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Singular limit problem for some elliptic systems

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1 Introduction

We consider the following singularly perturbed elliptic systems:

$$\epsilon^2\Delta u + f(u) - v = 0, \qquad \Delta v + g(u,v) = 0, \tag{1}$$

where $u = u(y)$ and $v = v(y)$ are real-valued functions on $y \in \mathbb{R}^2$; $\epsilon > 0$ is a positive constant; $f \in C^1(\mathbb{R})$ is a negative derivative of a double-equal-well potential $W \in C^2(\mathbb{R})$ satisfying $W(1) = W(-1) = 0 < W(s)\ {}^\forall s \in \mathbb{R}\backslash\{1,-1\}, W''(1)W''(-1) > 0$; and $g \in C^1(\mathbb{R}^2)$ is a smooth function such that $g(1,0) = 1 - m > 0$, $g(-1,0) = -m < 0$. Note that there hold $f(s) = -W'(s)$, $\int_{-1}^1 f(s)\,ds = 0$, and $f(i) = 0$, $f'(i) < 0$ $(i = \pm 1)$. A typical example of (f,g) is FitzHugh–Nagumo type, i.e., $f(s) = s - s^3$, $g(u,v) = \frac{1}{2}u - v$. The general case is referred to as the stationary activator-inhibitor system.

When the parameter ϵ is extremely small, very interesting patterns, such as stripes or spots, often appear. As a mathematical approach to understand this pattern formation, we consider the limit $\epsilon \to 0$. Then usually the domain is divided into two regions and the remaining part becomes a thin layer. In some cases, the width of the internal transition layer approaches 0 in the limit, and the discontinuity surface inside the domain, which is called sharp interface, appears. Recently very fine layered patterns of (1) have attracted a great deal of attention. See [5, 14, 15]. We consider this fine pattern which has the space scale of $\epsilon^{1/3}$ order. This is the unique scale that the driving force of v has the same order as that of the curvature of the sharp interface. See [12]. This scale also appeared in [5]. After rescaling $x = \frac{y}{\epsilon^{1/3}}$ and

$\varepsilon = \epsilon^{2/3}$, we obtain

$$\begin{cases} \Delta u + \dfrac{1}{\varepsilon^2}(f(u) - v) = 0, \\ \Delta v + \varepsilon g(u, v) = 0. \end{cases} \tag{2}$$

We consider the solutions of (2) subject to the homogeneous Neumann boundary condition:

$$\begin{cases} -\varepsilon^2 \Delta u = f(u) - v, & \text{in } \Omega, \\ -\Delta v = \varepsilon g(u, v), & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = \dfrac{\partial v}{\partial n} = 0, & \text{on } \partial\Omega, \end{cases} \tag{3}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain with the smooth bounday $\partial\Omega$; $\partial/\partial n$ is the outward normal derivative on $\partial\Omega$.

We shall formally deduce the reduced problem. If we assume $u \to u_0$ and $v \to v_0$ in the limit $\varepsilon \to 0$, we have $f(u_0) = v_0, \Delta v_0 = 0$ in Ω, $\frac{\partial v_0}{\partial n} = 0$ on $\partial\Omega$. Hence v_0 is a constant. Now assume that v_0 is close to 0 and $u_0 = f_1^{-1}(v_0)\mathbf{1}_{\Omega^+} + f_{-1}^{-1}(v_0)\mathbf{1}_{\Omega^-}$, where Ω^+, Ω^- are mutually disjoint open sets in Ω such that $\Gamma = \Omega \backslash (\Omega^+ \cup \Omega^-)$ is a curve embedded in Ω; $\mathbf{1}_{\Omega^\pm}$ denote the characteristic functions of $\Omega^\pm$; $u = f_{\pm 1}^{-1}(v)$ are the inverse functions of $v = f(u)$ near $u = \pm 1$ respectively. Here we call Γ sharp interface. We shall identify the profile of u near Γ.

It is known that there exists a constant $\tau > 0$, depending on f, such that for any $v \in (-\tau, \tau)$, the equation for u, $u_t = u_{xx} + f(u) - v$, has a traveling wave solution $u(x, t) = Q(x - ct\,; v)$ with the speed $c = c(v)$ and the profile $Q = Q(\xi; v)$. More precisely, $c(v)$ and $Q(\xi; v)$ for $v \in (-\tau, \tau), \xi \in \mathbb{R}$ satisfy

$$\begin{cases} \ddot{Q} + c(v)\dot{Q} + f(Q) - v = 0, & \text{in } \mathbb{R}, \\ \lim\limits_{\xi \to -\infty} Q(\xi; v) = f_1^{-1}(v), \\ \lim\limits_{\xi \to +\infty} Q(\xi; v) = f_{-1}^{-1}(v), \\ c(0) = 0. \end{cases}$$

Here dot means $d/d\xi$. See, for example, [4]. Near the sharp interface Γ, consider the function

$$u(x) = Q\left(\frac{d(x)}{\varepsilon}; v\right),$$

where $d = d(x)$ is the signed distance function from Γ such that $d(x) > 0$ if $x \in \Omega^-$ and $d(x) < 0$ if $x \in \Omega^+$. If the above function satisfy the first equation of (3) for each prescribed v, noting that $|\nabla d| = 1$, there holds $\ddot{Q} + \varepsilon(\Delta d)\dot{Q} + f(Q) - v = 0$. Since Δd is equal to the curvature κ of Γ on the interface Γ (here we choose the sign such that $\kappa > 0$ when Ω^+ is a disk), it follows that $c(v) = \varepsilon\kappa$ on Γ. Since $c(0) = 0$ by the assumption, we may assume that $v_0 = 0$ and $u_0 = \mathbf{1}_{\Omega^+} - \mathbf{1}_{\Omega^-}$.

Next we consider the higher order term. Assume $v = \varepsilon v_1 + O(\varepsilon^2)$. Then we obtain the reduced problem

$$\begin{cases} -\Delta v_1 = g(u_0, 0) = \mathbf{1}_{\Omega^+} - m, & \text{in } \Omega, \\ \dfrac{\partial v_1}{\partial n} = 0, & \text{on } \partial\Omega, \\ c'(0)v_1 = \kappa, & \text{on } \Gamma. \end{cases}$$

It is easily seen that there holds $c'(0) = -\frac{2}{\sigma} < 0$ with

$$\sigma = \int_{-1}^{1} \sqrt{2W(s)}\, ds.$$

Therefore, letting $\beta = 2/\sigma$, we finally obtain

$$\begin{cases} -\Delta v = \mathbf{1}_{\Omega^+} - m, & \text{in } \Omega, \\ \dfrac{\partial v}{\partial n} = 0, & \text{on } \partial\Omega, \\ \beta v + \kappa = 0, & \text{on } \Gamma. \end{cases} \tag{4}$$

Recall that $\Omega \subset \mathbb{R}^2$ is a bounded domain with the smooth boundary $\partial\Omega$; $\partial/\partial n$ is the normal derivative on $\partial\Omega$; Ω^+ is an open set in Ω; $\Gamma = \partial\Omega^+ \subset \Omega$ is a C^2-curve embedded in Ω; κ is the curvature of Γ; $m \in (0, 1)$ is a constant; and $\mathbf{1}_{\Omega^+}$ denotes the characteristic function of Ω^+.

The essentially same equation as (4) was obtained in [13] by using the matched expansion method. Once you have a "non-degenerate" solution of (4) in some sense, you can find a layered solution for the singularly perturbed elliptic problem (3). See [13]. For the reduction from the parabolic system to the sharp interface model, see [19].

In this résumé, we consider the problem to find a non-degenerate solution of (4) which does not necessarily correspond to the global minimizers. The radially symmetric case for the related problems is studied in [6, 7, 13, 17, 18, 20]. We do not assume any symmetry of the domain.

This résumé is organized as follows. In Section 2, we consider the existence of solutions. In Section 3, we consider the linearized non-degeneracy of the problem.

2 Existence

In order to state the result, we define the Green's function and its harmonic part.

Definition 2.1 For each $y \in \Omega$, let $G(x,y)$ be the solution to

$$\begin{cases} -\Delta_x G(x,y) = \delta(x-y) - \dfrac{1}{|\Omega|}, & x \in \Omega, \\ \dfrac{\partial G}{\partial n_x}(x,y) = 0, & x \in \partial\Omega, \\ \displaystyle\int_\Omega G(x,y)\,dx = 0. \end{cases}$$

Set

$$G(x,y) = -\frac{1}{2\pi}\log|x-y| + \frac{|x-y|^2}{4|\Omega|} + H(x,y), \quad x,y \in \Omega.$$

Then it is known that $H(x,y)$ is symmetric and harmonic in both x and y. Let $\mathcal{H}(x) = H(x,x)$.

We define the following two conditions.

(A1) $0 \in \Omega$ is a strict local minimum point of $\mathcal{H}$. More precisely, there exists a neighborhood U of 0 in Ω such that $\mathcal{H}(0) < \mathcal{H}(x)$ for all $x \in U\backslash\{0\}$.

(A2) $0 \in \Omega$ is a non-degenerate critical point of $\mathcal{H}$.

Remark. When $\Omega = \{x \in \mathbb{R}^2\,;\,|x| < 1\}$, $x = 0$ is a unique minimum point of $\mathcal{H}$ and both **(A1)** and **(A2)** are satisfied. Indeed, we have $\mathcal{H}(x) = -\frac{1}{2\pi}\log(1-|x|^2) + \frac{|x|^2}{2\pi} + \mathcal{H}(0)$, and hence $\frac{\partial^2\mathcal{H}}{\partial x_i \partial x_j}(0) = \frac{2}{\pi}\delta_{ij}$.

The regular part of Green's function subject to the homogeneous Dirichlet boundary condition has a unique non-degenerate minimum point when $\Omega \subset \mathbb{R}^2$ is convex (see [2]). On the other hand, the regular part of Green's function subject to the homogeneous Neumann boundary condition is studied in [8].

We denote by d_{H} the Hausdorff metric

$$d_{\mathrm{H}}(K_1, K_2) = \max[\sup\{\mathrm{dist}(x, K_2)\,;\, x \in K_1\}, \sup\{\mathrm{dist}(y, K_1)\,;\, y \in K_2\}],$$

$S_r(0) = \{x \in \mathbb{R}\,;\, |x| = r\}$, and $B_r(0) = \{x \in \mathbb{R}\,;\, |x| < r\}$.

Theorem 2.1 Assume that **(A1)** or **(A2)**. If $r_0 := \sqrt{\frac{m|\Omega|}{\pi}} < \mathrm{dist}(0, \partial\Omega)$, then there exists a constant $\beta_0 > 0$ such that (4) has a solution $(\Gamma, v, \Omega^+) = (\Gamma_\beta, v_\beta, \Omega^+_\beta)$ for all $\beta < \beta_0$ satisfying $d_{\mathrm{H}}(\Gamma_\beta, S_{r_0}(0)) \to 0$ as $\beta \to 0$.

2.1 Notations

We identify 2π-periodic functions on $\mathbb{R}$ with the functions on $S^1 = \{x \in \mathbb{R}^2\,;\, |x| = 1\} \cong \mathbb{R}/2\pi\mathbb{Z}$. For $q \in C^2(S^1)$, we use the following notations:

$$\dot{q}(\omega) = \frac{dq}{d\omega}(\omega) = \frac{d}{d\theta}q(\cos\theta, \sin\theta), \quad \omega = (\cos\theta, \sin\theta) \in S^1$$

and

$$\ddot{q}(\omega) = \frac{d^2q}{d\omega^2}(\omega) = \frac{d^2}{d\theta^2}q(\cos\theta, \sin\theta), \quad \omega = (\cos\theta, \sin\theta) \in S^1.$$

We set $X = C^2(S^1)$,

$$\|q\|_X = \max_{\omega\in S^1}|q(\omega)| + \max_{\omega\in S^1}|\dot{q}(\omega)| + \max_{\omega\in S^1}|\ddot{q}(\omega)|,$$

$Y = C(S^1)$, and

$$\|q\|_Y = \max_{\omega\in S^1}|q(\omega)|.$$

For $q_1, q_2 \in L^2(S^1)$, denote

$$\langle q_1, q_2\rangle = \int_{S^1} q_1(\omega)q_2(\omega)\,d\omega = \int_0^{2\pi} q_1(\cos\theta, \sin\theta)q_2(\cos\theta, \sin\theta)\,d\theta,$$

and $\|q_1\|^2 = \langle q_1, q_1\rangle$. Let $\Pi_{n^2} : L^2(S^1) \to L^2(S^1)$ denote the projections with respect to $\langle\cdot,\cdot\rangle$ onto $\mathrm{span}\{\cos i\theta, \sin i\theta\,;\, i = 0, 1, \cdots, n\}$ for $n = 0, 1, \cdots$. Let $\Pi^{\perp}_{n^2} = \mathbf{Id} - \Pi_{n^2}$.

Define $\Phi_0(\omega) = 1/\sqrt{2\pi}$, $\Phi_1(\omega) = \omega_1/\sqrt{\pi}$, and $\Phi_2(\omega) = \omega_2/\sqrt{\pi}$ for $\omega = (\omega_1, \omega_2) \in S^1$. Then $\Pi^{\perp}_0, \Pi^{\perp}_1$ are the projections onto the orthogonal complements of $\mathrm{span}\{\Phi_0\}$ and $\mathrm{span}\{\Phi_i\,;\, i = 0, 1, 2\}$ respectively.

2.2 Outline of Proof of Theorem 2.1

For brevity's sake, we assume that $r_0 = 1 < \mathrm{dist}(0, \partial\Omega)$. For $\ell > 0$, define $X_\ell = \{q \in X\,;\, \|q\|_X \leq \ell\}$. We can choose a constant $\delta \in (0, 1/2)$ such that $B_{1+\delta}(0) \subset \Omega$ by the assumption. For $q \in X_{\delta/2}$, define

$$\Gamma(q) = \{\sqrt{1+q(\omega)}\omega\,;\, \omega \in S^1\}, \quad \Omega^+(q) = \{r\omega\,;\, 0 \leq r \leq \sqrt{1+q(\omega)}, \omega \in S^1\}.$$

Note that there hold $\Gamma(q) \subset \Omega$ and $|\Omega^+(q)| = \pi$ for any $q \in X_{\delta/2} \cap \Pi_0^\perp X$. Let

$$L(t,p,s) = \frac{1 + t + \frac{3p^2}{4(1+t)} - \frac{1}{2}s}{\left[1 + t + \frac{p^2}{4(1+t)}\right]^{3/2}}$$

for $t > -1$, $p \in \mathbb{R}$, $s \in \mathbb{R}$. Then $K(q) = L(q, \dot{q}, \ddot{q})$ is the curvature of $\Gamma(q)$ for any $q \in X_{\delta/2}$. Let M_β be the map from $X_{\delta/2}$ to Y defined by

$$M_\beta(q)(\omega) = K(q)(\omega) + \beta \int_{\Omega^+(q)} G(\sqrt{1+q(\omega)}\omega, y)\, dy, \qquad \omega \in S^1$$

for $q \in X_{\delta/2}$. In order to prove Theorem 2.1, we need only show the following:

Proposition 2.1 Suppose either **(A1)** or **(A2)**. If $1 = \sqrt{m|\Omega|/\pi} < \mathrm{dist}(0, \partial\Omega)$, then there exists a constant $\beta_0 > 0$ such that $\Pi_0^\perp M_\beta(q) = 0$ has a solution $q = q_\beta \in X_{\delta/2} \cap \Pi_0^\perp X$ for all $\beta \in (0, \beta_0)$ satisfying $q_\beta \to 0$ in X as $\beta \to 0$. In addition, $\Gamma(q_\beta) = \boldsymbol{P}_\beta + \Gamma(\tilde{q}_\beta)$ for some $\boldsymbol{P}_\beta \in \Omega$, $\tilde{q}_\beta \in X$ such that $\boldsymbol{P}_\beta \to 0$, $\|\tilde{q}_\beta\|_X = O(\beta)$ as $\beta \to 0$.

Indeed, if $q \in X_{\delta/2} \cap \Pi_0^\perp X$ is a solution of $\Pi_0^\perp M_\beta(q) = 0$, then there exists a constant C_1 such that $M_\beta(q) \equiv C_1$. Now set

$$v(x) = \int_{\Omega^+(q)} G(x,y)\, dy - \frac{1}{\beta}C_1, \quad x \in \Omega.$$

Then v satisfies

$$\begin{cases} -\Delta v = \mathbf{1}_{\Omega^+(q)} - m, & \text{in } \Omega, \\ \dfrac{\partial v}{\partial n} = 0, & \text{on } \partial\Omega. \end{cases}$$

Hence we see that

$$\Gamma = \Gamma(q), \quad v(x) = \int_{\Omega^+(q)} G(x,y)\, dy - \frac{1}{\beta}C_1, \quad \Omega^+ = \Omega^+(q)$$

solves our equation (4) and completes the proof of Theorem 2.1.

3 Non-degeneracy

Throughout this section, we assume that there exists a compact subset $\mathcal{N} \subset \Omega$ satisfying $\mathrm{dist}(\mathcal{N}, \partial\Omega) > 1$. We linearize the equation around $\boldsymbol{P} + \Gamma(q) = \{\boldsymbol{P} + \sqrt{1+q(\omega)}\omega\,;\, \omega \in S^1\}$ for $\boldsymbol{P} \in \mathcal{N}$. Set

$$M_\beta(q\,;\boldsymbol{P})(\omega) := K(q)(\omega) + \beta \int_{\boldsymbol{P}+\Omega^+(q)} G(\boldsymbol{P} + \sqrt{1+q(\omega)}\omega, y)\,dy, \qquad \omega \in S^1$$

for $q \in X_{\delta/2}$, where $\boldsymbol{P} + \Omega^+(q)$ is the region surrounded by $\boldsymbol{P} + \Gamma(q)$.

Theorem 3.1 Suppose that

(B1) for every small $\beta > 0$, there exist $\tilde{q}_\beta \in X$ and $\boldsymbol{P} \in \mathcal{N}$ such that

$$(\Pi_4 - \Pi_1)M_\beta(\tilde{q}_\beta\,;\boldsymbol{P}) = 0,$$

(B2) $\|\tilde{q}_\beta\|_X = O(\beta)$ as $\beta \to 0$, and

(B3) the Hessian matrix $\left(\frac{\partial^2\mathcal{H}}{\partial x_i \partial x_j}(\boldsymbol{P})\right)_{1\le i,j\le 2}$ of $\mathcal{H}$ is non-degenerate for any $\boldsymbol{P} \in \mathcal{N}$.

Then for sufficiently small β, $\mathcal{L} = \Pi_0^\perp M_\beta'(\tilde{q}_\beta\,;\boldsymbol{P})$ is non-degenerate in the sense that $\mathcal{L}\zeta = 0$, $\int_{S^1} \zeta\,d\omega = 0$ implies that $\zeta = 0$.

Let q_β be a solution obtained in Proposition 2.1. Then there exist $\boldsymbol{P}_\beta \in \Omega$ and $\tilde{q}_\beta \in X$ such that $\Gamma(q_\beta) = \boldsymbol{P}_\beta + \Gamma(\tilde{q}_\beta)$, **(B1)** with $\boldsymbol{P} = \boldsymbol{P}_\beta$, and **(B2)** hold. Thus we have the following:

Corollary 3.1 Suppose **(A2)**. Then the solution obtained in Theorem 2.1 is non-degenerate in the sense of Theorem 3.1.

3.1 Outline of Proof of Theorem 3.1

For brevity's sake, we write $q = \tilde{q}_\beta$. Set

$$\begin{aligned} B(\zeta,\zeta) = & \int_{S^1} [-L_s(q,\dot{q},\ddot{q})\dot{\zeta}^2 + L_t(q,\dot{q},\ddot{q})\zeta^2]\, d\omega \\ & + \frac{\beta}{2}\int_{S^1}\int_{S^1} \zeta(\omega) G(P + \sqrt{1+q(\omega)}\omega, P + \sqrt{1+q(\hat{\omega})}\hat{\omega})\zeta(\hat{\omega})\, d\omega d\hat{\omega} \\ & + \frac{\beta}{2}\int_{S^1} d\omega \frac{\zeta(\omega)^2}{\sqrt{1+q(\omega)}} \int_{P+\Omega^+(q)} \omega\cdot\nabla_x G(P + \sqrt{1+q(\omega)}\omega, y)\, dy, \end{aligned}$$

for $\zeta \in H^1(S^1)$, where

$$L(t,p,s) = \frac{1 + t + \frac{3p^2}{4(1+t)} - \frac{1}{2}s}{\left[1 + t + \frac{p^2}{4(1+t)}\right]^{3/2}}$$

for $t > -1$, $p \in \mathbb{R}$, $s \in \mathbb{R}$. We regard $\mathcal{L}$ as the operator on $\Pi_0^\perp H^2(S^1)$ satisfying $B(\zeta,\zeta) = \langle \mathcal{L}\zeta, \zeta\rangle$ for all $\zeta \in \Pi_0^\perp H^2(S^1)$. Then we have the following two lemmas:

Lemma 3.1 Suppose **(B2)**. Let $\lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots$ be the eigenvalues of $\mathcal{L} : \Pi_0^\perp H^2(S^1) \to \Pi_0^\perp L^2(S^1)$ and $\zeta_i \in \Pi_0^\perp H^2(S^1)$ be the normalized eigenfunctions associated with λ_i. Then

$$\lambda_1 = \inf_{\zeta\in\Pi_0^\perp H^1(S^1), \|\zeta\|=1} B(\zeta,\zeta) = B(\zeta_1,\zeta_1) = O(\beta),$$

$$\lambda_2 = \inf_{\substack{\zeta\in\Pi_0^\perp H^1(S^1), \|\zeta\|=1 \\ \zeta\perp\zeta_1}} B(\zeta,\zeta) = B(\zeta_2,\zeta_2) = O(\beta),$$

$$\lambda_3 = \inf_{\substack{\zeta\in\Pi_0^\perp H^1(S^1), \|\zeta\|=1 \\ \zeta\perp \mathrm{span}\{\zeta_1,\zeta_2\}}} B(\zeta,\zeta) = B(\zeta_3,\zeta_3) = \frac{3}{2} + O(\beta).$$

Lemma 3.2 1. There hold $L_{ts}(0,0,0) = L_{tt}(0,0,0) = L_{pp}(0,0,0) = \frac{3}{4}$ and $L_{ss}(0,0,0) = L_{ps}(0,0,0) = L_{tp}(0,0,0) = 0$.

2. There hold

$$\int_{S^1} d\omega \Phi_j(\omega)\Phi_k(\omega)\omega\cdot\nabla_x H(P+\omega, P) = \frac{1}{2}\frac{\partial^2 H}{\partial x_j \partial x_k}(x,y)\Big|_{x=y=P}$$

and

$$\int_{S^1}\int_{S^1} \Phi_j(\omega) H(P+\omega, P+\hat{\omega})\Phi_k(\hat{\omega})\, d\omega d\hat{\omega} = \pi \frac{\partial^2 H}{\partial x_j \partial y_k}(x,y)\Big|_{x=y=P}$$

for each $j, k = 1, 2$.

3. Suppose **(B1)** and **(B2)**. Then

$$\lim_{\beta \to 0} \frac{1}{\beta} \langle \dot{q} \Phi_k, \dot{\Phi}_j \rangle = -\frac{\pi}{3} \frac{\partial^2 H}{\partial x_j \partial x_k}(x, y) \Big|_{x=y=P}$$

for each $j, k = 1, 2$.

Using these lemmas, we can show the following:

Lemma 3.3 Suppose **(B1)** and **(B2)**. Then there exists an orthogonal matrix $(c_{ij})_{i,j=1,2}$ such that for each $i = 1, 2$, $\zeta_i^R = \zeta_i - (c_{1i}\Phi_1 + c_{2i}\Phi_2)$ satisfies $\|\zeta_i^R\|^2 = O(\beta)$ as $\beta \to 0$. In addition, there holds

$$\sum_{k=1}^{2} \frac{\pi}{4} \frac{\partial^2 \mathcal{H}}{\partial x_j \partial x_k}(\boldsymbol{P}) c_{ki} = o(1) + \frac{\lambda_i}{\beta} c_{ji}$$

for each $i, j = 1, 2$.

Completion of the proof of Theorem 3.1. Assume by contrary that there exists a sequence ζ_β such that $\mathcal{L}\zeta_\beta = 0$, $\|\zeta_\beta\| = 1$, and $\int_{S^1} \zeta_\beta \, d\omega = 0$. This means that ζ_β is an eigenfunction of of $\mathcal{L}$ associated with the eigenvalue 0. We see that for sufficiently small β, either λ_1 or λ_2 is equal to 0. Then by Lemma 3.3, we have $\zeta_\beta = c_1\Phi_1 + c_2\Phi_2 + \zeta^R$ such that $(c_1, c_2) \in S^1$ and $\|\zeta^R\|^2 = O(\beta)$, and

$$\sum_{k=1}^{2} \frac{\partial^2 \mathcal{H}}{\partial x_j \partial x_k}(\boldsymbol{P}) c_k = o(1) \quad \text{for } j = 1, 2,$$

as $\beta \to 0$. Taking a subsequence if necessary, we may assume that $(c_1, c_2) \to (\hat{c}_1, \hat{c}_2) \in S^1$ and

$$\sum_{k=1}^{2} \frac{\partial^2 \mathcal{H}}{\partial x_j \partial x_k}(\boldsymbol{P}) \hat{c}_k = 0 \quad \text{for } j = 1, 2.$$

It follows from **(B3)** that $\hat{c}_1 = \hat{c}_2 = 0$. This is a contradiction and completes the proof.

参考文献

[1] J. Byeon and Y. Oshita, Existence of Multi-bump standing waves with a critical frequency for Nonlinear Schrödinger equations, *Comm. Partial Differential Equations* **29** (2004), no. 11-12, 1877–1904.

[2] L. A. Caffarelli and A. Friedman, Convexity of solutions of semilinear elliptic equations, *Duke Math. J.* **52** (1985), no. 2, 431–456.

[3] X. Chen, private communication.

[4] X. Chen, D. Hilhorst, and E. Logak, Asymptotic behavior of solutions of an Allen–Cahn equation with a nonlocal term, *Nonlinear Anal.* **28** (1997), no. 7, 1283–1298.

[5] X. Chen and Y. Oshita, Periodicity and uniqueness of global minimizers of an energy functional containing a long-range interaction, *SIAM J. Math. Anal.* **37** (2005), no. 4, 1299–1332.

[6] X. Chen and M. Taniguchi, Instability of spherical interfaces in a nonlinear free boundary problem, *Adv. Differential Equations* **5** (2000), no. 4-6, 747–772.

[7] M. A. del Pino, Radially symmetric internal layers in a semilinear elliptic system, *Trans. Amer. Math. Soc.* **347** (1995), no. 12, 4807–4837.

[8] T. Kolokolnikov, M. Titcombe, and M. Ward, Optimizing the Fundamental Neumann Eigenvalue for the Laplacian in a Domain with Small Traps, *European J. Applied Math.* **16** (2005), no.2, 161–200.

[9] Y. Y. Li, On a singularly perturbed elliptic equation, *Adv. Differential Equations* **2** (1997), no.6, 955–980.

[10] Y. Y. Li, On a singularly perturbed equation with Neumann boundary condition, *Comm. Partial Differential Equations* **23** (1998), no. 3-4, 487–545.

[11] Y. Y. Li and L. Nirenberg, The Dirichlet problem for singularly perturbed elliptic equations, *Comm. Pure appl. Math.* **51** (1998), no. 11-12, 1445–1490.

[12] Y. Nishiura and H. Suzuki, Nonexistence of higher-dimensional stable Turing patterns in the singular limit, *SIAM J. Math. Anal.* **29** (1998), no. 5, 1087–1105.

[13] Y. Nishiura and H. Suzuki, Higher dimensional SLEP equation and applications to morphological stability in polymer problems, *SIAM J. Math. Anal.* **36** (2004/05), no. 3, 916–966.

[14] Y. Oshita, On stable nonconstant stationary solutions and mesoscopic patterns for FitzHugh–Nagumo equations in higher dimensions, *J. Differential Equations* **188** (2003), no. 1, 110–134.

[15] Y. Oshita, Stable stationary patterns and interfaces arising in reaction-diffusion systems, *SIAM J. Math. Anal.* **36** (2004), no. 2, 479–497.

[16] Y. Oshita, Singular limit problem for some elliptic systems, preprint.

[17] K. Sakamoto and H. Suzuki, Spherically symmetric internal layers for activator-inhibitor systems. I. Existence by a Lyapunov-Schmidt reduction, *J. Differential Equations* **204** (2004), no. 1, 56–92.

[18] K. Sakamoto and H. Suzuki, Spherically symmetric internal layers for activator-inhibitor systems. II. Stability and symmetry breaking bifurcations, *J. Differential Equations* **204** (2004), no. 1, 93–122.

[19] P. Soravia and P. E. Souganidis, Phase-field theory for FitzHugh-Nagumo-type systems, *SIAM J. Math. Anal.* **27** (1996), no. 5, 1341–1359.

[20] M. Taniguchi, Multiple existence and linear stability of equilibrium balls in a nonlinear free boundary problem, *Quart. Appl. Math.* **58** (2000), no. 2, 283–302.