] and Wang [16], the variety corresponding to McKinsey axiom $\Box\Diamond p \to \Diamond\Box p$ is not complex, let alone canonical (failure of canonicity was proven earlier by Goldblatt) but is closed under completions.

Nevertheless, as our results show, a proof that a given variety is not closed under completions does not necessarily have to be harder than a proof of non-canonicity. The present paper uses fairly straightforward algebraic calculations. In particular, the precise way operations on the original algebra induce operations on its canonical extension is irrelevant here. Instead, we focus only on (possibly infinitely) equations that have to hold in the whole variety. This is yet another instance of a well-known mathematical paradox: a stronger theorem can be sometimes easier to prove. Intermediate cases — non-canonical varieties closed under completions — require more involved techniques.

The first group of examples in this paper is based on a generalization of a result from author's PhD Thesis [11]. We provide a generic theorem which allows to show that some well-known varieties of modal algebras are not closed under completions. These varieties include the variety of Löb (diagonalizable) algebras and Grzegorczyk algebras; thus, we generalize the known observations that these logics are not canoncial. Essentially, the varieties covered by our method correspond to logics determined by those classes of frames which contain frames of arbitrary finite size but no frames with either infinite ascending chains or two-point clusters. Thus, we call our base variety *acyclic* or *weakly Grzegorczyk.*

The second group of examples is taken from a recently submitted paper with Tomasz Kowalski, which is the basis of this report. Again, we provide a generic theorem allowing to show that several well-known varieties of FL-algebras are not closed under completions. More specifically, the background for our results is provided by the variety of GBL-algebras, defined first (as far as we know) in [10]. GBL-algebras form a rather wide class including, in particular, MV-algebras, BL-algebras, Brouwerian algebras and ℓ-groups. Our results generalize very broadly several already existing ones, for example Priestley and Gehrke [7] observation that MV-algebras are not canonical.

2. Acyclic algebras and their subvarieties

A *modal algebra* is a structure $\mathfrak{A} := \langle A, \wedge, \neg, \Diamond, \bot\rangle$, where $\langle A, \vee, \neg, \bot\rangle$ is a boolean algebra and $\Diamond$ preserves $\vee$ and $\bot$. Other operations ($\wedge$, $\to$, $\top$ and $\Box$) are defined in the standard way; $\Diamond^+ x := x \vee \Diamond x$, $\Box^+$ is its dual. A modal algebra is called *acyclic* or *weakly Grzegorczyk* (cf. [12]) if it satisfies the following inequality:

$$x \leq \neg\Box^+(x \to \Diamond(\neg x \wedge \Diamond x)).$$

Recall that *a Grzegorzyk algebra* satisfies

$$x \leq \neg\Box(x \to \Diamond(\neg x \wedge \Diamond x)).$$

Thus, every Grzegorczyk algebra is weakly Grzegorczyk. The reader may be more familiar with the definition of Grzegorczyk algebras as those satisfying

$$\Box(\Box(y \to \Box y) \to y) \leq y.$$

To see that these definitions are equivalent, negate both sides of the lower inequality, replace y with $\neg x$ and recall that in boolean algebras $a \to b = \neg b \to \neg a$. Another important class of acyclic algebras is the class of *Löb algebras* or *diagonalizable algebras* defined by the inequality

$$\Diamond x \leq \Diamond(x \wedge \neg\Diamond x),$$

or, dually,

$$\Box(\Box y \to y) \leq \Box y$$

Theorem 2.1. *Diagonalizable algebras are acyclic.*

Proof. It is perhaps easier here to work with dual forms. First, observe that diagonalizable algebras satisfy *transitivity*, i.e., $\Box y \leq \Box^2 y$ as

$$\Box y \leq \Box(\Box^2 y \wedge \Box y \to y) = \Box(\Box^2 y \wedge \Box y \to \Box y \wedge y) \leq \Box^2 y \wedge \Box y$$

(the first and second inequality holding by boolean laws, the last by the Löb inequality). Also, we have that

$$\Box(\Box(y \to \Box y) \to y) \leq \Box(\Box^2 \to y) \leq \Box(\Box y \to y) \leq \Box y$$

(the first inequality by boolean laws, the second by transitivity, the third by the Löb inequality). Finally,

$$\Box(y \to \Box y) \to y \leq \Box^2 y \to y \leq \Box y \to y$$

(the first inequality by boolean laws, the second by transitivity). In this way we get that in diagonalizable algebras

$$\Box^+(\Box(y \to \Box y) \to y) \leq \Box y \wedge (\Box y \to y) \leq y,$$

as desired. $\Box$

Recall that nontrivial Löb algebras are *not* Grzegorczyk.

3. GBL-ALGEBRAS AND THEIR SUBVARIETIES

A *residuated lattice* is an algebra $\mathfrak{A} := \langle A, \wedge, \vee, \cdot, \backslash, /, 1\rangle$ such that $\langle A, \wedge, \vee\rangle$ is a lattice, $\langle A, \cdot, 1\rangle$ is a monoid and the following *residuation law* is satisfied:

res: $x \cdot y \leq z$ iff $y \leq x\backslash z$ iff $x \leq z/y$.

The class of residuated lattices is equationally definable, cf. e.g. [10] for an equational basis. In connection to *substructural logics* (see, e.g. [14] for that connection), it is convenient to add another constant, called 0, to the type. This defines a class of algebras known as *FL-algebras*, i.e., algebras $\langle A, \wedge, \vee, \cdot, \backslash, /, 1, 0\rangle$ such that $\langle A, \wedge, \vee, \cdot, \backslash, /, 1\rangle$ is a residuated lattices. Residuated lattices as defined above are in this setting term-equivalent to the subvariety of FL-algebras defined by the equation $0 = 1$. Alternatively, they are also 0-free reducts of FL-algebras. These are essentially just two different ways of removing 0 from the signature, so we will not be particular about which way is preferred. We refer the reader to the forthcoming [4] for more of FL-algebras, residuated lattices and algebraic semantics for substructural logics.

Little of the theory of residuated lattices is needed for our results. One fact that we will use is that in any residuated lattice if $\bigvee X$ exists for some $X \subseteq A$, then for an arbitrary $y \in A$ the following hold

(1) $y \cdot \bigvee X = \bigvee\{yx \colon x \in X\}$,

(2) $\bigvee X \backslash y = \bigwedge\{x\backslash y \colon x \in X\}$.

Of importance, as we mentioned already, will be the variety of *Generalized BL-algebras* (see [5] for more on GBL). These are residuated lattices satisfying the following law:

GBL: $x[x\backslash(x \wedge y)] = x \wedge y = [(x \wedge y)/x]x$.

To get even more leeway with the extra constant 0, we will take as our base, the variety of *pointed GBL-algebras* defined as residuated lattices satisfying GBL and endowed with an additional constant 0 (proper residuated lattice setting), or as FL-algebras satisfying GBL (FL setting).

This is a rather comprehensive class of algebras. It contains, on the one hand, the variety of *lattice-ordered groups* (ℓ-groups) as a subvariety. Recall that ℓ-groups are (term-equivalent to) residuated lattices satisfying $x(x\backslash 1) = 1$ (by setting $x^{-1} = 1\backslash x$). Then, the elements $x\backslash 1$ and $1/x$ are, respectively, the left and right inverse of x, and they coincide, i.e., $x\backslash 1 = 1/x$ holds. On the other hand, it also contains the variety of *integral commutative* pointed GBL-algebras. A residuated lattice is *integral* if it satisfies $1 \geq x$, i.e., if the unit of the monoid is the greatest element. From integrality it easily follows that $x \cdot y \leq x \wedge y$. A residuated lattice is commutative if it satisfies $x \cdot y = y \cdot x$. In such a case / becomes redundant, as $x\backslash y = y/x$. Thus, for commutative residuated lattices, we leave / out of the signature and replace \ with $\rightarrow$. *Brouwerian algebras*[2] are residuated lattices satisfying $x \cdot y = x \wedge y$. The variety of Brouwerian algebras is precisely the class of 0-free subreducts of Heyting algebras, and thus Brouwerian algebras provide standard examples of integral commutative GBL-algebras.

Recall that a lattice is bounded if it has both the largest and the smallest element. Every bounded GBL-algebra is integral, in fact, every GBL-algebra is a direct product of an ℓ-group and an integral GBL-algebra ([1, 5]). An integral FL-algebra in which 0 is the lower bound is called *zero-bounded.* Zero-bounded integral pointed GBL-algebras that satisfy in addition

Lin: $(x \rightarrow y) \vee (y \rightarrow x) = 1$

are precisely Hajek's *BL-algebras* (cf. [9]).

Two subvarieties of BL-algebras are of particular interests for us. The first is the subvariety of *MV-algebras.* These are BL-algebras satisfying

MV: $(x \rightarrow 0) \rightarrow 0 = x$.

where $x \rightarrow 0$ is usually written as $\neg x$. The second is the subvariety of *product algebras* [3] defined by

Π: $\neg\neg x \leq (x \rightarrow xy) \rightarrow y(\neg\neg y)$.

[2]Again, an unfortunate clash of terminology. In some papers, Brouwerian algebras are defined to be the *duals* of Heyting algebras, not reducts without the bottom element. Given Brouwer's views on algebraic logic, it is rather ironic that his name is employed so often in the field.

The definition of a complete lattice is standard. Recall that every complete lattice is bounded. It is then very easy to conclude that some classes of residuated lattices do not contain any complete algebra.

Fact 3.1. *No non-trivial ℓ-group is complete.*

Proof. No non-trivial ℓ-group is bounded, as every such algebra contains $y > 1$ and multiplication is strictly order-preserving by cancellativity. □

So, lattice-incompleteness of ℓ-groups arises somehow trivially. In this case, a more interesting notion seems to be *local completeness.* A lattice L is locally complete if every interval of L is complete. Reals or integers (both examples of ℓ-groups) with standard order are locally complete but not complete. For bounded lattices, like BL-algebras or modal algebras, local completeness is equivalent to completeness. Notice also that by well-known Makinson's Theorem in modal logic, such a situation cannot arise for modal algebras. In order to obtain examples of varieties with no complete algebras, one needs boolean algebras with at least two diamonds; see [12] for details. It is curious that also in this case, one of operators is supposed to satisfy the weak Grzegorczyk axiom.

A class of algebras K is *closed under (local) completions* if every algebra from K can be embedded into a (locally) complete algebra from K. Let us stress once again: we *do not* require this embedding to preserve infinite meets and joins, only finitary operations. Canonical extensions, for example, do not preserve infinitary operations. Thus, when proving that closure under completions fails, we cannot use too much information about suprema and infima in an algebra witnessing this failure. As we said in the introduction, we only use laws that hold in the whole variety.

4. Acyclic varieties which are not closed under completions

First, let us provide a criterion of failure of closure under completions for weakly Grzegorczyk algebras:

Theorem 4.1. *No modal algebra $\mathfrak{L}$ containing a sequence $\{a_n\}_{n\in\omega} \in L$ with the following property for every n:*

$$\bot \neq a_n \leq \Diamond(\neg a_0 \wedge \cdots \wedge \neg a_n \wedge a_{n+1})$$

can be embedded in a complete acyclic algebra.

Proof. Define

$$b_n := \neg a_0 \wedge \cdots \wedge \neg a_{2n-1} \wedge a_{2n}, \quad c_n := \neg a_0 \wedge \cdots \wedge \neg a_{2n} \wedge a_{2n+1}.$$

and assume B and C are respective suprema of both sequences. It is straightforward to see that for every $m \in \omega$

(1) $\bot \neq b_m \leq B$,
(2) $b_m \wedge C = \bot$ and hence $B \leq \neg C$,
(3) $b_m \leq \Diamond c_m$, $c_m \leq \Diamond b_{m+1}$, and hence
(4) $B \leq \Diamond C \leq \Diamond(\neg B \wedge \Diamond B)$.

But then if we replace x with B in the weak Gregorczyk inequality, the right-hand side becomes equal to $\bot$, while left is greater than $\bot$, so the inequality does not hold. $\square$

For a subset $A \subseteq \omega$, let

$$\trianglerighteq A := \{n \in \omega \mid \exists a \in A. n \leq a\}, \quad \triangleright A := \{n \in \omega \mid \exists a \in A. n < a\}.$$

Let $\mathfrak{B}$ be the boolean algebra of finite-cofinite subsets of natural numbers. It is left to the reader to verify that:

(1) $\langle \mathfrak{B}, \trianglerighteq \rangle$ is a Grzegorczyk algebra;
(2) $\langle \mathfrak{B}, \triangleright \rangle$ is a diagonalizable algebra;
(3) in both cases, the sequence $\{\{n\}|n \in \omega\}$ satisfies the assumption of Theorem 4.1.

Thus, we get the following

Corollary 4.2. (1) *No variety of acyclic algebras containing $\langle \mathfrak{B}, \trianglerighteq \rangle$ is closed under completions. Hence, Grzegorczyk algebras are not closed under completions.*
(2) *No variety of acyclic algebras containing $\langle \mathfrak{B}, \triangleright \rangle$ is closed under completions. Hence, diagonalizable algebras are not closed under completions.*

This result can be generalized in several directions. For example. the weak Grzegorczyk inequality can be replaced with inequality expressing the fact that corresponding frames have no three-point cycles or no four-point cycles — or still weaker for arbitrary finite n. Alternatively, $\square^{+}$ in the inequality can be replaced with $\square^{\leq n}$ or any *possibility form* in the sense of Goldblatt [8].

5. Generic theorem for GBL-algebras

In order to introduce similiar criterion for GBL-algebras, we need a following

Definition 5.1. *Let $\mathfrak{L}$ be a GBL-algebra. Suppose there are subsets A and B of L with the following properties:*

1 *B is a nonprincipal ideal without supremum in L.*
2 *$B < A$, i.e., for each $a \in A$ and $b \in B$ we have $b < a$.*
3 *there are elements $w, u \in B$ with $w < u$ such that $b \backslash u \in A$ for all $b \in B$, and*
  a *for each $a \in A$ there is $b \in B$ with $b \backslash u \leq a$,*
  b *for each $b \in B$ there is $a \in A$ with $ba \leq w$.*

(B, A) is then called a residuation discontinuity *and $\mathfrak{L}$ itself — a* discontinuous GBL-algebra.

The choice of the name will become clear from the proof of the next theorem, where indeed it turns out that the function $x(x\backslash u)$ is not continuous at a certain point presumed to exist (namely b_∞). The reader is encouraged to look at pictures and proofs of corollaries in Section 6 to see concrete examples of such a discontinuity. Pictures are also meant to make the proof below appear less abstract.

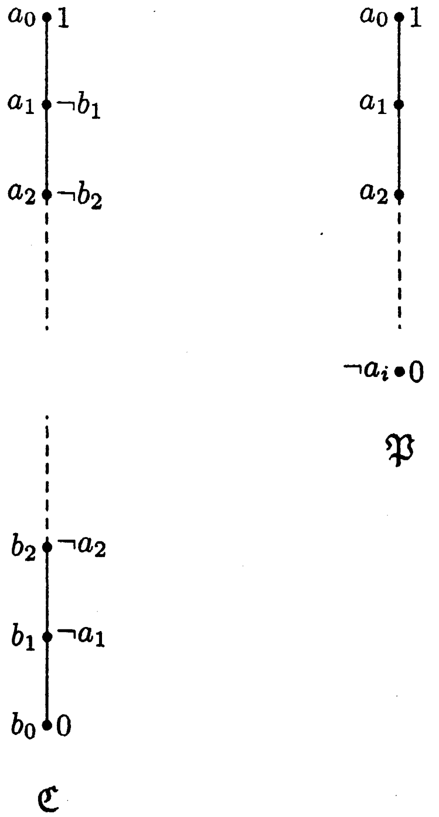

FIGURE 1. Chang's chain $\mathfrak{C}$ and product algebra $\mathfrak{P}$

Theorem 5.2 (Completion Discontinuity). *No variety of GBL-algebras containing a discontinuous GBL-algebra is closed under sectional completions.*

Proof. Suppose $\mathfrak{M}$ is a discontinuous GBL-algebra. Let (B, A) be a residuation discontinuity. Define $b_\infty = \bigvee B$ and $a_\infty = \bigwedge A$. Notice that we make no assumptions whatsoever about a_∞. It may well belong to L, or even to A.

Consider $b_\infty(b_\infty\backslash u)$. As $b_\infty > u$ we obtain $b_\infty(b_\infty\backslash u) = u$. On the other hand, $\bigvee B \cdot (\bigvee B\backslash u) = \bigvee B \cdot (\bigwedge\{b\backslash u\colon b \in B\})$. Since every $a \in A$ is minorised by $b\backslash u$ for some $b \in B$ and each $b\backslash u$ belongs to A, we have $\bigwedge\{b\backslash u\colon b \in B\} = a_\infty$. Therefore, $\bigvee B \cdot (\bigwedge\{b\backslash u\colon b \in B\}) = \bigvee B \cdot a_\infty = \bigvee\{b \cdot a_\infty\colon b \in B\}$. Now, for each $b \in B$ choose an $a_b \in A$ with $b \cdot a_b \leq w$. Since $a_\infty \leq a$ for all $a \in A$, we get $b \cdot a_\infty \leq b \cdot a_b \leq w < u$. Thus, $b_\infty(b_\infty\backslash u) < u$, a contradiction. □

6. Applications of The Generic Theorem

MV-algebras. A particular example of an MV-algebra is Chang's chain $\mathfrak{C}$ [2] whose universe is $C := \{a_n\}_{n\in\omega} \cup \{b_n\}_{n\in\omega}$ with $a_0 := 1$ and $b_0 := 0$. The lattice order is defined as follows: every b_n is below all a_n's, $b_n \leq b_m$ if $n \leq m$, $a_n \leq a_m$ if $n \geq m$. Multiplication is defined by

$$b_n \cdot b_m := 0, \quad a_n \cdot b_m := b_{m \dot{-} n}, \quad a_n \cdot a_m := a_{n+m},$$

where $\dot{-}$ stands for truncated subtraction. See Figure 2.

Lemma 6.1. *Chang's chain $\mathfrak{C}$ is a discontinuous GBL-algebra.*

Proof. Assume $\mathfrak{A}$ is a complete MV-algebra having a subalgebra $\mathfrak{C}'$ isomorphic to $\mathfrak{C}$; we can identify elements of $\mathfrak{C}'$ and $\mathfrak{C}$. Define $A = \{a_n\}_{n\in\omega}$, $B = \{b_n\}_{n\in\omega}$, $w = b_0$, $u = b_1$. Conditions 1 and 2 of Definition 5.1 are immediate. For 3, observe that for every i, $\neg b_i = a_i$ and $b_{i+1} \to b_1 = a_i$. □

Corollary 6.2. *No variety of zero-bounded GBL-algebras containing Chang's chain (in particular, MV-algebras and BL-algebras) is closed under completions.*

Another curious aside: for varieties of MV-algebras, closure under completions is actually equivalent to canonicity (we observed in the introduction that it is not always the case) and both in turn are equivalent to failure of finite generation. Let us prove it in more detail. The lemma below is folklore, but to make the paper self-contained we provide a proof.

Lemma 6.3. *A variety V of MV-algebras does not contain $\mathfrak{C}$ if and only if it is finitely generated.*

Proof. Since each finite MV-algebra satisfies the identity $x^{n+1} = x^n$ for some $n \in \omega$, we get that finitely generated V does not contain $\mathfrak{C}$. For the converse, suppose V is not finitely generated. Then for every $k \in \omega$ there is a subdirectly irreducible $\mathfrak{A}_k \in V$ falsifying $x^{k+1} = x^k$. Choose a suitable $a_k \in A_k$ witnessing that fact. Let $\mathfrak{A}$ be the ultraproduct $\prod_{k\in\omega} \mathfrak{A}_k/U$ for some nonprincipal U. Then, the element $a = (a_k : k \in \omega)/U$ has $a^{n+1} < a^n$ for every $n \in \omega$. Also, $\mathfrak{A}$ is linearly ordered, since all $\mathfrak{A}_k$ are. It is now easy to verify that the subalgebra of $\mathfrak{A}$ generated by a is isomorphic to $\mathfrak{C}$. □

The second lemma is due to Gehrke and holds essentially because canonical extensions commute with homomorphisms and Boolean products. Since every direct product can be rendered as a Boolean product of ultraproducts, and ultrapowers of finite algebras are these algebras themselves, we have:

Lemma 6.4. *A finitely generated variety is canonical.*

Although it was proven in the late 1980s, we give [6] as the reference as it is also an excellent general introduction to canonical extensions. Combining the lemmas together we obtain:

Theorem 6.5. *For a variety V of MV-algebras the following are equivalent:*

(1) *V is canonical,*
(2) *V is closed under completions,*
(3) *V is finitely generated,*
(4) *V is n-potent for some $n \in \omega$.*

Lattice-ordered groups. As we have already mentioned, ℓ-groups can be viewed as a subvariety of GBL algebras satisfying $x(x\backslash 1) = 1$. Mundici showed in [13] that there is a categorical equivalence between MV-algebras and ℓ-groups. The ℓ-group obtained from Chang's chain via this equivalence is, as should be expected, a good example that ℓ-groups are not closed under local completions. This algebra is isomorphic to the lexicographic product of integers with themselves. To be precise, we

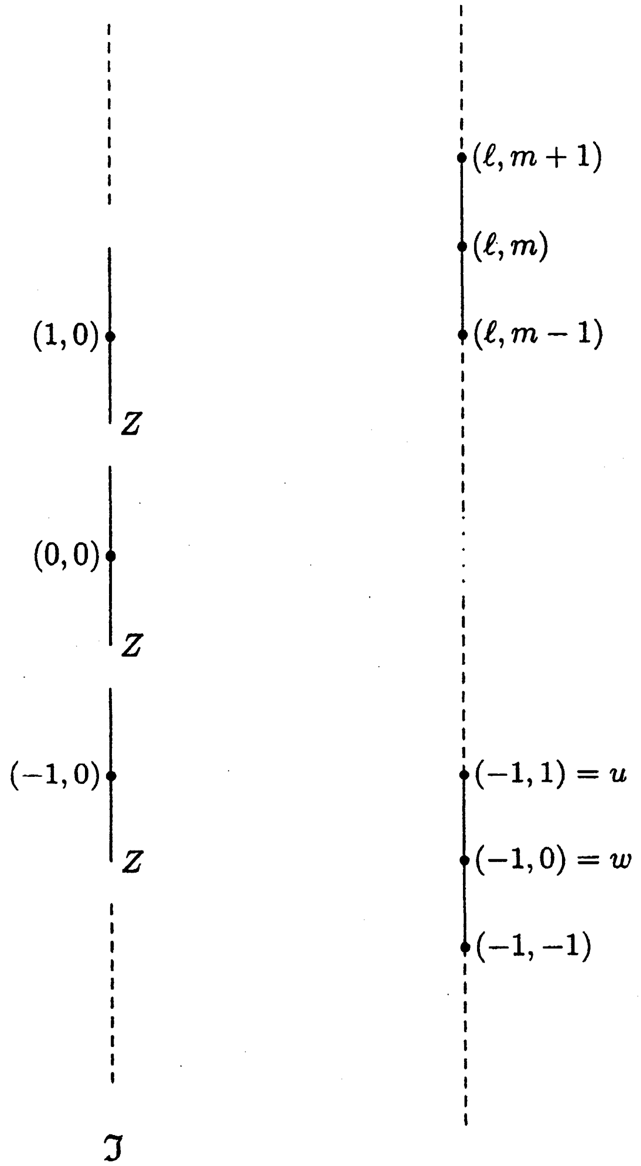

FIGURE 2. Algebra $\mathfrak{I}$ (left) and its magnified detail (right)

define the algebra $\mathfrak{I} = \langle Z \times Z, \wedge, \vee, \cdot, ^{-1}, (0,0)\rangle$, where $\wedge$ and $\vee$ are respectively the minimum and maximum with respect to the lexicographic order, and multiplication and inverse are defined pointwise, as addition and unary minus.

Theorem 6.6. *The algebra $\mathfrak{I}$ is a discontinuous GBL-algebra. Thus, no non-trivial variety of ℓ-groups is closed under (local) completions.*

Proof. Define $A = \{(\ell, m) \in I \colon \ell \geq 0\}$ and $B = I \setminus A = \{(k, n) \in I \colon k \leq -1\}$. So defined A and B satisfy conditions 1 and 2. Now putting $u = (-1,1)$ and $w = (-1,0)$ we can verify that conditions 3a and 3b hold as well. As all these verifications are routine, we will only show 3a. Take any $a = (\ell, m) \in A$. There are two cases to consider.

If $\ell > 0$, we take the element $b = (x, y)$ with $x = -\ell$ and y arbitrary. Observe that b belongs to B. We have $b\backslash u = -(-\ell, y) + (-1, 1) = (\ell - 1, 1 - y) < (\ell, m) = a$ since $\ell - 1 < \ell$.

If $\ell = 0$, we take the element $b = (x, y)$ with $x = -1$ and $y = m - 1$. This again belongs to B. Then, we have $b\backslash u = -(-1, 1 - m) + (-1, 1) = (0, m) = a$. □

Product algebras. The variety of product algebras is generated by the real interval $[0, 1]$ as a residuated lattice under natural order and natural multiplication. In particular, the algebra $\mathfrak{P}$ best described as the upper half $\{a_n : n \in \omega\}$ of Chang's chain with an additional zero element at the bottom, is a product algebra. See Figure 2 for a picture of $\mathfrak{P}$; notice that for any $i \in \omega$ we have $\neg a_i = 0$. The algebra $\mathfrak{P}$ is subdirectly irreducible with A being its monolith congruence filter. In fact, $\mathfrak{P}$ also generates the whole variety of product algebras. Unlike $\mathfrak{C}$, the algebra $\mathfrak{P}$ is complete itself. It will be our example of a complete algebra whose ultrapower is not embeddable in any complete algebra from the variety it generates. Take an ultrapower $\mathfrak{R} = \mathfrak{P}^I/U$ for a nonprincipal U.

Theorem 6.7. *The algebra $\mathfrak{R}$ is a discontinuous GBL-algebra. Thus, the variety of product algebras is not closed under completions. The same holds for every larger variety of GBL-algebras.*

Proof. The algebra $\mathfrak{R}$ is subdirectly irreducible, with the monolith congruence filter isomorphic to A. Pick elements $w, u \in R \setminus A$ with $0 < w \prec u$. This is always possible by properties of ultraproducts, moreover, every element $e \in R \setminus A$ has an immediate successor e' (i.e., a cover with respect to the natural ordering). Define inductively $u_0 = u$ and $u_{n+1} = u'_n$. Let B be the downward closure of $\{u_k : k \in \omega\}$. Clearly, B is an ideal without supremum in R. Notice that $\mathfrak{P}$ has the following property:

- for every x and y, if $x > 0$ and y is the k-th successor of x, then $y \to x$ is the k-th predecessor of 1, i.e., equals a_k.

The property above is first-order expressible, and thus carries over to $\mathfrak{R}$. Therefore, $u_k \to u = a_k \in A$. Since every $b \in B$ is either smaller than u or equal to u_k for some $k \in \omega$, we have $b \to u \in A$ for every $b \in B$. Further, for each $j \in \omega$ we get $a_j = u_j \to u$. This takes care of all the required properties up to 3(a). To show that 3(b) holds as well, note first that if $b \leq w$, then $xb \leq w$ for any x whatsoever, so it remains to show that for each $j \in \omega$ there is an $a \in A$ with $au_j \leq w$. To this end, observe that $\mathfrak{P}$ has the following, first-order expressible, property.

- for every x and y, if $x > 0$ and y is the k-th predecessor of x, then the $k + 1$-th successor of 1, i.e., a_{k+1} has $a_{k+1} \cdot y \prec x$.

We conclude that $a_{k+1} \cdot u_k = w \prec u$ holds in $\mathfrak{R}$. This completes the list of requirements. The conclusion now follows by Theorem 5.2. □

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