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Mass quantization in equilibrium self-gravitating fluid

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1 Exponential nonlinearity revisied

Emden-Fowler equation with exponential nonlinearity,

$$-\Delta v = \sigma e^{v} \quad \text{in } \Omega, \qquad v = 0 \quad \text{on } \partial\Omega \tag{1}$$

arises in the theories of thermonic emission, isothermal stationary gas sphere, and gas combustion [20, 10, 4], where $\Omega \subset \mathbf{R}^{m}$ is a bounded domain with smooth boundary $\partial\Omega$ and $\sigma > 0$ is a parameter. In the case of $m = 2$, it is also associated with the theories of turbulence and self-dual gauge [30, 28, 55, 47]. Actually, this equation with $m = 2$ is provided with complex and geometric structures, which results in the mass quantization of the blowup family of solutions [43, 45].

Complex Structure

Putting $u = v + \log\sigma$, we obtain

$$-\Delta u = e^{u} \qquad \text{in } \Omega. \tag{2}$$

If we identify $x = (x_1, x_2) \in \Omega$ to $z = x_1 + \imath x_2 \in \mathbf{C}$, then (2) means

$$u_{z\overline{z}} = -\frac{1}{4}e^{u}$$

for $\overline{z} = x_1 - \imath x_2$. This implies

$$s_{\overline{z}} = u_{zz\overline{z}} - u_z u_{z\overline{z}} = -\frac{1}{4}e^{u}u_z + \frac{1}{4}u_z e^{u} = 0$$

for

$$s = u_{zz} - \frac{1}{2}u_z^2, \tag{3}$$

and therefore, $s = s(z)$ is a holomorphic function of $z \in \Omega \subset \mathbf{C}$.

Regarding (3) as a Riccati equation of u, we obtain

$$\varphi_{zz} + \frac{1}{2}s\varphi = 0 \tag{4}$$

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for $\varphi = e^{-u/2}$. Here, we take $x^* = (x_1^*, x_2^*) \in \Omega$, and define a fundamental system of solutions to the linear equation (4), denoted by $\{\varphi_1(z), \varphi_2(z)\}$, such that

$$\varphi_1\big|_{z=z^*} = \frac{\partial \varphi_2}{\partial z}\bigg|_{z=z^*} = 1 \qquad \text{and} \qquad \frac{\partial \varphi_1}{\partial z}\bigg|_{z=z^*} = \varphi_2\big|_{z=z^*} = 0, \tag{5}$$

where $z^* = x_1^* + \imath x_2^*$. This $\{\varphi_1(z), \varphi_2(z)\}$ is composed of analytic functions of $z \in \Omega$, and it holds that

$$\varphi = e^{-u/2} = \overline{f_1}(\overline{z})\varphi_1(z) + \overline{f_2}(\overline{z})\varphi_2(z) \tag{6}$$

for some functions $\overline{f_1}$ and $\overline{f_2}$ of $\overline{z}$.

These $\overline{f_1}(\overline{z})$, $\overline{f_2}(\overline{z})$ are prescribed by the Wronskian. Since

$$W(\varphi_1, \varphi_2) \equiv \varphi_1\varphi_{2z} - \varphi_{1z}\varphi_2 = 1,$$

it holds that

$$\begin{aligned} \overline{f_1}(\overline{z}) &= W(\varphi, \varphi_2) = \varphi\varphi_{2z} - \varphi_z\varphi_2 \\ \overline{f_2}(\overline{z}) &= W(\varphi_1, \varphi) = \varphi_1\varphi_z - \varphi_{1z}\varphi, \end{aligned}$$

and the left-hand side is independent of z. Taking $z = z^*$ in the right-hand side, therefore, we obtain

$$\overline{f_1}(\overline{z}) = \varphi(z^*, \overline{z}) \qquad \text{and} \qquad \overline{f_2}(\overline{z}) = \varphi_z(z^*, \overline{z}). \tag{7}$$

Since φ is real-valued, it holds that

$$\varphi_{\overline{z}\overline{z}} + \frac{1}{2}\overline{s}\varphi = 0 \tag{8}$$

for $\overline{s} = \overline{s}(\overline{z})$ defined by $\overline{s}(\overline{z}) = \overline{s(z)}$. This relation is valid to $\varphi = \overline{f_1}(\overline{z}), \overline{f_2}(\overline{z})$ formulated by (7), while $\{\overline{\varphi}_1, \overline{\varphi}_2\}$ forms a fundamental system of solutions satisfying

$$\overline{\varphi}_1\big|_{\overline{z}=\overline{z^*}} = \frac{\partial \overline{\varphi}_2}{\partial \overline{z}}\bigg|_{\overline{z}=\overline{z}^*} = 1 \qquad \text{and} \qquad \frac{\partial \overline{\varphi_1}}{\partial \overline{z}}\bigg|_{\overline{z}=\overline{z}^*} = \overline{\varphi_2}\big|_{\overline{z}=\overline{z}^*} = 0.$$

Thus, $\overline{f_1}(\overline{z})$ and $\overline{f_2}(\overline{z})$ are linear combinations of $\overline{\varphi}_1(\overline{z})$ and $\overline{\varphi}_2(\overline{z})$.

If the above prescribed $x^* = (x_1^*, x_2^*) \in \Omega$ is a critical point of u, then it holds that

$$\begin{aligned} \overline{f_1}(\overline{z}^*) &= \varphi(z^*, \overline{z}^*) = e^{-u/2}\Big|_{x=x^*} \\ \frac{\partial \overline{f_1}}{\partial \overline{z}}(\overline{z}^*) &= \varphi_{\overline{z}}(z^*, \overline{z}^*) = \frac{\partial}{\partial \overline{z}} e^{-u/2}\bigg|_{x=x^*} = 0 \\ \overline{f_2}(\overline{z}^*) &= \varphi_{\overline{z}}(z^*, \overline{z}^*) = \frac{\partial}{\partial \overline{z}} e^{-u/2}\bigg|_{x=x^*} = 0 \\ \frac{\partial \overline{f_2}}{\partial \overline{z}}(\overline{z}^*) &= \varphi_{z\overline{z}}(z^*, \overline{z}^*) = \frac{1}{4}\Delta e^{-u/2}\bigg|_{x=x^*} = -\frac{1}{8}e^{-u/2}\Delta u\bigg|_{x=x^*} \\ &= \frac{1}{8}e^{u/2}\bigg|_{x=x^*}, \end{aligned}$$

and therefore, we obtain $\overline{f_1}(\overline{z}) = c\overline{\varphi_1}(\overline{z})$ and $\overline{f_2}(\overline{z}) = \frac{1}{8}c^{-1}\overline{\varphi_2}(\overline{z})$ for $c = e^{-u/2}\big|_{x=x^*}$. This means $f_1 = c\varphi_1$ and $f_2 = \frac{c^{-1}}{8}\varphi_2$, and therefore, it holds that

$$e^{-u/2} = c\,|\varphi_1|^2 + \frac{c^{-1}}{8}\,|\varphi_2|^2 \tag{9}$$

by (6). Writing $\psi_1 = c^{1/2}8^{1/4}\varphi_1$ and $\psi_2 = c^{-1/2}8^{-1/4}\varphi_2$, we have

$$\begin{aligned} W(\psi_1,\psi_2) &= W(\varphi_1,\varphi_2) = 1 \\ \left(\frac{1}{8}\right)^{1/2} e^{u/2} &= \left\{ c\left(\frac{1}{8}\right)^{-1/2} |\varphi_1|^2 + c^{-1}\left(\frac{1}{8}\right)^{1/2} |\varphi_2|^2 \right\}^{-1} \\ &= \frac{1}{|\psi_1|^2 + |\psi_2|^2}, \end{aligned}$$

and therefore,

$$\frac{|F'|}{1+|F|^2} = \frac{W(\psi_1,\psi_2)}{|\psi_1|^2 + |\psi_2|^2} = \left(\frac{1}{8}\right)^{1/2} e^{u/2} \tag{10}$$

for $F = \psi_2/\psi_1$. This means that (1) is reduced to finding an analytic function $F = F(z)$ of $z \in \Omega \subset \mathbf{C}$ such that

$$\rho(F)|_{\partial\Omega} = \left(\frac{\sigma}{8}\right)^{1/2} \tag{11}$$

by $u = v + \log\sigma$ and $v|_{\partial\Omega} = 0$, where

$$\rho(F) = \frac{|F'|}{1+|F|^2}.$$

The above defined $F = F(z)$ is a quotient of two linearly independent solutions to (4), and therefore, it holds that

$$\{F; z\} = -\frac{1}{2}s,$$

where

$$\{F; z\} = \frac{3}{4}\left(\frac{F''}{F'}\right)^2 - \frac{1}{2}\frac{F'''}{F'}$$

is the Schwarzian derivative.

Geometric Structure

It is known that $\rho(F)$ describes the spherical derivative of the meromorphic function $F = F(z)$. More precisely, if $d\Sigma^2$ denote the standard metric of the Riemannian sphere $\overline{\mathbf{C}}$ with the south pole $(0,0,0)$ and the north pole $(0,0,1)$, and if $\tau : \overline{\mathbf{C}} \to \mathbf{C}\cup\{\infty\}$ denotes the stereographic projection, then the conformal transformation $\overline{F} = \tau^{-1} \circ F$ induces the relation

$$\frac{d\Sigma}{ds} = \rho(F), \tag{12}$$

where $ds^2 = dx_1^2 + dx_2^2$ denotes the Euclidean metric on $\mathbf{C} \cong \mathbf{R}^2$. In particular, $\rho(F)$ is invariant under $O(3)$ transformation of $\overline{\mathbf{C}}$.

If $\omega \subset\subset \Omega$ is a sub-domain, then the immersed length of $\overline{F}(\partial\omega)$ and the immersed area of $\overline{F}(\omega)$ on $\overline{\mathbf{C}}$ are defined by

$$\ell_1(\partial\omega) = \int_{\partial\omega} \rho(F)ds \qquad \text{and} \qquad m_1(\omega) = \int_\omega \rho(F)^2 dx,$$

respectively, and therefore, it follows that

$$\ell_1(\partial\omega)^2 \geq 4m_1(\omega)\left(\pi - m_1(\omega)\right) \tag{13}$$

from the isoperimetric inequality. Putting

$$\begin{aligned} \ell(\partial\omega) &= \int_{\partial\omega} p^{1/2} ds = 8^{1/2} \int_{\partial\omega} \rho(F)ds \\ m(\omega) &= \int_\omega p dx = 8 \int_\omega \rho(F)^2 dx \end{aligned}$$

with $p = e^u$, we obtain

$$\ell(\partial\omega)^2 \geq \frac{1}{2} m(\omega)\left(8\pi - m(\omega)\right) \tag{14}$$

by (10) and (13).

Relation (14) is a form of Bol's inequality on the surface $\mathcal{M}$ with the Gaussian curvature less than or equal to 1/2. More precisely, (14) describes this inequality for non-parametric $\mathcal{M}$, where $p = p(x) > 0$ is a C^2 function defined on the domain $\Omega \subset \mathbf{R}^2$ of which boundary is composed of a finite number of Jordan curves,

$$-\Delta \log p \leq p \qquad \text{in } \Omega, \tag{15}$$

and $\omega \subset\subset \Omega$ is a sub-domain with the boundary $\partial\omega$ locally homeomorphic to a line. This geometric isoperimetric inequality induces an analytic isoperimetric inequality [2] concerning the first eigenvalue of the Laplace-Beltrami operator, by spherically decreasing rearrangement with respect to $d\Sigma = p(x)^{1/2}ds$, i.e.,

$$\lambda \equiv \int_\Omega p < 8\pi \quad \Rightarrow \quad \nu_1(p, \Omega) \geq \nu_1(p^*, \Omega^*), \tag{16}$$

where

$$\nu_1(p, \Omega) = \inf\left\{ \int_\Omega |\nabla v|^2 dx \mid v \in H_0^1(\Omega),\ \int_\Omega v^2 p dx = 1 \right\}, \tag{17}$$

is the first eigenvalue of

$$-\Delta\varphi = \nu p \varphi \quad \text{in } \Omega, \qquad \varphi = 0 \quad \text{on } \partial\Omega, \tag{18}$$

$p^* = \sigma^* e^{v^*}$, $\Omega^* = \left\{ x \in \mathbf{R}^2 \mid |x| < 1 \right\}$, and

$$\begin{aligned} &-\Delta v^* = \sigma^* e^{v^*} \quad \text{in } \Omega^*, \qquad v^* = 0 \quad \text{on } \partial\Omega^* \\ &\int_{\Omega^*} \sigma^* e^{v^*} = \lambda. \end{aligned} \tag{19}$$

This (σ^*, v^*) exists uniquely for each $\lambda \in (0, 8\pi)$.

Radial Solutions

From the general theory [21], any classical solution to (1) is radially symmetric if Ω is the unit ball. To classify such a solution for $n = 2$, first, we study

$$v'' + \frac{1}{r}v' + \sigma e^v = 0 \quad (0 < r < \infty), \qquad v'(0) = 0. \tag{20}$$

If $v_0 = v_0(r)$ is a solution to this problem, then so is

$$v(r) = v_0(e^{\alpha/2}r) + \alpha$$

for $\alpha \in \mathbf{R}$. Thus, we shall assign a special solution $v_0(r)$ to (20) and chose α by the boundary condition, i.e.,

$$v_0(e^{\alpha/2}) + \alpha = 0. \tag{21}$$

For this purpose, we deduce

$$\frac{d^2}{ds^2}(v + 2s) + \sigma e^{v+2s} = 0$$

from (20) using $s = \log r$, and obtain the one-dimensional case,

$$u'' + \sigma e^u = 0 \quad (-\infty < s < \infty), \tag{22}$$

where $u = v + 2s$. This equation implies

$$\left\{u'' - \frac{1}{2}(u')^2\right\}' = 0,$$

and we take the case

$$u'' - \frac{1}{2}(u')^2 = -2. \tag{23}$$

Actually, (23) is reduced to the logistic equation

$$\ell' = (1 - \ell)\ell$$

by $\ell = \frac{2-u'(s/2)}{4}$, and we can assign a solution

$$\ell(s) = \frac{1}{2}\left(1 + \tanh\frac{s}{2}\right).$$

This $\ell = \ell(s)$ induces $v_0 = v_0(r)$ defined by

$$v_0 + 2s = -2\log\cosh s + \log\frac{2}{\sigma},$$

i.e.,

$$v_0(r) = \log\left\{\frac{8/\sigma}{(r^2+1)^2}\right\}$$

as a special solution to (20). Then, (21) is reduced to the algebraic equation

$$\frac{8}{\sigma} = \frac{(e^\alpha + 1)^2}{e^\alpha},$$

and thus, we have classified the solutions to (1) for

$$\Omega = \Omega^* \equiv \left\{x \in \mathbf{R}^2 \mid |x| < 1\right\}.$$

Actually, they are described explicitly, i.e.,

$$v = v_{\pm}^{*\sigma}(x) = \log\left\{\frac{8\beta_\pm/\sigma}{\left(1+\beta_\pm |x|^2\right)^2}\right\}, \quad \beta_\pm = \frac{4}{\sigma}\left\{1-\frac{\sigma}{4}\pm\left(1-\frac{\sigma}{2}\right)^{1/2}\right\}.$$

It holds that

$$v_+^{*\sigma} = v_-^{*\sigma} = 2\log\frac{2}{1+|x|^2}$$

for $\sigma = 2$, and the number of solutions to (1) $\Omega = \Omega^*$ is two, one, and zero according to $0 < \sigma < 2$, $\sigma = 2$, and $\sigma > 2$, respectively. Total set of solutions $\mathcal{C}^* = \{(\sigma, v)\}$, on the other hand, forms a one-dimensional manifold in $\mathbf{R}_+ \times C(\overline{\Omega}^*)$. We obtain

$$\lim_{\sigma\downarrow 0} v_-^{*\sigma}(x) = 0 \qquad \text{uniformly in } x \in \overline{\Omega}^*$$

$$\lim_{\sigma\downarrow 0} v_+^{*\sigma}(x) = 4\log\frac{1}{|x|} \quad \text{locally uniformly in } x \in \overline{\Omega}^* \setminus \{0\},$$

and therefore, the endpoints of $\mathcal{C}^*$ are $(0,0)$ and $(0, v_*)$ with $v_* = 4\log\frac{1}{|x|}$. Thus, $v_* = v_*(x)$ is a singular limit of the solution.

It is convenient to write these radially symmetric solutions as

$$\left(\frac{\sigma}{8}\right)^{1/2} e^{v/2} = \left(\frac{e^u}{8}\right)^{1/2} = \frac{\mu^{1/2}}{|x|^2+\mu} \tag{24}$$

with

$$\mu^{1/2} = \mu_\pm^{1/2} = \left(\frac{2}{\sigma}\right)^{1/2}\left\{1 \mp \sqrt{1-\frac{\sigma}{2}}\right\}.$$

In fact, in this case their Liouville integrals (10) are described by $F(z) = Cz$ with

$$C = \mu^{-2} = C_\pm = \left\{\frac{1}{\sigma}\left\{4-\sigma\pm 2\sqrt{4-2\sigma}\right\}\right\}^{1/2},$$

and the length of $\overline{F}(\partial\Omega^*)$ and the area of $\overline{F}(\Omega^*)$ are equal to

$$\ell_1(\partial\Omega^*) = \int_{\partial\Omega^*}\left(\frac{e^u}{8}\right)^{1/2} ds = 2\pi\left(\frac{\sigma}{8}\right)^{1/2}$$

and

$$m_1(\Omega^*) = \int_{\Omega^*}\frac{e^u}{8}dx = \frac{1}{8}\int_{\Omega^*} p\,dx = \frac{\lambda}{8},$$

respectively. Therefore, λ grows from 0 to 8π monotonously along the branch $\mathcal{C}^*$ between $(0,0)$ and $(0, v_*)$. Furthermore, the bending point $\sigma = 2$ of $\mathcal{C}^*$ corresponds to $\lambda = 4\pi$, while σ increases first from 0 to 2, and then decreases from 2 to 0.

If we take λ as a control parameter and eliminate σ in (1), then it follows that

$$-\Delta v = \frac{\lambda e^v}{\int_\Omega e^v} \quad \text{in } \Omega, \qquad v = 0 \quad \text{on } \partial\Omega. \tag{25}$$

This λ casts a physical parameter derived from the inverse temperature and the coupling constant in turbulence and self-dual guage, respectively [8, 9, 30, 45, 46, 55].

Laplace-Beltrami Operator

From the general theory of bifurcation [16, 43], the above described profile of C^* guarantees the linearized stability of $v_-^{*\sigma}$ as a solution to (1) for $\Omega = \Omega^*$. This means that the first eigenvalue of the self-adjoint operator

$$L_-^{*\sigma} = -\Delta - \sigma e^{v_-^{*\sigma}}$$

in $L^2(\Omega^*)$ with the domain $(H^2 \cap H_0^1)(\Omega^*)$ is positive if $0 < \sigma < 2$, and zero if $\sigma = 2$. These properties are equivalent to $\nu_1(p^*, \Omega) > 1$ and $\nu_1(p^*, \Omega^*) = 1$ for $0 < \sigma < 2$ and $\sigma = 2$, respectively, where $\nu_1 = \nu_1(p^*, \Omega^*)$ denotes the first eigenvalue of

$$-\Delta\varphi = \nu p^* \varphi \quad \text{in } \Omega^*, \qquad \varphi = 0 \quad \text{on } \partial\Omega^* \tag{26}$$

for $p^* = \sigma e^{v_-^{*\sigma}}$.

This $\nu_1(p^*, \Omega^*)$ is equal to $\nu_1(p, \Omega)$ of (17) for $(p, \Omega) = (p^*, \Omega^*)$, and by the above consideration it holds that

$$0 < \lambda < 4\pi \;\Rightarrow\; \nu_1(p^*, \Omega^*) > 1, \tag{27}$$

where

$$p^*(x) = \frac{8\mu}{\left(|x|^2 + \mu\right)^2} \tag{28}$$

with $\mu > 0$ determined by

$$\lambda = \int_{\Omega^*} p^*(x)dx. \tag{29}$$

Thus, we obtain

$$0 < \lambda < 4\pi \;\Rightarrow\; \nu_1(p, \Omega) > 1 \tag{30}$$

if $\Omega \subset \mathbf{R}^2$ is a domain with smooth boundary $\partial\Omega$, $p = p(x) > 0$ is a C^2 function on $\overline{\Omega}$ satisfying (15), and $\int_\Omega p dx = \lambda$.

We can confirm, on the other hand, (27) directly, using the associated Legendre equation. More precisely, putting

$$\varphi(x) = \Phi(\xi)e^{\imath m\theta}, \quad x = re^{\imath\theta}, \quad \xi = \frac{\mu - r^2}{\mu + r^2}, \quad \Lambda = 1/\nu,$$

we obtain the associated Legendre equation [2]

$$\begin{aligned} &\left[(1-\xi^2)\Phi_\xi\right]_\xi + \left[2/\Lambda - m^2/(1-\xi^2)\right]\Phi = 0 \quad (\xi_\mu < \xi < 1) \\ &\Phi(1) = 1, \qquad \Phi(\xi_\mu) = 0 \end{aligned} \tag{31}$$

by (26), where $\xi_\mu = (\mu - 1)/(\mu + 1)$. Thus, if $\Phi = \Phi(\xi)$ denotes a solution to the first equation of (31) for $\Lambda = 1$, $m = 0$, and $\Phi(1) = 1$, then $\nu_1(p^*, \Omega^*) > 1$ is equivalent to

$$\Phi(\xi) > 0 \qquad (\xi_\mu < \xi < 1).$$

Since such Φ is given by $P_0(\xi) = \xi$, this means $\xi_\mu > 0$, and therefore, we can reproduce (27) by

$$\lambda < 4\pi \quad \Leftrightarrow \quad \mu > 1 \quad \Leftrightarrow \quad \xi_\mu > 0.$$

The associated Legendre equation arises when one adopts the polar coordinate to sustain the eigenvalues of three dimensional Laplacian written in the Cartesian coordinate. To understand the reason why this equation arises in the study of (26), we recall that $p^* = p^*(x)$ of (28) is associated with the Liouville integral $F(z) = \mu^{-2}z$ by $(p^*/8)^{1/2} = \rho(F)$. Using the stereographic projection: $\tau : \overline{\mathbf{C}} \to \mathbf{C} \cup \{\infty\}$, therefore, $\overline{\varphi} = \varphi \circ \tau$ satisfies

$$-\Delta_{\overline{\mathbf{C}}}\overline{\varphi} = \frac{\nu}{8}\overline{\varphi} \quad \text{in } \hat{\omega}, \qquad \varphi = 0 \quad \text{on } \partial\hat{\omega}, \tag{32}$$

where $\varphi = \varphi(x)$ is a solution to (26). Here, $\Delta_{\overline{\mathbf{C}}}$ is the Laplace-Beltrami operator and $\hat{\omega} \subset \overline{\mathbf{C}}$ is a disc with the center $(0,0,0)$. In other words, $\nu_1(p^*, \Omega^*)$ defined by (28)-(29) is nothing but the first eivenvalue of the Laplace-Beltrami operator $-\Delta_{S^2}$ defined on $\omega \subset S^2$ with $\cdot|_{\partial\omega} = 0$, where S^2 and $\omega \subset S^2$ denote the round sphere with total area 8π and an immersed disc with total area $\lambda \in (0, 8\pi)$, respectively. Then, we obtain the associated Legendre equation using separation of variables to (32).

Spherically decreasing rearrangement used in the proof of (16) is reformulated as a Schwarz symmetrization on the round sphere in this context. Thus, given a positive C^2 function $p = p(x)$ defined on a domain $\Omega \subset \mathbf{R}^2$ with continuous extension to $\overline{\Omega}$ satisfying (15) and $\int_\Omega p(x)dx = \lambda \in (0, 8\pi)$, we take an immersed disc $\omega \subset S^2$ with total area λ. Let $\varphi = \varphi(x)$ be a non-negative C^2 function defined on $\overline{\Omega}$ satisfying $\varphi|_{\partial\Omega} = 0$. Then, we put

$$\varphi^*(x) = \sup\{t \mid x \in \omega_t\} \tag{33}$$

for $x \in \omega$, where ω_t denotes the concentric disc of ω satisfying

$$\int_{\omega_t} dv = \int_{\{\varphi > t\}} pdx \tag{34}$$

and dv is the area element of S^2.

Spherically Harmonic Functions

We recall that if $\Omega \subset \mathbf{R}^2$ is a domain, then (2) is equivalent to (10), i.e., $\rho(F) = (e^u/8)^{1/2}$, where $F = F(z)$ is an analytic function. This is regarded as an analogy of the harmonic case, that is, $\Delta u = 0$ in Ω if and only if $u = \mathrm{Re}\ F$, where $F = F(z)$ is an analytic function. In fact, we can derive the mean value theorem for this type of functions described below, and this property guarantees a Harnack type inequality [41]. In this sense, the function $u = u(x)$ satisfying $-\Delta u \le e^u$ and $\Delta u \le e^u$ may be called spherically sub-harmonic and super-harmonic, respectively [43, 45], i.e., $-\Delta u \le e^u$ in Ω if and only if

$$u(x_0) \le \frac{1}{|\partial B(x_0,R)|}\int_{\partial B(x_0,R)} uds - 2\log\left\{1 - \frac{1}{8\pi}\int_{B(x_0,R)} e^u dx\right\}_+$$

for any $B(x_0, R) \subset\subset \Omega$, and similarly, $\Delta u \le e^u$ in Ω if and only if

$$u(x_0) \ge \frac{1}{|\partial B(x_0,R)|}\int_{\partial B(x_0,R)} uds - 2\log\left\{1 + \frac{1}{8\pi}\int_{B(x_0,R)} e^u dx\right\}$$

for any $B(x_0, R) \subset\subset \Omega$.

The first inequality implies the following fact, called Bandle's mean value theorem [1]: If $p = p(x)$ is continuous on $\overline{B}$, C^2 in B, and satisfies

$$-\Delta \log p \leq p \quad \text{in } B, \qquad \int_B p \leq 4\pi,$$

then it holds that

$$\frac{p(0)}{1 + r^2 p(0)/8} \leq \frac{1}{|\partial B_r|} \int_{\partial B_r} p^{1/2} ds, \tag{35}$$

where $B = B(0, R) \subset \mathbf{R}^2$, $B_r = B(0, r)$, and $r \in (0, R)$.

Duality

Problem (25) is the Euler-Lagrange equation of the functional

$$\mathcal{J}_\lambda(v) = \frac{1}{2} \|\nabla v\|_2^2 - \lambda \log \left(\int_\Omega e^v \right) + \lambda \log \lambda - \lambda \tag{36}$$

defined for $v \in H_0^1(\Omega)$. The Trudinger-Moser inequality [32] guarantees that this functional is C^1, and is bounded from below if $\lambda =' 8\pi$. Actually, there are several inequalities of this type [44].

This $\mathcal{J}_\lambda = \mathcal{J}_\lambda(v)$ is regarded as the dual form of a physically important functional, Helmholtz' free energy

$$\mathcal{F}(u) = \int_\Omega u(\log u - 1) - \frac{1}{2} \langle (-\Delta_D)^{-1} u, u \rangle$$

defined for $u \geq 0$ and $\int_\Omega u = \lambda$ ([45]). The equilibrium with respect to $\mathcal{F}(u)$ is described by

$$(-\Delta_D)^{-1} u = \log u + \text{constant} \quad \text{in } \Omega, \qquad \|u\|_1 = \lambda. \tag{37}$$

We define, on the other hand, the Lagrangian by

$$L(u, v) = \int_\Omega u(\log u - 1) + \frac{1}{2} \|\nabla v\|_2^2 - \langle v, u \rangle .$$

First, if $v \in H_0^1(\Omega)$ is a solution to (25) then $u = \frac{\lambda e^v}{\int_\Omega e^v}$ is a solution to (37), and conversely, if $u \geq 0$ is a solution to (37) then $v = (-\Delta_D)^{-1} u$ is a solution to (25). Next, there are unfolding Legendre transformation and the minimality in accordance with the Lagrangian formulated by

$$L|_{v=(-\Delta_D)^{-1}u} = \mathcal{F} \qquad \text{and} \qquad L|_{u=\frac{\lambda e^v}{\int_\Omega e^v}} = \mathcal{J}_\lambda \tag{38}$$

and

$$L(u, v) \geq \max \{ \mathcal{F}(u), \mathcal{J}_\lambda(v) \}, \tag{39}$$

respectively, where $u \geq 0$, $\|u\|_1 = \lambda$, and $v \in H_0^1(\Omega)$. We have, more precisely,

$$\begin{aligned} &\inf \{ L(u, v) \mid v \in H_0^1(\Omega) \} = \mathcal{F}(u) \\ &\inf \{ L(u, v) \mid u \geq 0, \ \|u\|_1 = \lambda \} = \mathcal{J}_\lambda(v), \end{aligned}$$

and in particular,

$$\inf_{u,v} L(u,v) = \inf_{v} \mathcal{J}_\lambda(v) = \inf_{u} \mathcal{F}(u).$$

These profiles are the consequence of the abstract structure of Toland duality [48, 49] observed in wide areas [45].

Decrease of this free energy together with mass conservation, on the other hand, is realized by the model (B) equation [23, 24]. This is the Smoluchowski-Poisson equation concerning material transport under the self-attractive force, or a simplified system of chemotaxis in the context of mathematical biology, i,e.,

$$\begin{aligned} & u_t = \nabla \cdot (\nabla u - u\nabla v), \quad -\Delta v = u \qquad \text{in } \Omega \times (0,T) \\ & \frac{\partial u}{\partial \nu} - u\frac{\partial v}{\partial \nu} = v = 0 \qquad \text{on } \partial\Omega \times (0,T). \end{aligned} \tag{40}$$

In fact, this system is described by

$$u_t = \nabla \cdot (u\nabla \delta\mathcal{F}(u)), \qquad u\frac{\partial}{\partial \nu}\delta\mathcal{F}(u)\Big|_{\partial\Omega} = 0$$

and hence it follows that

$$\begin{aligned} & \frac{d}{dt}\int_\Omega u = -\int_{\partial\Omega} u\frac{\partial}{\partial\nu}\delta\mathcal{F}(u) = 0 \\ & \frac{d}{dt}\mathcal{F}(u) = -\int_\Omega u\,|\nabla\delta\mathcal{F}(u)|^2 \le 0. \end{aligned}$$

This means that the stationary state of (40) is defined by

$$u \ge 0, \quad \|u\|_1 = \lambda, \quad \delta\mathcal{F}(u) = \text{constant},$$

that is, (37), equivalent to (25) by the above mentioned transformation.

Collapse Formation

There is a quantized blowup mechanism in the nonlinear eigenvalue problem (25) derived from (1), which is the origin of the formation of collapse with quantized mass to (40). A typical example of such a profile is the following theorem [44].

Theorem 1 *If the solution to*

$$\begin{aligned} & u_t = \nabla \cdot (\nabla u - u\nabla v), \quad -\Delta v = u - \frac{1}{|\Omega|}\int_\Omega u \qquad \text{in } \Omega \times (0,T) \\ & \frac{\partial u}{\partial \nu} - u\frac{\partial v}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 \qquad \text{on } \partial\Omega \times (0,T) \end{aligned}$$

blows-up at $t = T < +\infty$, then it holds that

$$u(x,t)dx \rightharpoonup \sum_{x_0 \in \mathcal{S}} m_*(x_0)\delta_{x_0}(dx) + f(x)dx$$

in $\mathcal{M}(\overline{\Omega})$ as $t \uparrow T$, where $\Omega \subset \mathbf{R}^2$ is a bounded domain with smooth boundary $\partial\Omega$,

$$\begin{aligned} \mathcal{S} = \{ x_0 \in \overline{\Omega} \mid & \textit{ there exist } x_k \to x_0 \textit{ and } t_k \uparrow T \textit{ such that} \\ & u(x_k, t_k) \to +\infty \} \end{aligned}$$

is the blowup set, $0 \leq f = f(x) \in L^1(\Omega) \cap C(\overline{\Omega} \setminus \mathcal{S})$, *and*

$$m_*(x_0) = \begin{cases} 8\pi & (x_0 \in \Omega) \\ 4\pi & (x_0 \in \partial\Omega). \end{cases}$$

Thus, we obtain

$$2\sharp(\mathcal{S} \cap \Omega) + \sharp(\mathcal{S} \cap \partial\Omega) \leq \|u_0\|_1 / (4\pi). \tag{41}$$

We can show also that the equality in (41) is excluded [45]. See [34] for later developments.

2 Two-Dimensional Mass Quantization

The quantized blowup mechanism to (1) is described as follows [33].

Theorem 2 *Let* $\Omega \subset \mathbf{R}^2$ *be a bounded domain with smooth boundary* $\partial\Omega$, *and* $\{(\sigma_k, v_k)\}_k$ *be a solution sequence to (1) such that* $\sigma_k \downarrow 0$. *Then, passing to a subsequence,* $\lambda_k = \int_\Omega \sigma_k e^{v_k} dx \to 8\pi\ell$ *with some* $\ell = 0, 1, \ldots, +\infty$. *According to this value, the solution behaves as follows:*

1. $\ell = 0$: *uniform convergence to* 0, *i.e.,* $\|v_k\|_\infty \to 0$.

2. $0 < \ell < +\infty$: *n-point blowup, i.e., there exist* $x_j^* \in \Omega$ $(j = 1, \ldots, \ell)$ *and* $v_0 = v_0(x)$ *such that* $v_k \to v_0$ *locally uniformly in* $\overline{\Omega} \setminus \mathcal{S}$ *for* $\mathcal{S} = \{x_1^*, \ldots, x_\ell^*\}$. *We obtain*

$$v_0(x) = 8\pi \sum_{j=1}^{\ell} G(x, x_j^*)$$
$$\frac{1}{2}\nabla R(x_j^*) + \sum_{i \neq j} \nabla_x G(x_i^*, x_j^*) = 0 \qquad (1 \leq j \leq \ell), \tag{42}$$

where $G = G(x, x')$ *is the Green's function:*

$$-\Delta G(\cdot, x') = \delta_{x'}(dx) \quad \text{in } \Omega, \qquad G(\cdot, x') = 0 \quad \text{on } \partial\Omega$$

defined for $x' \in \Omega$, *and*

$$R(x) = \left[G(x, x') + \frac{1}{2\pi} \log |x - x'| \right]_{x'=x}$$

is the Robin function.

3. $\ell = +\infty$: *entire blowup, i.e.,* $v_k \to +\infty$ *locally uniformly in* Ω.

The second case is crucial in the above theorem. Using (25), we can reformulate it as follows.

Theorem 3 *Let $\Omega \subset \mathbf{R}^2$ be a bounded domain with smooth boundary $\partial\Omega$, and $\{(\lambda_k, v_k)\}$ be a solution seqence to (25) satisfying $\lambda_k \to \lambda_0 \in (0, +\infty)$. Then, $\lambda_0 = 8\pi\ell$ with $\ell \in \mathbf{N}$, and passing to a subsequence, we obtain $v_k \to v_0$ locally uniformly in $\overline{\Omega} \setminus \mathcal{S}$ with $v_0 = v_0(x)$ satisfying (42) for $\mathcal{S} = \{x_1^*, \ldots, x_\ell^*\}$.*

To prove this case, first, we note that $\|v_k\|_{W^{1,q}} = O(1)$ holds by the L^1-estimate [39, 7], where $1 \leq q < 2 = \frac{n}{n-1}$ from $n = 2$. This implies a uniform boundary estimate indicated by

$$\|v_k\|_{L^\infty(\omega)} = O(1), \tag{43}$$

using the reflection argument combined with the Kelvin transformation [21, 17], where $\omega = \hat{\omega} \cap \overline{\Omega}$ and $\hat{\omega}$ is an open set satisfying $\partial\Omega \subset \hat{\omega}$.

The original proof uses the complex structure [33]. In fact, we obtain

$$e^{-v/2} = c\,|\varphi_1|^2 + \frac{\sigma c^{-1}}{8}\,|\varphi_2|^2$$

for (1), using $s(z)$ of (3) with u replaced by v, where $\{\varphi_1(z), \varphi_2(z)\}$ is the fundamental system of solutions to (4) defined by (5) with z^* corresponding to a critical point of v, denoted by $x^* \in \Omega$. Thus, there is a family of holomorphic functions $\{s_k(z)\}$ defined by (3) for $u = v_k$, and this family is uniformly bounded on $\overline{\Omega}$ by (43). Passing to a subsequence, therefore, we obtain $s_k \to s_0$ locally uniformly in Ω.

Introducing the fundamental system of solutions $\{\varphi_{1k}(z), \varphi_{2k}(z)\}$ to (4) for $s = s_k(z)$, we take $x^* = x_k^*$ as a maximum point of v_k. Passing to a subsequnece, the convergence $s_k \to s$ mentioned above guarantees those of $\varphi_{1k} \to \varphi_{10}$ and $\varphi_{2k} \to \varphi_{20}$ locally uniformly as analytic functions in Ω, because $\{x_k^*\}$ is in $\Omega \setminus \hat{\omega}$. Then, it holds that $c_k = \exp\left(-\|v_k\|_\infty/2\right) \to 0$ in the analogous relation to (9),

$$e^{-v_k/2} = c_k\,|\varphi_{1k}|^2 + \frac{\sigma_k c_k^{-1}}{8}\,|\varphi_{2k}|^2. \tag{44}$$

Since $\{v_k\}$ is bounded in $W^{1,q}(\Omega)$ for $1 \leq q < 2$, any blowup point of $\{v_k\}$ must be zero of the analytic function φ_{10}, and therefore, each blowup point is isolated. We obtain finiteness of the blowup points in this way, while classification of the singular limit, (42), is derived by residue analysis, more precisely, singularity vanishing of $s_0(z) = v_{0zz} - \frac{1}{2}v_{0z}^2$.

This proves the theorem, but here we obtain $\sigma_k c_k^{-1} \approx 1$, i.e., $\|v_k\|_\infty \approx -2\log\sigma_k$ as $k \to \infty$. From the proof of Theorem 5 described below, on the other hand, each x_j^* takes a sequence $x_k^j \to x_j^*$, where x_k^j is a local maximum point of v_k. Thus, we can reformulate $x^* = x_k^j$ in (44), and conseqeuently, the rates of blowup $v_k(x_k^j) \to +\infty$ $(j = 1, \ldots, \ell)$ are proportional each other.

The other proof of the above theorem uses Theorems 4-5 described in the following paragraph and the Pohozaev identity instead of the complex structure [29]. This argument is valid to the non-homogeneous coefficient case.

The second equality of (42) means that $(x_1^*, \ldots, x_\ell^*) \in \Omega \times \ldots \times \Omega$ is a critical point of

$$H = H(x_1, \ldots, x_\ell) = \frac{1}{2}\sum_i R(x_i) + \sum_{i<j} G(x_i, x_j).$$

If it is non-degenerate, then there is a local branch of solutions taking $(\sigma, v) = (0, v_0)$ as an endpoint for $v_0 = v_0(x)$ defined by the first relation of (42) ([3]). First, the complex structure was used for this purpose, assuming that Ω is simply-connected and $\ell = 1$ ([53, 31, 42]).

Theorem 3 guarantees that the total degree of the solution set is constant in each component of $[0, +\infty) \setminus 8\pi\mathbf{N}$ ([26]). It is actually determined by the genus of Ω and explicit formula is given by a detailed blowup analysis [11, 12, 13]. For example, if $\ell = 1$, then it holds that

$$\|v_k\|_\infty = -2\log\sigma_k + 2\log 8 - 8\pi R(x_0) + o(1).$$

We obtain also

$$v(x) = \sum_{i=1}^{2} \frac{a_i x_i}{8 + |x|^2} + b \cdot \frac{8 - |x|^2}{8 + |x|^2}$$

if $v = v(x)$ is a uniformly bounded solution to the linearized entire problem

$$-\Delta v = \frac{v}{\left\{1 + \frac{|x|^2}{8}\right\}^2} \qquad \text{in } \mathbf{R}^2,$$

where $a_i, b \in \mathbf{R}$ ([22]).

Blowup Analysis

Self-similarity is observed in many equations in mathematical physics. Concerning (2) derived from the vorticity equation and the abelian Higgs theory, it is invariant under the transformation

$$u^\mu(x) = u(\mu x) + 2\log\mu,$$

where $\mu > 0$ is a constant. This causes the lack of compactness of the family of (approximate) solutions, and this mechanism is clarified by the blowup analysis of which ingredients are summarized as follows:

1. scaling invariance of the problem.
2. classification of the entire solution.
3. control at infinity of the rescaled solution.
4. hierarchical argument.

The following theorem, free from the boundary condition, is useful in such a study, because the effect of boundary conditions is usually lost in the scaling argument. It also deals with the nonhomogeneous coefficient case with the lack of the complex structure. Theorem 2 is associated with this theorem by $v_k - \log\sigma_k$. Actually, we obtain $v > 0$ in (1) ([6, 27]).

Theorem 4 *If $\Omega \subset \mathbf{R}^2$ is a bounded domain and $v_k = v_k(x)$ $(k = 1, 2, \ldots)$ is a solution sequence to*

$$-\Delta v_k = V_k(x)e^{v_k} \qquad \text{in } \Omega$$

with

$$0 \leq V_k(x) \leq C_1 \quad \text{in } \Omega, \qquad \int_\Omega e^{v_k} \leq C_2, \tag{45}$$

where C_1, C_2 are constants, then, passing to a subsequence, there arises the following alternatives:

1. *$\{v_k\}$ is locally uniformly bounded in Ω.*

2. *$v_k \to -\infty$ locally uniformly in Ω.*

3. *There is a finite set $\mathcal{S} = \{x_j^*\} \subset \Omega$ and $m_j \geq 4\pi$ such that $v_k \to -\infty$ locally uniformly in $\Omega \setminus \mathcal{S}$ and*

$$V_k(x)e^{v_k}dx \rightharpoonup \sum_j m_j \delta_{x_j^*}(dx) \qquad \text{in } \mathcal{M}(\Omega). \tag{46}$$

Furthermore, $\mathcal{S}$ is the blowup set of $\{v_k\}$ in Ω.

Theorem 5 *In the third case of the above theorem, we obtain $m_j = 8\pi n_j$ for some $n_j \in \mathbf{N}$, provided that $V_k \to V$ uniformly on $\overline{\Omega}$.*

Boundedness of the Palais-Smale sequence relative to the Trudinger-Moser inequality does not follow always. Then, the above theorem is applied to compensate this difficulty in constructing non-trivial solutions to the mean field equation [40, 18].

There are several differences between the energy quantization described in the previous chapter. First, the above blowup mechanism occurs only to the quantized values of mass, realized as the eigenvalue λ. Thus, we obtain the residual vanishing, the disappearance of the regular part of the limit measure in (46) under the control (45) of the additive constant. Second, global structure described by the compactness of the domain manifold or the boundary condition excludes multiple bubbles, prescribing the location of blowup points.

The proof of these theorems are described in [47, 45]. First, we perform the prescaled analysis to prove Theorem 4. In more precise, if $\Omega \subset \mathbf{R}^2$ is a bounded domain, $f \in L^1(\Omega)$, and

$$-\Delta v = f(x) \quad \text{in } \Omega, \qquad v = 0 \quad \text{on } \partial\Omega,$$

then it holds that

$$\int_\Omega \exp\left(\frac{4\pi - \delta}{\|f\|_1} |v(x)|\right) dx \leq \frac{4\pi^2}{\delta} (\text{diam } \Omega)^2,$$

where $0 < \delta < 4\pi$. This implies ε-regularity, described by the following lemma, and then Theorem 4 is obtained by a standard argument.

Lemma 2.1 *Let $\Omega \subset \mathbf{R}^2$ be a bounded domain, $K \subset \Omega$ a compact set, $c_1, c_2 > 0$, and $\varepsilon_0 \in (0, 4\pi)$. Then, there is $C > 0$ such that*

$$-\Delta v = V(x)e^v, \quad 0 \leq V(x) \leq c_1 \quad \text{in } \Omega$$
$$\|v^+\|_1 \leq c_2, \quad \int_\Omega V(x)e^v \leq \varepsilon_0$$

implies $\|v^+\|_{L^\infty(K)} \leq C$.

Once Theorem 4 is proven, then Theorem 5 is reduced to the following case, where $B = B(0, R) \subset \mathbf{R}^2$ and $B_r = B(0, r)$.

Theorem 6 *If*

$$\begin{aligned}
&-\Delta v_k = V_k(x)e^{v_k}, \quad V_k(x) \geq 0 \qquad in\ B \\
&V_k \to V \qquad in\ C(\overline{B}) \\
&\max_{\overline{B}} v_k \to +\infty \\
&\max_{\overline{B}\setminus B_r} v_k \to -\infty \qquad (0 < r < R) \\
&\lim_{k\to\infty} \int_B V_k(x)e^{v_k} = \alpha \\
&\int_B e^{v_k} \leq C_0,
\end{aligned}$$

then it holds that $\alpha \in 8\pi\mathcal{N}$.

There is actually the case of $\alpha = 8\pi\ell$ with $\ell \geq 2$ in the above theorem [15]. However, the conclusion $\alpha = 8\pi$ arises, provided that

$$\max_{\partial B} v_k - \min_{\partial B} v_k \leq C \qquad \text{and} \qquad \|\nabla V_k\|_\infty \leq C. \tag{47}$$

We obtain, furthermore,

$$\left| v_k(x) - \log \frac{e^{v_k(0)}}{\left(1 + \frac{V_k(0)}{8} e^{v_k(0)} |x|^2\right)^2} \right| \leq C \tag{48}$$

for $k = 1, 2, \dots$ and $x \in B$ in this case [26]. If (47) holds for $B = \Omega$, then $n_j = 1$ for any j in Theorem 5 and furthermore, the blowup points $x_1^*, \dots, x_\ell^*$ are prescribed by

$$\frac{1}{2}\nabla R(x_j^*) + \sum_{i\neq j} \nabla_x G(x_i^*, x_j^*) + \frac{1}{8\pi}\nabla \log V(x_j^*) = 0 \qquad (1 \leq j \leq \ell)$$

similarly to the second equation of (42) ([29]). See [35, 36, 37, 44] for related results.

Theorem 6 is proven by the blowup anlaysis. Thus, we take $x_k \in B$ satisfying $v_k(x_k) = \|v_k\|_\infty$ with $x_k \to 0$, and put

$$\begin{aligned}
&\tilde{v}_k(x) = v_k(\delta_k x + x_k) + 2\log\delta_k \\
&\delta_k = e^{-v_k(x_k)/2} \to 0.
\end{aligned}$$

Then, it holds that

$$\begin{aligned}
&-\Delta\tilde{v}_k = V_k(\delta_k x + x_k)e^{\tilde{v}_k}, \quad \tilde{v}_k \leq 0 = \tilde{v}_k(0) = 0 \qquad \text{in } B(0, R/2\delta_k) \\
&\int_{B(0,R/2\delta_k)} e^{\tilde{v}_k} \leq C_0,
\end{aligned}$$

and Theorem 4 is applicable to this $\{\tilde{v}_k\}$. Thus, $\{\tilde{v}_k\}$ is locally uniformly bounded in $\mathbf{R}^2$, and passing to a subsequence, we obtain $\tilde{v}_k \to \tilde{v}$ locally uniformly in $\mathbf{R}^2$ with

$$\begin{aligned}
&-\Delta\tilde{v} = V(0)e^{\tilde{v}}, \quad \tilde{v} \leq 0 = \tilde{v}(0) \qquad \text{in } \mathbf{R}^2 \\
&\int_{\mathbf{R}^2} e^{\tilde{v}} \leq C_0
\end{aligned}$$

by the elliptic regularity. From this we can infer $V(0) > 0$ and hence assume

$$a \leq V_k(x) \leq b \qquad (x \in B)$$

for $k = 1, 2, \ldots$ without loss of generality, where $a, b > 0$ are constants. We obtain, furthermore, $\tilde{v} = \tilde{v}(|x|)$ by the method of moving plane [14], which results in

$$\tilde{v}(x) = \log \frac{1}{\left(1 + \frac{V(0)}{8}|x|^2\right)^2}, \qquad \int_{\mathbf{R}^2} V(0)e^{\tilde{v}} = 8\pi. \tag{49}$$

Sup + Inf Inequality

We have detected the principal collapse formed at the origin in the proof of Theorem 6. Now, we have to show the vanishing of residual parts by collecting the other collapses. This is done by the sup + inf inequality proven by Alexandroff's inequality originally. Alexandroff's inequality is also an isoperimetric inequality on surface described by its Gaussian curvatue, regarded as a refinement of Bol's inequality [2]. Thus, we can show the following lemma [38].

Lemma 2.2 *Let $B = B(0,1) \subset \mathbf{R}^2$ and $a, b > 0$ be constants. Then, there are $C_0 > 0$ and $\alpha_0 > 4\pi$ such that*

$$-\Delta v = V(x)e^v, \quad a \leq V(x) \leq b \qquad \text{in } B$$
$$\int_B V(x)e^v \leq \alpha_0$$

implies $v(0) \leq C_0$.

This lemma is regarded as a refinement of Lemma 2.1 under the cost of $V(x) \geq a$. If $V = V(x)$ is restricted to a compact family in $C(\overline{\Omega})$, which is sufficient for later arguments, then we can apply the blowup analysis for the proof. In this case, the above α_0 can be arbitrary in $\alpha_0 < 8\pi$ and furthermore, the case $a = 0$ is permitted.

Using the above lemma and the scaling invariance of the equation, next, we show the following lemma [38].

Lemma 2.3 *If $\Omega \subset \mathbf{R}^2$ is a bounded domain, $K \subset \Omega$ is a compact set, and $a, b > 0$ are constants, then there are*

$$c_1 = c_1(a,b) \geq 1 \qquad \text{and} \qquad c_2 = c_2(a, b, \text{dist}\,(K, \partial\Omega)) > 0$$

such that

$$-\Delta v = V(x)e^v, \quad a \leq V(x) \leq b \qquad \text{in } \Omega$$

implies

$$\sup_K v + c_1 \inf_\Omega v \leq c_2. \tag{50}$$

In the other version of (50) proven by the blowup analysis [5], the condition $c_1 = 1$ is achieved under the cost of $\|\nabla V\|_\infty \leq C$. In any case, this sup + inf inequality induces the key estimate, again by the scaling [27].

Lemma 2.4 *Given $a, b > 0$ and $C_1 > 0$, we obtain $\gamma > 0$, $C_2 > 0$ independent of $0 < R_0 \leq R/4$ such that*

$$\begin{aligned} &-\Delta v = V(x)e^v, \quad a \leq V(x) \leq b \qquad \text{in } B_R \\ &v(x) + 2\log|x| \leq C_1 \qquad \text{in } B_R \setminus \overline{B_{R_0}} \end{aligned}$$

implies

$$e^{v(x)} \leq C_2 e^{-\gamma v(0)} \cdot |x|^{-2(\gamma+1)}$$

for $2R_0 \leq |x| \leq R/2$.

Residual Vanishing

To complete the proof of Theorem 6, first, we recall the blowup argument to detect the principal collapse. Using the diagonal argument, this process is refined. More precisely, we obtain $r_k^0 \to 0$ satisfying $r_k^0/\delta_k^0 \to +\infty$ and

$$\int_{B(x_k^0, 2r_k^0)} V_k(x)e^{v_k} \to 8\pi,$$

where $\|v_k\|_\infty = v_k(x_k^0) \to +\infty$, $x_k^0 \to 0$, and $\delta_k^0 = e^{-v_k(x_k^0)/2} \to 0$.

If

$$\sup\left\{v_k(x) + 2\log\left|x - x_k^0\right| \mid x \in B \setminus B(x_k^0, r_k^0)\right\} < +\infty, \tag{51}$$

then Lemma 2.4 is applicable. We have

$$e^{v_k(x)} \leq Ce^{-\gamma v_k(x_k^0)}\left|x - x_k^0\right|^{-2(\gamma+1)}$$

for $x \in B_{R/2} \setminus B(x_k^0, r_k^0)$, and therefore,

$$\begin{aligned} \int_{B_{R/2}\setminus B(x_k^0, r_k^0)} V_k(x)e^{v_k} &\leq b \cdot C \cdot (\delta_k^0)^{2\gamma} \cdot 2\pi \cdot \int_{r_k^0}^{+\infty} r^{-2(\gamma+1)} r dr \\ &= \frac{\pi bC}{\gamma}(\delta_k^0/r_k^0)^{2\gamma} \to 0. \end{aligned}$$

This implies

$$\int_B V_k(x)e^{v_k} \to 8\pi$$

because $v_k \to -\infty$ locally uniformly in $\overline{B} \setminus \{0\}$, and hence $\alpha = 8\pi$.

If (51) is not the case, then there is $x_k^1 \in \overline{B}$ such that

$$\sup_{x \in B \setminus B(x_k^0, r_k^0)} \left\{v_k(x) + 2\log\left|x - x_k^0\right|\right\} = v_k(x_k^1) + 2\log\left|x_k^1 - x_k^0\right| \to +\infty.$$

This implies $v_k(x_k^1) \to +\infty$ and $x_k^1 \to 0$. Furthermore, $\sigma_k^1 = d_k/\delta_k^1 \to +\infty$ for

$$d_k = \left|x_k^1 - x_k^0\right| \qquad \text{and} \qquad \delta_k^1 = e^{-v_k(x_k^1)/2}.$$

Given $|x| \leq \sigma_k^1/2$, we have

$$\left|\delta_k^1 x + x_k^1 - x_k^0\right| \geq \left|x_k^1 - x_k^0\right| - \delta_k^1|x| \geq \frac{1}{2}\left|x_k^1 - x_k^0\right|$$

and therefore,

$$\begin{aligned}
\tilde{v}_k^1(x) &\equiv v_k(\delta_k^1 x + x_k^1) + 2\log\delta_k^1 \\
&\leq v_k(x_k^1) + 2\log\left|x_k^1 - x_k^0\right| - 2\log\left|\delta_k^1 x + x_k^1 - x_k^0\right| + 2\log\delta_k^1 \\
&\leq v_k(x_k^1) + 2\log\delta_k^1 + 2\log\left|x_k^1 - x_k^0\right| - 2\log\frac{1}{2}\left|x_k^1 - x_k^0\right| \\
&= 2\log 2.
\end{aligned}$$

This implies

$$\begin{aligned}
&-\Delta\tilde{v}_k^1 = V_k(\delta_k^1 x + x_k^1)e^{\tilde{v}_k^1}, \quad \tilde{v}_k^1 \leq 2\log 2 \qquad \text{in } B_{\sigma_k^1/2} \\
&\tilde{v}_k^1(0) = 0,
\end{aligned}$$

and passing to a subsequence, we obtain $\tilde{v}_k^1 \to \tilde{v}^1$ in $C_{loc}^{1,\alpha}(\mathbf{R}^2)$ with $\tilde{v}^1 = \tilde{v}^1(x)$ satisfying

$$\begin{aligned}
&\tilde{v}^1(x) = \log\frac{a^2}{\left(1 + \mu^2 a^2\left|x - \overline{x}\right|^2\right)^2}, \quad \tilde{v}^1(x) \leq 2\log 2 \qquad (\text{for } x \in \mathbf{R}^2) \\
&\tilde{v}^1(0) = 0
\end{aligned}$$

for some $\mu, a > 0$ and $\overline{x} \in \mathbf{R}^2$.

This convergence allows us to reformulate x_k^1 and δ_k^1 by

$$v_k(x_k^1) = \|v_k\|_{L^\infty(B(x_k^1, 2r_k^1))} \to +\infty,$$

$\delta_k^1 = e^{-v_k(x_k^1)/2} \to 0$, and $r_k^1/\delta_k^1 \to +\infty$, where $r_k^1 = d_k/4$. Similarly to the above case, it follows that

$$\int_{B(x_k^1, 2r_k^1)} V_k(x)e^{v_k} \to 8\pi$$

with $B(x_k^1, 2r_k^1) \cap B(x_k^0, 2r_k^0) = \emptyset$.

We shall show $\alpha = 16\pi$, if

$$\sup\left\{v_k(x) + 2\log\min_{j=0,1}\left|x - x_k^j\right| \mid x \in B \setminus \bigcup_{j=0}^{1} B(x_k^j, r_k^j)\right\} < +\infty. \tag{52}$$

is satisfied. It suffices to prove

$$\int_{B(x_k^0, 2d_k)} V_k(x)e^{v_k} \to 16\pi \tag{53}$$

because

$$\int_{B\setminus B(x_k^0, 2d_k)} V_k(x)e^{v_k} \to 0$$

follows from (52) similarly. For this purpose, we take $\tilde{v}_k(x) = v_k(d_k x + x_k^0) + 2\log d_k$ and obtain

$$-\Delta\tilde{v}_k = V_k(d_k x + x_k^0)e^{\tilde{v}_k} \qquad \text{in } d_k^{-1}\left(B - \left\{x_k^0\right\}\right).$$

We put, furthermore,

$$\tilde{x}_k^j = \frac{x_k^j - x_k^0}{d_k}, \qquad \tilde{\delta}_k^j = e^{-\tilde{v}_k(\tilde{x}_k^j)/2} = \frac{\delta_k^j}{d_k}, \qquad \tilde{r}_k^j = \frac{r_k^j}{d_k}$$

for $j = 0, 1$, and then it holds that

$$\begin{aligned}
&\frac{\tilde{r}_k^j}{\tilde{\delta}_k^j} = \frac{r_k^j}{\delta_k^j} \to +\infty \\
&B(\tilde{x}_k^0, 2\tilde{r}_k^0) \cap B(\tilde{x}_k^1, 2\tilde{r}_k^1) = \emptyset \\
&\sup\left\{ \tilde{v}_k(x) + 2\log \min_{j=0,1} \left| x - \tilde{x}_k^j \right| \mid x \in B_{R/d_k} \setminus \bigcup_{j=0}^{1} B(\tilde{x}_k^j, \tilde{r}_k^j) \right\} < +\infty \\
&\int_{B(\tilde{x}_k^j, 2\tilde{r}_k^j)} \tilde{V}_k(x) e^{\tilde{v}_k} \to 8\pi \qquad (j = 0, 1)
\end{aligned} \tag{54}$$

for $\tilde{V}_k(x) = V_k(d_k x + x_k^0)$.

We obtain $\tilde{x}_k^0 = 0$ and $\left|\tilde{x}_k^1 - \tilde{x}_k^0\right| = 1$, and therefore, $\tilde{x}_k^1 \to \tilde{x}^1$ with $\left|\tilde{x}^1\right| = 1$, passing to a subsequence. The third relation of (54) and Theorem 4 now imply $\tilde{v}_k \to -\infty$ locally uniformly in $\mathbf{R}^2 \setminus \left\{0, \tilde{x}^1\right\}$. Therefore, if $\tilde{r}_k^1 \to \tilde{r}^1 > 0$, passing to a subsequence, then

$$\int_{B(\tilde{x}_k^1, 1/2)} \tilde{V}_k(x) e^{\tilde{v}_k} \to 8\pi \tag{55}$$

and hence (53). If $\tilde{r}_k^1 \to 0$, we apply the scaling around $\tilde{x}_k^1$. Then, it holds that (55) by the third relation of (54).

If (52) is not the case, we continue the process and obtain $x_k^2 \to 0$ and $r_k^2 \to 0$ satisfying $v_k(x_k^2) = \|v_k\|_{L^\infty(B(x_k^2, 2r_k^2))} \to +\infty$, $r_k^2/\delta_k^2 \to 0$, $B(x_k^i, 2r_k^i) \cap B(x_k^j, 2r_k^j) = \emptyset$ for $0 \le i < j \le 2$, and

$$\int_{B(x_k^2, 2r_k^2)} V_k(x) e^{v_k} \to 8\pi,$$

where $\delta_k^2 = e^{-v_k(x_k^2)/2}$. To show that $\alpha = 24\pi$ in the case of

$$\sup\left\{ v_k(x) + 2\log \min_{0 \le j \le 2} \left| x - x_k^j \right| \mid x \in B \setminus \bigcup_{j=0}^{2} B(x_k^j, r_k^j) \right\} < +\infty,$$

we classify the rate $d_{i,j} = \left| x_k^i - x_k^j \right|$ of concetration to the origin for $0 \le i < j \le 2$. First, we show the residual vanishing inside the ball containing $B(x_k^j, 2r_k^j)$ with a proportional rate. These balls are contained in a larger ball, where the residual vanishing occurs similarly. We end-up this procedure in finitely many times, and obtain the conclusion. □

Profile of $v_k = v_k(x)$ in the outer region $x \in B \setminus B(x_0, \delta_k)$ is almost similar to that of the Kelvin transformation of $v_k = v_k(x)$ on $B(x_0, \delta_k)$, under the assumption of (47). This is actually proven by the method of moving plane, and then (48) is obtained [26]. See [25] for the other argument using the Pohozaev identity.

3 Higher-Dimensional Mass Quantization

Free Boundary Problem

Putting $w = v + \log\lambda - \log\int_\Omega e^v$ in (25), we obtain

$$-\Delta w = e^w \quad \text{in } \Omega, \qquad w = \text{constant on } \Gamma = \partial\Omega, \qquad \int_\Omega e^w = \lambda. \tag{56}$$

Conversely, if $w = w(x)$ solves (56), then $v = w - w_\Gamma$ is a solution to (25). By Theorem 3, we can show the quantized blowup mechanism to (56).

Theorem 7 *If $\Omega \subset \mathbf{R}^2$ is a bounded domain with smooth boundary $\partial\Omega$ and $\{(\lambda_k, w^k)\}$ is a solution sequence to (56) satisfying $\lambda_k \to \lambda_0$, then passing to a subsequence the following alternatives hold:*

1. $\|w^k\|_\infty = O(1)$.
2. $\sup_\Omega w^k \to -\infty$.
3. *$\lambda_0 = 8\pi\ell$ for some $\ell \in \mathbf{N}$, and there exist $x_j^* \in \Omega$ $(j = 1, \ldots, \ell)$ satisfying the second relation of (42) and $x_k^j \to x_j^*$, such that $x = x_k^j$ is a local maximum point of $w^k = w^k(x)$, $w^k(x_k^j) \to +\infty$, $w^k \to -\infty$ locally uniformly in $\overline{\Omega} \setminus \{x_1^*, \ldots, x_\ell^*\}$, and*
$$e^{w^k} dx \rightharpoonup \sum_j 8\pi\delta_{x_j^*}(dx) \qquad \text{in } \mathcal{M}(\overline{\Omega}).$$
Thus, $S = \{x_1^*, \ldots, x_\ell^*\}$ is the blowup set of $\{w^k\}$.

Higher-Dimensional Case

The problem (56) is regarded as a free boundary problem associated with plasma confinement, where $\{w > 0\}$ indicates the plasma region [19, 45]. Higher-dimensional mass quantization is observed in an analogous problem

$$-\Delta w = w_+^q \quad \text{in } \Omega, \qquad w = \text{constant on } \Gamma, \qquad \int_\Omega w_+^q = \lambda, \tag{57}$$

where $\Omega \subset \mathbf{R}^m$ $(m \geq 3)$ is a bounded domain with smooth boundary $\partial\Omega = \Gamma$, and $q = \frac{m}{m-2}$. Furthermore, we can formulate it as the equilibrium self-gravitating fluid equation described by the field component [45].

Similarly to Theorem 7, we can prove the quantized blowup mechanism, where the quantized value $m_* > 0$ is defined by

$$m_* = \int_B U^q$$

for $U = U(x)$ satisfying

$$-\Delta U = U^q, \quad U > 0 \quad \text{in } B, \qquad U = 0 \quad \text{on } \partial B$$

with $B = B(0, R)$. This U is radially symmetric and exists uniquely for each $R > 0$, while m_* is independent of $R > 0$. In the following theorem, $G = G(x, x')$

denotes the Green's function of $-\Delta$ on Ω with the Dirichlet boundary condition and

$$R(x) = [G(x,x') - \Gamma(x-x')]_{x'=x},$$

where

$$\Gamma(x) = \frac{1}{\omega_m (m-2) |x|^{m-2}}$$

is the fundamental solution to $-\Delta$ and ω_m is the $(m-1)$ dimensional volume of the boundary of the unit ball in $\mathbf{R}^m$.

Theorem 8 *If $\Omega \subset \mathbf{R}^m$ $(m \geq 3)$ is a bounded domain with smooth boundary $\partial\Omega$ and $\{(\lambda_k, w^k)\}$ is a solution sequence to (57) with $q = \frac{m}{m-2}$ satisfying $\lambda_k \to \lambda_0$, then passing to a subsequence the following alternatives hold:*

1. $\|w^k\|_\infty = O(1)$.
2. $\sup_\Omega w^k \to -\infty$.
3. *$\lambda_0 = m_* \ell$ for some $\ell \in \mathbf{N}$, and there exist $x_j^* \in \Omega$ and $x_k^j \to x_j^*$ $(j = 1, \ldots, \ell)$, where $\mathcal{S} = \{x_1^*, \ldots, x_\ell^*\} \subset \Omega$ coincides with the blowup set of $\{w^k\}$ on $\overline{\Omega}$ satisfying the second relation of (42), $x = x_k^j$ is a local maximum point of $w^k = w^k(x)$, $w^k(x_k^j) \to +\infty$, $w^k \to -\infty$ locally uniformly in $\overline{\Omega} \setminus \mathcal{S}$, and*

$$w^k(x)_+^q dx \rightharpoonup \sum_j m_* \delta_{x_j^*}(dx) \qquad \text{in } \mathcal{M}(\overline{\Omega}).$$

The reverse result, actual existence of the solution sequence described in the above theorem, is obtained by [52]. It is comparable to [3] for the two-dimensional case. See also [51, 54]. Similarly to the two-dimensional case, we obtain $v_k = w^k - w_\Gamma^k \geq 0$ in Ω by the maximum principle. This $\{v_k\}$, furthermore, satisfies the boundary estimate. In fact, $v \in [0,\infty) \mapsto f(v) = (v + w_\Gamma)_+^q$ is locally Lipschitz continuous in (57), and also if the first two equations hold for $\Omega = B^c$ with $B = B(0,1)$, then we obtain

$$-\Delta v = |y|^{-2} v_+^{\frac{m}{m-2}} \quad \text{in } B, \qquad v = \text{constant on } \partial B$$

and also

$$\int_B |y|^{-2} v_+^q dy \leq \int_B |y|^{-m} v_+^q dy = \int_{B^c} w_+^q dx$$

by the Kelvin transformation $v(y) = |x|^{m-2} w(x)$ with $y = x/|x|^2$. These structures are sufficient to guarantee $\|v_k\|_{L^\infty(\omega)} \leq C$, where ω is an $\overline{\Omega}$-neighbourhood of $\partial\Omega$.

Local version comparable to Theorems 4-5 also holds. Actually, there are ε-regularity, self-similarity, classification of the entire solution, and sup + inf inequality, and these structures guarantee the following theorem. A slight difference to Theorem 7 is that the entire solution

$$-\Delta w = w_+^q, \quad w \leq w(0) = 1 \quad \text{in } \mathbf{R}^m, \qquad \int_{\mathbf{R}^m} w_+^q < +\infty \tag{58}$$

has a compact support, which, makes the later argument simpler [50].

Theorem 9 *If $\Omega \subset \mathbf{R}^m$ $(m \geq 3)$ is a bounded domain and $w = w^k$ $(k = 1, 2, \ldots)$ satisfies*

$$-\Delta w = w_+^q \quad \text{in } \Omega, \qquad \int_\Omega w_+^q \leq C$$

for $q = \frac{m}{m-2}$ and $C > 0$, then, passing to a subsequence, we obtain the following alternatives.

1. *$\{w^k\}$ is locally uniformly bounded in Ω.*

2. *$w^k \to -\infty$ locally uniformly in Ω.*

3. *There exist $\ell \in \mathbf{N}$, x_j^* $(j = 1, \ldots, \ell)$, and $x_k^j \to x_j^*$ such that $x = x_k^j$ is a local maximum point of $w^k = w^k(x)$, $w^k(x_k^j) \to +\infty$, $w^k \to -\infty$ locally uniformly in $\Omega \setminus \{x_1^*, \ldots, x_\ell^*\}$, and*

$$w^k(x)_+^q dx \rightharpoonup \sum_j m_* n_j \delta_{x_j^*}(dx) \qquad \text{in } \mathcal{M}(\Omega),$$

where $n_j \in \mathbf{N}$.

Proof of Theorem 8: We have only to show the third case, assuming $w_\Gamma^k \to -\infty$ and $\|v_k\|_\infty \to +\infty$ for $v_k = w^k - w_\Gamma^k \geq 0$. Henceforth, we drop k for simplicity. Then, it holds that

$$w(x)_+^q dx \rightharpoonup \sum_{x_0 \in \mathcal{S}} m(x_0) \delta_{x_0}(dx)$$

in $\mathcal{M}(\overline{\Omega})$, where $\mathcal{S} = \{x_1^*, \ldots, x_\ell^*\} \subset \Omega$ is a set prescribed in the third case of Theorem 9, and $m(x_0) \in m_* \mathbf{N}$ for each $x_0 \in \mathcal{S}$. Thus, we have only to show $m(x_0) = m_*$ and the second relation of (42).

For this purpose, we apply the method of duality and scaling [45]. Thus, we take $u = w_+^q \geq 0$ and obtain

$$\int_\Omega u = \lambda, \qquad w - w_\Gamma = \int_\Omega G(\cdot, x') u(x') dx'.$$

This implies

$$\nabla w(x) = \int_\Omega \nabla_x G(x, x') u(x') dx'$$

and therefore,

$$\int_\Omega (\psi \cdot \nabla w) u = \int \int_{\Omega \times \Omega} \psi(x) \cdot \nabla_x G(x, x') u(x) u(x') dx dx'$$

for $\psi \in C_0^\infty(\Omega)^m$, where the left-hand side is equal to

$$\int_\Omega (\psi \cdot \nabla w) u = \frac{1}{q+1} \int_\Omega \psi \cdot \nabla w_+^{q+1} = -\frac{1}{q+1} \int_\Omega w_+^{q+1} \nabla \cdot \psi. \tag{59}$$

Henceforth, $\varphi = \varphi_{x_0,R}$ denotes a smooth function supported by $B(x_0, R)$ and is equal to 1 on $\overline{B(x_0, R/2)}$. We put $\psi(x) = (x - a)\varphi(x)$ for $a \in \mathbf{R}^m$ and $\varphi = \varphi_{x_0,R}$, where $x_0 \in \mathcal{S}$, $B(x_0, 2R) \subset \Omega$, and $B(x_0, 2R) \cap \mathcal{S} = \{x_0\}$.

Then, it holds that

$$\nabla \cdot \psi = m\varphi + (x-a)\cdot\nabla\varphi,$$

and therefore,

$$\int_\Omega (\psi\cdot\nabla w)u = -\frac{m}{q+1}\int_\Omega w_+^{q+1}\varphi + o(1)$$

by (59). Thus, we obtain

$$\frac{m}{q+1}\int_\Omega w_+^{q+1}\varphi + \iint_{\Omega\times\Omega} \psi(x)\cdot\nabla_x G(x,x')u(x)u(x')dxdx' = o(1). \tag{60}$$

Using $\hat{\varphi} = \varphi_{x_0,2R}$, the second term of the left-hand side of (60) is equal to

$$\begin{aligned}
&\iint_{\Omega\times\Omega} \psi(x)\cdot\nabla_x G(x,x')u(x)u(x')dxdx' \\
&= \iint_{\Omega\times\Omega} \psi(x)\cdot\nabla_x G(x,x')u(x)\hat{\varphi}(x)u(x')dxdx' \\
&= \int_{\Omega\times\Omega} \psi(x)\cdot\nabla_x G(x,x')u(x)\hat{\varphi}(x)u(x')\hat{\varphi}(x')dxdx' \\
&\quad + \iint_{\Omega\times\Omega} \psi(x)\cdot\nabla_x G(x,x')u(x)\hat{\varphi}(x)u(x')(1-\hat{\varphi}(x'))dxdx'.
\end{aligned}$$

The second term of the right-hand side of the above equality is equal to

$$m(x_0)(x_0-a)\cdot\sum_{x_0'\in\mathcal{S}\setminus\{x_0\}} m(x_0')\nabla_x G(x_0,x_0') + o(1),$$

while the method of symmetrization [44] is applied to the first term. Using

$$K(x,x') = G(x,x') - \Gamma(x-x'),$$

this term is thus equal to

$$\begin{aligned}
&\frac{1}{2}\iint_{\Omega\times\Omega} \rho_\psi^0(x,x')u^0(x)u^0(x')dxdx' \\
&\quad + \iint_{\Omega\times\Omega} \psi(x)\cdot\nabla_x K(x,x')u^0(x)u^0(x')dxdx',
\end{aligned}$$

for $u^0 = u\hat{\varphi}$ and $\rho_\psi^0(x,x') = (\psi(x)-\psi(x'))\cdot\nabla\Gamma(x-x')$.

Since $K = K(x,x') \in C^{2,\theta}(\overline{\Omega}\times\Omega\bigcup\Omega\times\overline{\Omega})$, it holds that

$$\begin{aligned}
&\iint_{\Omega\times\Omega} \psi(x)\cdot\nabla_x K(x,x')u^0(x)u^0(x')dxdx' \\
&\qquad = m(x_0)^2(x_0-a)\cdot\nabla_x K(x_0,x_0) + o(1).
\end{aligned}$$

We have, on the other hand,

$$\rho_\psi^0(x,x') = -(m-2)\Gamma(x-x') \qquad \text{in } B(x_0,R/2)\times B(x_0,R/2)$$

and therefore,

$$\rho_\psi^0(x,x')u^0(x)u^0(x') = -(m-2)\Gamma(x-x')\tilde{u}^0(x)\tilde{u}^0(x')$$
$$+\rho_\psi^0(x,x')(1-\tilde{\varphi}(x))\tilde{\varphi}(x')u^0(x)u^0(x') + \rho_\psi^0(x,x')\tilde{\varphi}(x)(1-\tilde{\varphi}(x'))u^0(x)u^0(x'),$$

where $\tilde{\varphi} = \varphi_{x_0,R/2}$ and $\tilde{u}^0 = u\tilde{\varphi}$. Here,

$$\left|\rho_\psi^0(x,x')\right| \leq C\Gamma(x-x')$$

and it holds that

$$\begin{aligned} 0 &\leq \int\int_{\Omega\times\Omega} \Gamma(x-x')(1-\tilde{\varphi}(x'))u^0(x)u^0(x')dxdx' \\ &= \left\langle \Gamma * u^0, (1-\tilde{\varphi})u^0 \right\rangle. \end{aligned}$$

This term is $o(1)$ because $\left\|(1-\tilde{\varphi})u^0\right\|_\infty \to 0$ and $\left\|\Gamma * u^0\right\|_1 = O(1)$ by $\|u\|_1 = O(1)$. Thus, we obtain

$$\begin{aligned} &\frac{1}{2}\int\int_{\Omega\times\Omega} \rho_\psi^0(x,x')u^0(x)u^0(x')dxdx' \\ &= -\frac{m-2}{2}\int\int_{\Omega\times\Omega} \Gamma(x-x')\tilde{u}^0(x)\tilde{u}^0(x')dxdx' + o(1), \end{aligned}$$

and (60) is reduced to

$$\begin{aligned} &\frac{m}{q+1}\int_\Omega w_+^{q+1}\varphi - \frac{m-2}{2}\int\int_{\Omega\times\Omega} \Gamma(x-x')\tilde{u}^0(x)\tilde{u}^0(x')dxdx' \\ &+m(x_0)(x_0-a)\cdot \sum_{x_0'\in\mathcal{S}\setminus\{x_0\}} m(x_0')\nabla_x G(x_0,x_0') \\ &+m(x_0)^2(x_0-a)\cdot\nabla_x K(x_0,x_0) = o(1). \end{aligned}$$

Since

$$\frac{m}{q+1}\int_\Omega w_+^{q+1}\varphi = \frac{m-2}{\gamma}\int_\Omega u^\gamma\varphi = \frac{m-2}{\gamma}\int_\Omega (\tilde{u}^0)^\gamma + o(1)$$

for $\gamma = 1 + \frac{1}{q} = 2 - \frac{2}{m}$, it holds that

$$\begin{aligned} &(m-2)\mathcal{F}_0(\tilde{\varphi}u) \\ &+m(x_0)(x_0-a)\cdot\left\{\sum_{x_0'\in\mathcal{S}\setminus\{x_0\}} m(x_0')\nabla_x G(x_0,x_0') + m(x_0)\nabla_x K(x_0,x_0)\right\} \\ &= o(1), \end{aligned} \tag{61}$$

for any $a \in \mathbf{R}^m$. Here and henceforth,

$$\mathcal{F}_0(u) = \frac{1}{\gamma}\int_{\mathbf{R}^m} u^\gamma - \frac{1}{2}\langle\Gamma * u, u\rangle$$

and 0-extension is taken to u where it is not defined. Since a is arbitrary, this implies

$$\frac{m(x_0)}{2}\nabla R(x_0) + \sum_{x_0'\in\mathcal{S}\setminus\{x_0\}} m(x_0')\nabla_x G(x_0,x_0') = 0 \tag{62}$$

and also

$$\mathcal{F}_0(\tilde{\varphi}u) = o(1). \tag{63}$$

Using $q = \frac{m}{m-2}$, we obtain

$$\mathcal{F}_0(u_\mu) = \mu^{m-2}\mathcal{F}_0(u),$$

where $\mu > 0$ and $u_\mu(x) = \mu^m u(\mu x + x_0)$.

Beginning the blowup analysis, now we prescribe the suffix k again. First, there is a maximum point $x = x_k^0$ of $u_k = u_k(x)$ in $B(x_0, 2R)$ such that $x_k^0 \to x_0$ from the proof of Theorem 9. Then, the rescaled $\tilde{u}_k(x) = \mu_k^m u(\mu_k x + x_k^0)$ is associated with $\tilde{w}_k(x) = \mu_k^{m-2} w_k(\mu_k x + x_k^0)$, and passing to a subsequence, $\tilde{w}_k \to \tilde{w}$ locally uniformly in $\mathbf{R}^m$, where $\mu_k = u_k(x_k^0)^{-1/m}$ and

$$-\Delta\tilde{w} = \tilde{w}_+^q, \quad \tilde{w} \leq \tilde{w}(0) = 1 \quad \text{in } \mathbf{R}^m, \qquad \int_{\mathbf{R}^m} \tilde{w}_+^q < +\infty.$$

This entire solution $\tilde{w}$ of (58) is radially symmetric, compactly supported on $\overline{B}$ for some $B = B(0, L)$, and

$$\int_{\mathbf{R}^m} \tilde{w}_+^q = m_*.$$

Here, we reformulate $\tilde{u}_k = \tilde{u}_k(x)$ by $\tilde{u}_k(x) = \mu_k^m(\tilde{\varphi}u_k)(\mu_k x + x_k^0)$ and obtain

$$\mathcal{F}_0(\tilde{u}_k) = \mu_k^{m-2}\mathcal{F}_0(\tilde{\varphi}u_k) \to 0. \tag{64}$$

by (63). It still holds that $\tilde{u}_k \to \tilde{u} \equiv \tilde{w}_+^q$ locally uniformly in $\mathbf{R}^m$, and therefore, $\nabla\tilde{w} = \nabla\Gamma * \tilde{u}$. This implies

$$\mathcal{F}_0(\tilde{u}) = \frac{1}{\gamma}\int_{\mathbf{R}^m} \tilde{u}^\gamma - \frac{1}{2}\langle\Gamma * \tilde{u}, \tilde{u}\rangle = 0 \tag{65}$$

similarly to (60), i.e.,

$$\frac{m}{q+1}\int_{\mathbf{R}^m} \tilde{w}_+^q + \int\int_{\mathbf{R}^m\times\mathbf{R}^m} x \cdot \nabla\Gamma(x - x')\tilde{u}(x)\tilde{u}(x')dxdx' = 0.$$

We have, on the other hand,

$$\langle\Gamma * \tilde{u}_k, \tilde{u}_k\rangle \to \langle\Gamma * \tilde{u}, \tilde{u}\rangle \tag{66}$$

passing to a subsequence, because $\{\tilde{u}_k\}$ is bounded in $(L^1 \cap L^\infty)(\mathbf{R}^m)$. Thus, $\int_{\mathbf{R}^m} \tilde{u}_k^\gamma \to \int_{\mathbf{R}^m} \tilde{u}^\gamma$ by (64)-(66), and therefore,

$$\tilde{u}_k \to \tilde{u} \qquad \text{in } L^\gamma(\mathbf{R}^m). \tag{67}$$

From the proof of Theorem 9, if $m(x_0) > m_*$, then there exist a local maximum point $x = x_k^1$ of $u_k = u_k(x)$ and $r_k^0, r_k^1 \to 0$ such that $x_k^1 \neq x_k^0$, $x_k^1 \to x_0$,

$$\int_{B(x_k^0, r_k^0)} u_k \to m_*, \qquad \int_{B(x_k^1, r_k^1)} u_k \to m_*,$$

and $B(x_k^0, 2r_k^0) \cap B(x_k^1, 2r_k^1) = \emptyset$. Furthermore, the connected components of the support of u_k containing x_k^0 and x_k^1 are contained in $B(x_k^0, 2r_k^0)$ and $B(x_k^1, 2r_k^1)$, respectively, for k large. In the rescaled variables, this means

$$\int_{B(0,L')} \tilde{u}_k \to m_*, \qquad \int_{B(x_k', r_k')} \tilde{u}_k \to m_* \tag{68}$$

and $B(0,2L')\cap B(x'_k,2r'_k)=\emptyset$ for some $L'>L$, x'_k, and r'_k, where the connected components of the support of $\tilde{u}_k$ containing 0 and x'_k are contained in $B(0,2L')$ and $B(x'_k,2r'_k)$, respectively. Here, it holds that $|x'_k|\to+\infty$, because $\tilde{u}_k\to 0$ locally uniformly in $B(0,L')^c$.

The second rescaling is defined by $\tilde{u}'_k(x)=(\mu'_k)^m\tilde{u}_k(\mu'_k x+x'_k)$ with $\mu'_k=\tilde{u}_k(x'_k)^{-1/m}\geq 1$. Passing to a subsequence, now we shall show $\mu'_k\to+\infty$, which implies also $r'_k\to+\infty$ by (68).

In fact, if this is not the case, then it holds that $\mu'_k=\tilde{u}_k(x'_k)^{-1/m}\approx 1$. We obtain $\tilde{u}''_k=\tilde{u}_k(\cdot+x'_k)\to\tilde{u}''=a\tilde{w}''^q_+$ locally uniformly in $\mathbf{R}^n$, where $a>0$ is a constant and

$$-\Delta\tilde{w}''=a\tilde{w}''^q_+,\quad 0<\tilde{w}''(0)=\max_{\mathbf{R}^m}\tilde{w}'',\quad \text{in } \mathbf{R}^m,\qquad \int_{\mathbf{R}^m}\tilde{w}''^q_+<+\infty.$$

This implies $r'_k\approx 1$ with

$$\int_{\mathbf{R}^m}\tilde{u}''=\int_{\mathbf{R}^m}a\tilde{w}''^q_+=m_*,$$

and therefore, it holds that

$$\lim_k\int_{B(0,2L)^c}\tilde{u}_k^\gamma\geq\lim_k\int_{B(x'_k,2r'_k)}(\tilde{u}'_k)^\gamma>0.$$

Since this contradicts to (67), we obtain $\mu'_k\to+\infty$, or equivalently, $\frac{u_k(x_k^0)}{u_k(x_k^1)}\to+\infty$.

Now, we replace the roles of x_k^0 and x_k^1, and repeat the above argument. Changing notations, this means $\frac{u_k(x_k^0)}{u_k(x_k^1)}\to 0$ and therefore, $\{\tilde{u}_k\}$ concentrates around $x'_k\in B(0,L')^c$. We obtain

$$\int_{B(x'_k,1)}\tilde{u}_k\to m_*$$

by $|x'_k|\to+\infty$, while $\int_{B(x'_k,\delta)}\tilde{u}_k^\gamma\to 0$ follows because (67) is obtained similarly. This is a contradiction again, and thus, $m(x_0)=m_*$ for each $x_0\in\mathcal{S}$. Then, (42) follows from (62). $\square$

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