数理解析研究所講究録
1526 巻 2006 年 164-172


# 橋梁マネジメントにおける有限要素曲面回帰

# Finite Element Surface Fitting Algorithms for Bridge Management

陳小君 木村拓馬
Xiaojun Chen Takuma Kimura
弘前大学理工学部数理科学科
Department of Mathematical Sciences, Hirosaki University

## 1. Introduction

In the study of bridge deterioration, it is important not only to consider structural evaluations and traffic load but also environmental factors such as climate and frequency of earthquakes. However, such studies require extensive data sets and methods for handling them. In the United States, the National Bridge Inventory Database (NBI) contains data on over 600,000 bridges for a span of 33 years. In the past few years, there has been a growing interest in the study of data mining methods for efficiently using NBI database with Geographic Information System (GIS) database to analyze and predict bridge deterioration [1]. The safety of bridges in Japan is also crucial for the national traffic network. Japan has more than 1,000 islands which form a long chain-like island arc surrounded by the Pacific Ocean and the Sea of Japan. In Japan, there exist extensive database that contains detailed historical data on over 672,000 bridges. Moreover, environmental factors have great impact to bridges. To utilize the extensive bridge database combined with very large geographic data sets, we have to develop efficient surface fitting methods.

Geographic datasets are often made at irregularly spaced points and have measurement errors. To analyze the irregularly spaced noisy data, many interpolation methods and least-squares smooth fitting methods have been developed. Our aim is to efficiently use very large data sets. In this paper, we study finite element discretization with a preconditioned Newton method to find an approximation of a smooth function which minimizes a sum of data residuals and the second derivative in $H^2(\Omega)$ under some constraints on data. The discretization results a large scale constrained quadratic program which involves large sparse matrices. We show the quadratic program has a unique solution and propose a preconditioned Newton method to find the solution.

Aomori Prefecture is located on the northernmost tip of Hunshu Island, Japan. It is bordered by the Sea of Japan(west), the Pacific Ocean(east), and the Tsugaru Channel(north). There are 1646 bridges in Aomori Region, in which 734 bridges are managed by Aomori Prefecture. See Figure 1. Environmental factors have significant impact to these bridges. We use the proposed method to form surfaces over Aomori Region by geographic databases, such that every bridge became associated with snowfall value, rainfall value, earthquake magnitude, etc. Using these expanded data sets, we can investigate the relation between environmental factors and bridge deterioration by data mining methods [1].

## 2. Finite element surface fitting

Let $\Omega \subset \mathbb{R}^2$ be a convex bounded domain. The given data are measurement points $\mathbf{x}_i = (x_{i,1}, x_{i,2}) \in \Omega$ and corresponding real values $y_i$, $i = 1, 2, \ldots n$. In a cartesian coordinate system, the $x_{i,1}$ and $x_{i,2}$ coordinates reflect the longitude and latitude, while the real value $y_i$ may reflect rainfall value or earthquake magnitude at point $\mathbf{x}_i$. We assume that $\mathbf{x}_i, i = 1, 2, \ldots, n$ are not collinear (i.e., they do not all lie on a line in $\Omega$). Let $S = \{\mathbf{x}_i, i = 1, 2, \ldots, n\}$ be the set of all measurement points, and $S_0 \subset S$ be a subset of $S$.

We consider the following minimization problem,

$$\begin{aligned} \min \quad & \frac{1}{n}\sum_{i=1}^{n}(f(\mathbf{x}_i) - y_i)^2 + \alpha |f|^2_{H^2(\Omega)} \\ \text{s.t.} \quad & f(\mathbf{x}_i) \geq \tilde{y}_i, \qquad \mathbf{x}_i \in S_0 \end{aligned} \tag{1}$$

over all functions $f$ in the Sobolev space $H^2(\Omega)$. Here $\tilde{y}_i$ are input data related to $y_i$ and $\alpha$ is a fixed positive parameter. The minimizer of (1) not only depends on the given data $\mathbf{x}_i$,$y_i$ and $\tilde{y}_i$, but also on the parameter $\alpha$. An appropriate choice of $\alpha$ depends on the size of the data.

In the case where $S_0$ is empty, the minimizer of (1) is called a thin plate spline. It has been shown by Duchon[6] that there exists a unique thin plate spline in $H^2(\Omega)$, when $\Omega = \mathbb{R}^2$. Moreover, numerical methods for finding approximate thin plate splines using simple finite element spaces in $H^2(\Omega)$ have been studied. However, there are some technical problems to use standard thin plate splines for applications that have large data sets. To deal with large data sets, Christen, Roberts, Hegland and Altas [5, 7] proposed a method to find a finite element thin plate spline in $H^1(\Omega)$. In their method, only first order derivatives occur, so that simple finite element spaces in $H^1(\Omega)$ can be used to discretize the problem.

In many applications, some constraints on data are required. For instance, at measurement points, snowfall values can not be negative or less than certain values. Furthermore, in some case, we do not know the exact values at some measurement points. Only upper or lower bounds of these values are available. Hence it is necessary to study the case where $S_0$ is nonempty. In this paper, we generalize the finite element approximation technique for the thin plate spline in [7] and define a discretization problem of (1). We introduce new vector variable $\mathbf{u} = (u_1, u_2)$ which represents the gradient of the function $f$ in (1), that is,

$$\nabla f = \mathbf{u}. \tag{2}$$

Moreover, we generalize the normalization condition

$$\sum_{i=1}^{n}(f(\mathbf{x}_i) - y_i) = 0$$

for the case $S_0 = \emptyset$ to

$$\sum_{\mathbf{x}_i \notin S_0} (f(\mathbf{x}_i) - y_i) = 0, \;\; f(\mathbf{x}_i) - \tilde{y}_i \geq 0, \;\; \mathbf{x}_i \in S_0. \tag{3}$$

Suppose that $S_0 = \{\mathbf{x}_{n-r+1}, \dots, \mathbf{x}_n\}$, where $r \geq 0$ ( $r = 0$ means $S_0 = \emptyset$).

Now we consider an associated function $u \in H^2(\Omega)$ satisfies

$$f(\mathbf{x}) = u(\mathbf{x}) + \mu \quad \text{and} \quad \sum_{i=1}^{n-r} u(\mathbf{x}_i) = 0$$

where $\mu$ is a constant. From (3) we have $\mu = \frac{1}{n-r}\sum_{i=1}^{n-r} y_i$. This leads to the following minimization problem:

$$\begin{aligned} \min \quad & \frac{1}{n}\sum_{i=1}^{n}(u(\mathbf{x}_i) + \mu - y_i)^2 + \alpha(|u_1|^2_{H^1(\Omega)} + |u_2|^2_{H^1(\Omega)}) \\ \text{s.t.} \quad & (\nabla u, \nabla v)_{L^2(\Omega)^2} = (\mathbf{u}, \nabla v)_{L^2(\Omega)^2}, \quad \forall v \in H^1(\Omega) \\ & \sum_{i=1}^{n-r} u(\mathbf{x}_i) = 0 \\ & u(\mathbf{x}_i) + \mu \geq \tilde{y}_i, \qquad i = 1, 2, \dots, r \end{aligned} \tag{4}$$

over all functions $u, u_1, u_2 \in H^1(\Omega)$. In this problem, $u_1$ and $u_2$ are approximations of $\dfrac{\partial u}{\partial x_1}$ and $\dfrac{\partial u}{\partial x_2}$, respectively. The purpose in introducing auxiliary functions $u_1$ and $u_2$ is for the use of simple finite element spaces in $H^1(\Omega)$. In particular, we use simple continuous piecewise polynomial spaces $\Omega^h \subset H^1(\Omega)$ associated with a finite element mesh over the domain $\Omega$.

Let $\mathbf{b}(\mathbf{x}) = (b_1(\mathbf{x}), \dots, b_m(\mathbf{x}))^T$ denote a vector of basis functions for $\Omega^h$. Then functions $u, u_1, u_2$ are given by

$$u(\mathbf{x}) = \mathbf{b}(\mathbf{x})^T c, \quad u_1(\mathbf{x}) = \mathbf{b}(\mathbf{x})^T g_1, \quad u_2(\mathbf{x}) = \mathbf{b}(\mathbf{x})^T g_2,$$

where the vectors $c, g_1, g_2 \in \mathbb{R}^m$ represent the linear combination coefficients in the basis $\mathbf{b}$. Using the following matrix $N = (b_i(x_j)) \in \mathbb{R}^{n \times m}$, we can define the values of $u, u_1, u_2$ at points $\mathbf{x}_i, i = 1, 2, \dots, n$, by

$$u(\mathbf{x}_i) = (Nc)_i, \quad u_1(\mathbf{x}_i) = (Ng_1)_i, \quad u_2(\mathbf{x}_i) = (Ng_2)_i.$$

Let matrices $A, B_1, B_2 \in \mathbb{R}^{m \times m}$ be given by

$$A_{i,j} = \int_\Omega \left(\frac{\partial b_i}{\partial x_1}\frac{\partial b_j}{\partial x_1} + \frac{\partial b_i}{\partial x_2}\frac{\partial b_j}{\partial x_2}\right) d\mathbf{x}, \;\; (B_l)_{i,j} = \int_\Omega \frac{\partial b_i}{\partial x_l} b_j d\mathbf{x}, \;\; l = 1, 2.$$

Set $P = (0_{r\times(n-r)}, I_{r\times r}) \in \mathbb{R}^{r\times n}$. Let $e_n \in R^n$ and $e_r \in R^r$ be the vectors whose elements are all ones. For the corresponding data $y_i$, we set $\mathbf{y} = (y_1 \cdots y_n)^T \in \mathbb{R}^n$, $\tilde{\mathbf{y}} = (\tilde{y}_1 \cdots \tilde{y}_r)^T \in \mathbb{R}^r$. Now, we can write the finite element surface fitting problem as a constrained optimization problem in $\mathbb{R}^{3m}$,

$$\begin{aligned}
\min \quad & \frac{1}{n}\|Nc + \mu e_n - \mathbf{y}\|_2^2 + \alpha(g_1^T A g_1 + g_2^T A g_2) \\
\text{s.t.} \quad & Ac = B_1 g_1 + B_2 g_2 \\
& (e_n^T - e_r^T P)Nc = 0 \\
& PNc + \mu e_r \geq \tilde{\mathbf{y}}
\end{aligned} \tag{5}$$

It is worth noting that the number of variables $(c, g_1, g_2) \in \mathbb{R}^{3m}$ depends on the discretization of a finite element mesh, but not on the size of data. Moreover, matrices $A, B_1, B_2 \in \mathbb{R}^{m\times m}$ and $N \in \mathbb{R}^{n\times m}$ are sparse.

## 3. Preconditioned Newton method

In this section, we show that problem (5) has a solution and present a preconditioned Newton method for solving (5).

Let

$$W = \begin{pmatrix} A \\ (e_n^T - e_r^T P)N \end{pmatrix}.$$

For any given $g_1, g_2 \in \mathbb{R}^m$, $c$ can be obtained by the equality constraints in problem (5) as

$$c = W^+ \begin{pmatrix} B_1 g_1 + B_2 g_2 \\ 0 \end{pmatrix},$$

where $W^+$ is the generalized inverse of $W$ [2] which satisfies $W^+ W = I \in \mathbb{R}^{m\times m}$. Let $W_1^+ \in \mathbb{R}^{m\times m}$ be the submatrix of $W^+$ whose entries lie in the first $m$ columns. Then we have $c = W_1^+(B_1 g_1 + B_2 g_2)$. Let us denote

$$g = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}, \quad G = NW_1^+(B_1, B_2), \quad H = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}.$$

Then problem (5) reduces to the following optimization problem

$$\begin{aligned}
\min \quad & \tfrac{1}{n}\|Gg + \mu e_n - \mathbf{y}\|_2^2 + \alpha g^T H g \\
\text{s.t.} \quad & PGg + \mu e_r \geq \tilde{\mathbf{y}}.
\end{aligned} \tag{6}$$

The objective function in (6) can be rewritten as

$$\frac{1}{n} g^T (G^T G + n\alpha H) g - \frac{2}{n} g^T G^T (\mathbf{y} - \mu e_n) + \frac{1}{n}(\mathbf{y} - \mu e_n)^T (\mathbf{y} - \mu e_n).$$

Let $Z = W_1^+ N^T N W_1^+$. The matrix

$$Q := G^T G + n\alpha H = \begin{pmatrix} B_1^T Z B_1 + n\alpha A & B_1^T Z B_2 \\ B_2^T Z B_1 & B_2^T Z B_2 + n\alpha A \end{pmatrix}$$

is symmetric positive definite. Therefore, the objective function in (6) is strongly convex, and problem (6) has a unique solution.

By convex optimization theory, problem (6) is equivalent to the following system of nonsmooth equations

$$F(z) = \begin{pmatrix} Qg - G^T P^T \lambda - G^T(\mathbf{y} - \mu e_n) \\ \min(\lambda, PGg + \mu e_r - \tilde{\mathbf{y}}) \end{pmatrix} = 0 \tag{7}$$

where $z = (g, \lambda)$ and $\lambda \in R^r$ is the Lagrange multiplier.

To solve (7), we use the generalized Newton method,

$$z^{k+1} = z^k - V_k^{-1} F(z^k) \tag{8}$$

where $V_k$ is an element in the generalized Jacobian of $F$ at $z^k$ [4]. Moreover, at each iteration of (8), we use a preconditioned Uzawa method [3] to solve the system of linear equations

$$V_k d = -F(z^k). \tag{9}$$

We can show that this method superlinearly converges to a solution $z^*$ of (7), that is,

$$\lim_{k\to\infty} \frac{\|z^{k+1} - z^*\|}{\|z^k - z^*\|} = 0. \tag{10}$$

Moreover, it can be shown that there is a constant $\kappa$ such that for any $z \in R^{2m+r}$,

$$\|z - z^*\| \leq \kappa \|F(z)\|. \tag{11}$$

This error bound can be used to examine how measurement errors in the data affect predicated values.

## 4. Numerical experiment

In the United States, data mining methods have been used to efficiently access bridge inventory database and geographic information for the study of bridge deterioration [1]. The KITACON company inspects bridges in Aomori Region regularly and has structural evaluations and traffic records for all bridges managed by Aomori Prefecture. In order to use efficient data mining methods for bridge management, we need environmental data, such as snowdepth values, rainfall values, earthquake magnitude, etc, on every bridge. We used the finite element surface fitting method proposed in previous sections and climate data from Japan Meteorological Agency to predicate some environmental data on every bridge in Aomori Region.

Aomori Region is surrounded by water on all three sides, the Pacific Ocean to the east, the Tsugaru Channel to the north and the sea of Japan to the west. Climate factors have great impact to bridges in Aomori Region. We report some numerical results on temperature, snowdepth, rainfall in Figures 2-4.

To verify the accuracy of the predicated environmental data on every bridge, we use these predicated data and the finite element surface fitting method to define testing values at the measurement points of Japan Meteorological Agency. Table 1 shows the relative error of the testing values to the true values at the measurement points.

Finally, we use the "peaks" function from Matlab to show that our finite element surface fitting method can handle very large data sets. The peaks function is formed as a linear combination of several scaled and translated Gaussian distributions. Table 2 reports the error $= \frac{1}{n}\sum_{i=1}^{n}|f(\mathbf{x}_i) - y_i|$ for a data set of $n$ points with a finite element grid of $m$ by $m$.

Preliminary numerical results indicate that the finite element surface fitting method is promising for handling very large data sets. The generalized Newton method is efficient for solving (7). Computation time for solving (7) are shown in Table 1 and Table 2. Computations were performed by using Matlab 7.0 on a IBM PC with 1GB memory and Pentium 4(3GHz).

## References


[1] S. B. Chase and E.P.Small, An in-depth analysis of the national bridge inventory database utilizing data mining, GIS and advanced statistical methods, Transportation Research Circular 498(C6), Presentations of the 8th International Bridge Management Conference, Denver, Colorado, 1999, 1-17.

[2] X. Chen, M.Z. Nashed and L. Qi, Convergence of Newton's method for singular smooth and nonsmooth equations using outer inverses, SIAM J. Optim., 7(1997) 445-462.

[3] X. Chen, Global and superlinear convergence of inexact Uzawa methods for saddle point problems with nondifferentiable mappings, SIAM J. Numer. Anal. 35(1998) 1130-1148.

[4] F. H. Clarke, Optimization and Nonsmooth Analysis, Wiley, New York, 1983.

[5] P. Christen, M.Hegland, S.Roberts and I.Altas, A scalable parallel FEM surface fitting algorithm for data mining, Technical Report TR-CS-01-01, The Australian National University 2001.

[6] J. Duchon, Splines minimizing rotation-invariant semi-norms in Sobolev spaces, in Constructive Theory of Functions of Several Variables, Lecture Notes in Math. 571, 1977, 85-100.

[7] S. Roberts, M.Helgland and I.Altas, Approximation of a thin plate spline smoother using continuous piecewise polynomial functions, SIAM J. Numer. Anal. 41(2003) 208-234.

Table 1: Environmental data in Aomori (n=734, $\alpha = 10^{-10}$)

| | m | relative error | | CPU(sec) |
|---|---|---|---|---|
| | | mean | max | |
| temperature | 380 | 7.71E-04 | 6.25E-03 | 1.31 |
| | 575 | 5.24E-04 | 5.38E-03 | 4.14 |
| | 992 | 2.74E-04 | 3.10E-03 | 20.33 |
| rainfall | 380 | 2.98E-03 | 3.43E-02 | 1.23 |
| | 575 | 2.24E-03 | 2.12E-02 | 4.22 |
| | 992 | 1.01E-03 | 1.57E-02 | 20.43 |
| snowdepth | 361 | 7.90E-03 | 1.33E-01 | 1.11 |
| | 576 | 5.29E-03 | 7.09E-02 | 4.19 |
| | 961 | 3.13E-03 | 4.69E-02 | 18.52 |

Table 2: Numerical results using the peaks function ($\alpha = 10^{-11}$)

| n | m | r | absolute error | | CPU(sec) |
|---|---|---|---|---|---|
| | | | mean | max | |
| $10^3$ | 324 | 0 | 3.25E-02 | 0.244 | 1.34 |
| | | 30 | 3.28E-02 | 0.241 | 2.55 |
| | 400 | 0 | 2.27E-02 | 0.207 | 2.30 |
| | | 40 | 2.26E-02 | 0.206 | 3.55 |
| $10^4$ | 324 | 0 | 3.43E-02 | 0.348 | 1.30 |
| | | 30 | 3.47E-02 | 0.346 | 2.44 |
| | 400 | 0 | 2.70E-02 | 0.280 | 2.25 |
| | | 40 | 2.75E-02 | 0.279 | 3.56 |
| $10^5$ | 324 | 0 | 3.49E-02 | 0.450 | 1.98 |
| | | 30 | 3.61E-02 | 0.448 | 2.02 |
| | 400 | 0 | 2.80E-02 | 0.529 | 3.39 |
| | | 40 | 2.72E-02 | 0.351 | 3.66 |
| $10^6$ | 400 | 0 | 2.73E-02 | 0.371 | 2.30 |
| | | 40 | 2.82E-02 | 0.368 | 5.28 |
| | 500 | 0 | 1.97E-02 | 0.289 | 5.08 |
| | | 50 | 2.00E-02 | 0.288 | 9.56 |

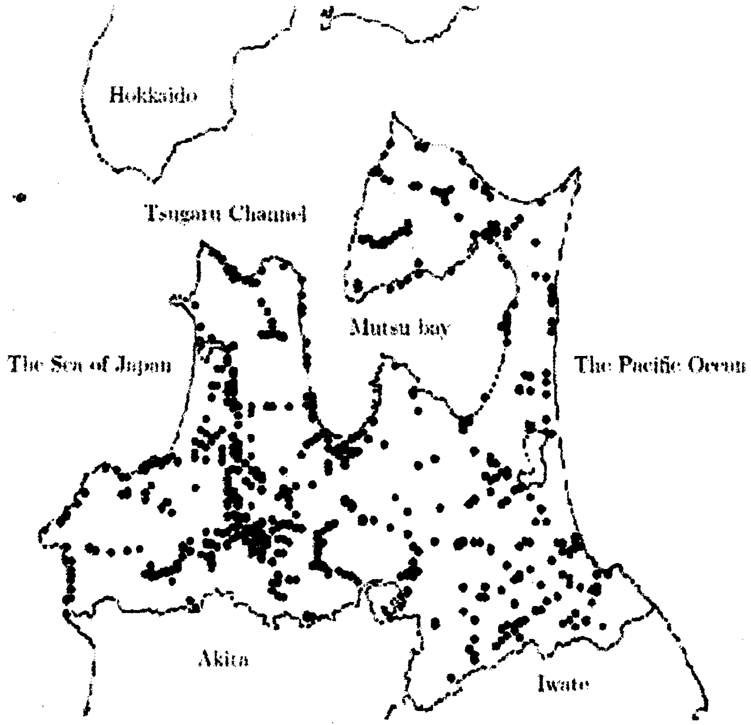


Figure 1: Distribution of bridges in Aomori

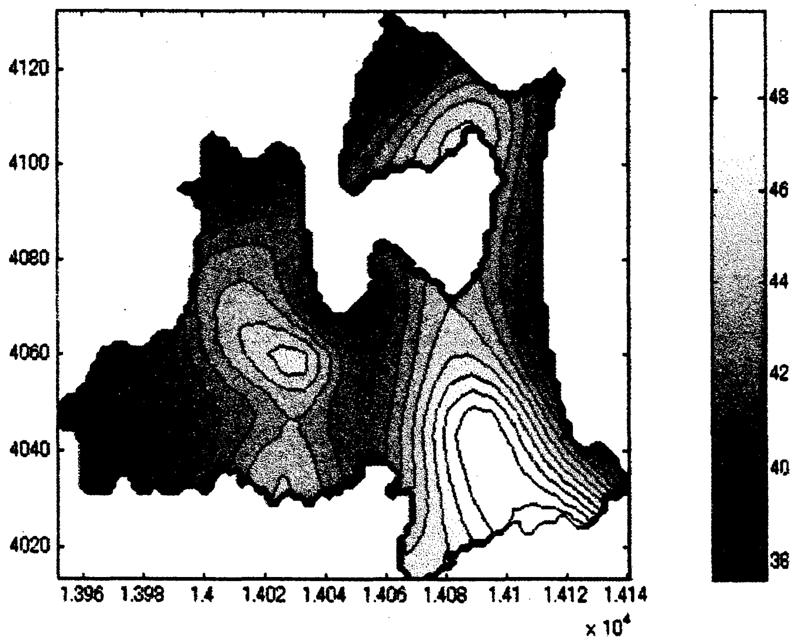


Figure 2: Values of maximum difference in temperature in 2005 (°C)

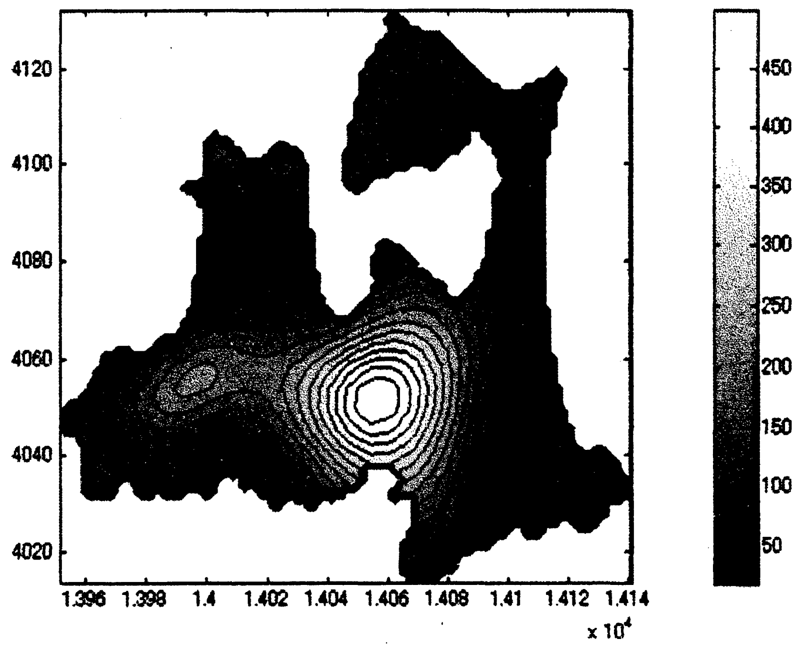


Figure 3: Values of highest snowdepth in 2005 (cm)

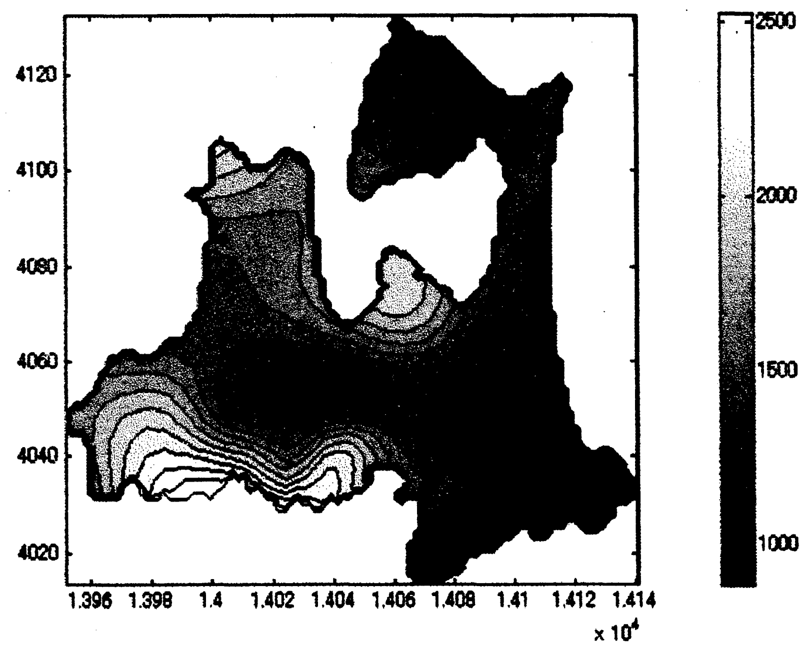


Figure 4: Values of total rainfall in 2005 (mm)