数理解析研究所講究録
1528巻 2007年 117-127

A PARABOLIC FREE BOUNDARY PROBLEM WITH BERNOULLI TYPE CONDITION ON THE FREE BOUNDARY

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ABSTRACT. In the brilliant paper [1], H.W Alt and L.A. Caffarelli proved that close to flat points the free boundary of certain weak solutions of the Bernoulli free boundary problem

$$\Delta u - \partial_t u = 0 \text{ in } \{u > 0\}\,,\ |\nabla u| = 1 \text{ on } \partial\{u > 0\}\,.$$

is analytic.
The result is related to the theory of harmonic measures (see [10], [11], [12]).
For a realistic class of solutions, containing for example *all* limits of the singular perturbation problem

$$\Delta u_\varepsilon - \partial_t u_\varepsilon = \beta_\varepsilon(u_\varepsilon) \text{ as } \varepsilon \to 0,$$

we prove that one-sided flatness of the free boundary implies regularity.
In particular, we show that the topological free boundary $\partial\{u > 0\}$ can be decomposed into an *open* regular set (relative to $\partial\{u > 0\}$) which is locally a surface with Hölder-continuous space normal, and a closed singular set.
Our result extends the main theorem in the paper by H.W. Alt-L.A. Caffarelli (1981) to more general solutions as well as the time-dependent case. Our proof uses methods developed in H.W. Alt-L.A. Caffarelli (1981), however we replace the core of that paper, which relies on non-positive mean curvature at singular points, by an argument based on scaling discrepancies, which promises to be applicable to more general free boundary or free discontinuity problems.

1. INTRODUCTION

This note contains an announcement as well as heuristics for the paper with the same title to appear, but no rigorous proofs.
The parabolic free boundary problem

$$\Delta u - \partial_t u = 0 \text{ in } \{u > 0\}\,,\ |\nabla u| = 1 \text{ on } \partial\{u > 0\} \tag{1.1}$$

2000 *Mathematics Subject Classification.* Primary 35R35, Secondary 35K55.

Key words and phrases. Free boundary, Bernoulli type, parabolic, regularity, flatness improvement.

J. Andersson has been partially supported by a fellowship of the Max Planck Society. G.S. Weiss has been partially supported by the Grant-in-Aid 15740100/18740086 of the Japanese Ministry of Education, Culture, Sports, Science and Technology and partially supported by a fellowship of the Max Planck Society. Both authors thank the Max Planck Institute for Mathematics in the Sciences for the hospitality during their stay in Leipzig.

has originally been derived as singular limit from a model for the propagation of equidiffusional premixed flames with high activation energy ([3]); here $u = \lambda(T_c - T)$, T_c is the flame temperature, which is assumed to be constant, T is the temperature outside the flame and λ is a normalization factor.

Let us shortly summarize the mathematical results directly relevant in this context, beginning with the limit problem (1.1): in the brilliant paper [1], H.W. Alt and L.A. Caffarelli proved via minimization of the energy $\int(|\nabla u|^2 + \chi_{\{u>0\}})$ – here $\chi_{\{u>0\}}$ denotes the characteristic function of the set $\{u > 0\}$ – existence of a stationary solution of (1.1) in the sense of distributions. They also derived regularity of the free boundary $\partial\{u > 0\}$ up to a set of vanishing $n-1$-dimensional Hausdorff measure. By [16] existence of singular minimizers implies the existence of singular minimizing cones. L.A. Caffarelli-D. Jerison-C. Kenig showed that singular minimizing cones do not exist in dimension 3 ([6]). Moreover it is known that singular minimizing cones exist for $n \geq 7$ ([9]). *Non-minimizing* singular cones appear already for $n = 3$ (see [1, example 2.7]). Moreover it is known, that solutions of the Dirichlet problem in two space dimensions are not unique (see [1, example 2.6]). C.E. Kenig-T. Toro ([10], [11], [12]) extended the result in [1] to VMO-coefficients and applied it to abstract harmonic measures.

For the time-dependent (1.1), both "trivial non-uniqueness" (the positive solution of the heat equation is always another solution of (1.1)) and "non-trivial uniqueness" (see [14]) occur. Even for flawless initial data, classical solutions of (1.1) develop singularities after a finite time span; consider e.g. the example of two colliding traveling waves

$$\begin{aligned} u(t,x) = \; & \chi_{\{x+t>1\}}(\exp(x+t-1)-1) \\ & + \chi_{\{-x+t>1\}}(\exp(-x+t-1)-1) \text{ for } t \in [0,1) \end{aligned} \tag{1.2}$$

(see Figure 1).

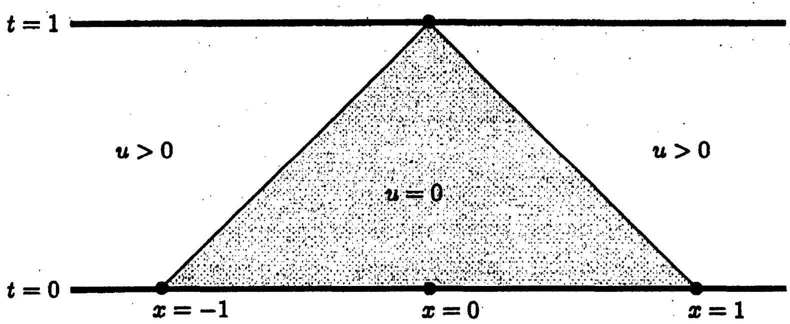

FIGURE 1. Colliding traveling waves

There are several approaches concerning the construction of a solution of the time-dependent problem, all of which are based in some form on the convergence of the solution

u_ε of the reaction-diffusion equation

$$\Delta u_\varepsilon - \partial_t u_\varepsilon = \beta_\varepsilon(u_\varepsilon) \tag{1.3}$$

to (1.1) as $\varepsilon \to 0$; here $\beta_\varepsilon(z) = \frac{1}{\varepsilon}\beta(\frac{z}{\varepsilon})$, $\beta \in C_0^1([0,1])$, $\beta > 0$ in $(0,1)$ and $\int \beta = \frac{1}{2}$.
L.A. Caffarelli and J.L. Vazquez proved in [7] uniform estimates for (1.3) and a convergence result: for initial data u^0 that are strictly mean concave in the interior of their support, a sequence of ε-solutions converges to a solution of (1.1) in the sense of distributions.
Let us also mention several results on the corresponding two-phase problem, which are relevant as solutions of the one-phase problem are automatically solutions of the corresponding two-phase problem. In [5] and [4], L.A. Caffarelli, C. Lederman and N. Wolanski prove convergence to a barrier solution in the case that the limit function satisfies $\{u = 0\}^\circ = \emptyset$.
Then, there is the convergence to a solution in the sense of domain variations [15] which seems to contain more information than the barrier solutions in [5] and [4]. For more general two-phase problems see [17]. Domain variation solutions play an important rule in this paper and will be discussed in more detail in Section 3.

Here let it suffice to say that domain variation solutions are pairs (u, χ) where the order parameter χ shares many properties with the characteristic function $\chi_{\{u>0\}}$ but does not necessarily coincide with it. By [15], *all limits* of the singular perturbation problem (1.3) are domain variation solutions, so all results in the present paper hold for all limits of (1.3).
Our main result Theorem 8.1 states – leaving out inessential assumptions – that if $(0, \rho^2)$ is a point on the topological free boundary and if the set $\{\chi > 0\}$ is flat enough, i.e.

$$\chi(x,t) = 0 \text{ when } (x,t) \in Q_\rho \text{ and } x_n \geq \sigma\rho,$$

for some $\sigma \leq \sigma_0$ (see Figure 2), then the free boundary $Q_{\rho/4} \cap \partial\{u > 0\}$ is a surface with Hölder-continuous space normal.
As a consequence we obtain that the regular set is open relative to $\partial\{u > 0\}$ (Corollary 8.2, cf. Figure 3).
Note that even in the stationary case our result extends the result in [1] as our assumptions *do not exclude* degenerate points or cusps close to the origin (excluded by the definition of weak solutions [1, 5.1]), *our result does that.*
In the proof of our result we use ingenious tools developed in [1]: We prove that flatness on the side of $\{\chi = 0\}$ implies flatness on the side of $\{\chi > 0\}$ which in turn yields uniform convergence of an inhomogeneously scaled sequence of free boundaries.
However we replace the core in the method of H.W. Alt-L.A. Caffarelli, relying on non-positive mean curvature of $\partial\{u > 0\}$ at singularities, by a method based on *scaling discrepancies* (Proposition 7.1). This original component gives hope that the method

may now be applicable to more general free boundary or free discontinuity problems, in particular two-phase free boundary problems.
Note that the time-dependent problem (1.1) is related to caloric measures (see [8] where the topic of the present paper has been mentioned as open problem).

2. Notation

Throughout this article $\mathbf{R}^n$ will be equipped with the Euclidean inner product $x \cdot y$ and the induced norm $|x|$, $B_r(x_0)$ will denote the open n-dimensional ball of center x_0, radius r and volume $r^n\omega_n$, $B'_r(0)$ the open $n-1$-dimensional ball of center 0 and radius r, and e_i the i-th unit vector in $\mathbf{R}^n$. We define $Q_r(x_0,t_0) := B_r(x_0) \times (t_0 - r^2, t_0 + r^2)$ to be the cylinder of radius r and height $2r^2$, $Q_r^-(x_0,t_0) := B_r(x_0) \times (t_0 - r^2, t_0)$ its "negative part" and $T_r^-(t_0) := \mathbf{R}^n \times (t_0 - 4r^2, t_0 - r^2)$ the horizontal layer from $t_0 - 4r^2$ to $t_0 - r^2$. Let us also introduce the parabolic distance $\mathrm{pardist}((t,x),A) := \inf_{(s,y)\in A}\sqrt{|x-y|^2 + |t-s|}$. Considering a function $\phi \in H^{1,2}_{\mathrm{loc}}(\mathbf{R}^n;\mathbf{R}^n)$ we denote by $\operatorname{div}\phi := \sum_{i=1}^n \partial_i\phi_i$ the space divergence and by

$$D\phi := \begin{pmatrix} \partial_1\phi_1 & \dots & \partial_n\phi_1 \\ & \dots & \\ \partial_1\phi_n & \dots & \partial_n\phi_n \end{pmatrix}$$

the matrix of the spatial partial derivatives.
Given a set $A \subset \mathbf{R}^n$, we denote its interior by A° and its characteristic function by χ_A. In the text we use the n-dimensional Lebesgue-measure $\mathcal{L}^n$ and the m-dimensional Hausdorff measure $\mathcal{H}^m$. When considering a given set $A \subset \mathbf{R}^n$, let

$$\partial_M A := \{x \in \mathbf{R}^n \; : \; \limsup_{r\to 0} \frac{\mathcal{L}^n(B_r(x)\cap A)}{\mathcal{L}^n(B_r)} > 0 \text{ and } \limsup_{r\to 0} \frac{\mathcal{L}^n(B_r(x) - A)}{\mathcal{L}^n(B_r)} > 0\}$$

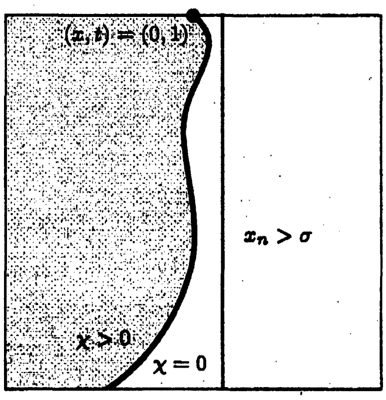

FIGURE 2. One-sided flatness in the case $\rho = 1$

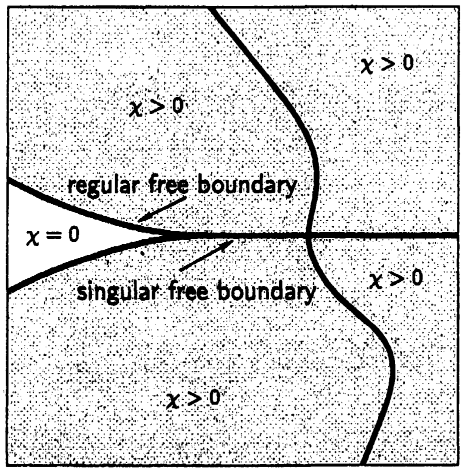

FIGURE 3. Example of the set of regular free boundary points (stationary)

be the measure-theoretic boundary of A, let $\partial^* A := \{x \in \mathbf{R}^n \ : \ \text{there is } \nu(x) \in \partial B_1(0) \text{ such that } r^{-n}\int_{B_r(x)} |\chi_A - \chi_{\{y:(y-x)\cdot\nu(x)<0\}}| \to 0 \text{ as } r \to 0\}$ (by [18, Corollary 5.6.8] $\partial^* A$ coincides $\mathcal{H}^{n-1}$-a.e. with the reduced boundary of a set of finite perimeter defined in [18, Definition 5.5.1]), and let $\nu : \partial^* A \to \partial B_1(0)$ denote this measure theoretic outward normal to ∂A. We shall often use abbreviations for inverse images like $\{u > 0\} := \{x \in \Omega : u(x) > 0\}$, $\{x_n > 0\} := \{x \in \mathbf{R}^n : x_n > 0\}$, $\{s = t\} := \{(s, y) \in \mathbf{R}^{n+1} : s = t\}$ etc. as well as $A(t) := A \cap \{s = t\}$ for a set $A \subset \mathbf{R}^{n+1}$, and occasionally we employ the decomposition $x = (x', x_n)$ of a vector $x \in \mathbf{R}^n$ as well as the corresponding decompositions of the gradient and the Laplace operator,

$$\nabla u = (\nabla' u, \partial_n u) \text{ and } \Delta u = \Delta' u + \partial_{nn} u \, .$$

Finally, $\mathbf{C}^{\beta,\mu} := \mathbf{H}^{\mu,\beta}$ denotes the parabolic Hölder-space defined in [13].

3. Notion of solution and Preliminaries

In this section we gather some results from [15]. As degenerate points are unavoidable in the parabolic problem (see the introduction of [15] for examples), an extension of the *weak solutions* in [1] does not seem to be the right choice. Instead we use the solutions of [15, Definition 6.1], which are, roughly speaking, solutions in the sense of domain variations. The advantage is that the class of solutions defined in [15, Definition 6.1] is closed under the blow-up process. Moreover, *all* limits of the singular perturbation problem discussed in [7] *are* domain variation solutions and satisfy [15, Definition 6.1] (see [15, Section 6]). Let us recall the definition of solutions and the monotonicity formula used therein:

Theorem 3.1 (Monotonicity Formula, cf. [15, Theorem 5.2]). *Let* $(x_0, t_0) \in \mathbf{R}^n \times (0,\infty)$, $T_r^-(t_0) = \mathbf{R}^n \times (t_0 - 4r^2, t_0 - r^2)$, $0 < \rho < \sigma < \frac{\sqrt{t_0}}{2}$ *and*

$$G_{(x_0,t_0)}(x,t) = 4\pi(t_0-t)\, |4\pi(t_0-t)|^{-\frac{n}{2}-1} \exp\left(-\frac{|x-x_0|^2}{4(t_0-t)}\right) .$$

Then

$$\Psi_{(x_0,t_0)}(r) = r^{-2} \int_{T_r^-(t_0)} \left(|\nabla u|^2 + \chi\right) G_{(x_0,t_0)} - \tfrac{1}{2} r^{-2} \int_{T_r^-(t_0)} \tfrac{1}{t_0-t}\, u^2\, G_{(x_0,t_0)}$$

satisfies the monotonicity formula

$$\Psi_{(x_0,t_0)}(\sigma) - \Psi_{(x_0,t_0)}(\rho)$$

$$\geq \int_\rho^\sigma r^{-1-2} \int_{T_r^-(t_0)} \frac{1}{t_0-t} \left(\nabla u \cdot (x-x_0) - 2(t_0-t)\partial_t u - u\right)^2 G_{(x_0,t_0)}\, dr \geq 0 .$$

Definition 3.2 (cf. [15, Definition 6.1]). We call (u,χ) a solution in $\Omega_0 := \mathbf{R}^n \times (0,\infty)$ (in which case we set $\tau := 0$) or $\Omega_1 := \mathbf{R}^n \times (-\infty,\infty)$ (in which case we set $\tau := 1$), if:
1) $u \in \mathbf{C}^{1,\frac{1}{2}}_{\text{loc}}(\Omega_\tau) \cap C^2(\Omega_\tau \cap \{u>0\}) \cap H^{1,2}_{\text{loc}}(\Omega_\tau)$ and $\chi \in L^1((-\tau R, R); BV(B_R(0)))$ for each $R \in (0,\infty)$. For each $R \in (0,\infty)$ and $\delta \in (0,1)$ there exists $C_1 < \infty$ such that for $Q_r(x_0,t_0) \subset \Omega_\tau \cap Q_R(0)$

$$\int_{Q_r(x_0,t_0)} |\nabla \chi| \leq C_1\, r^{n+1},$$

$$\int_{Q_r(x_0,t_0)} |\partial_t u|^2 \leq C_1\, r^n, \text{ and}$$

$$\int_{B_r(x_0)\times(t_0+S_1r^2, t_0+S_2r^2)} |\partial_t(|\nabla u|^2 + \chi) * \phi_{r\delta}| \leq C_1 \sqrt{S_2 - S_1}\, r^n$$

for $0 < S_1 < S_2 < \infty$; here the mollifier $(\phi_\delta)_{\delta\in(0,1)}$ should be non-negative and satisfy $\phi_\delta(\cdot) = \frac{1}{\delta^n}\phi(\frac{\cdot}{\delta}), \phi \in C^{0,1}_0(\mathbf{R}^n)$, $\int \phi = 1$ and supp $\phi \subset B_1(0)$.
Moreover, $\chi \in \{0,1\}$ a.e. in Ω_τ and $\chi_{\{u>0\}} \leq \chi$ a.e. in Ω_τ.
2) The solution u satisfies the monotonicity formula Theorem 3.1 (in the case of $\tau = 1$ for $(x_0,t_0) \in \mathbf{R}^{n+1}$ and $\sigma \in (0,\infty)$).

$$3)\ 0 = \int_{-\infty}^{\infty} \int_{\mathbf{R}^n} [-2\partial_t u\, \nabla u \cdot \xi + (|\nabla u|^2 + \chi) \operatorname{div} \xi - 2\nabla u D\xi \nabla u]$$

for every $\xi \in C^{0,1}_0(\Omega_\tau; \mathbf{R}^n)$.
4) The solution u is non-negative.
5) The solution u attains the initial data $u^0 \in C^{0,1}_0(\mathbf{R}^n)$ in $L^2_{\text{loc}}(\mathbf{R}^n)$ in the case that $\tau = 0$.
6) For each $\kappa > 0$ there is $\delta > 0$ such that $Q_r(x_0,t_0) \subset \Omega_\tau$ and $\|\frac{u(x_0+rx, t_0+r^2t)}{r} -$

$\theta |x_n| \| _{C^0(Q_1(0))} < \delta$ imply $\theta < 1 + \kappa$.
7) For $\delta \in (0,1)$, $\psi_\delta \in C_0^{0,1}(\{|y|^2 + s^2 < \delta^2\})$, $u_r(y,s) := \frac{u(t_0+r^2 s, x_0+ry)}{r}$ and $\chi_r(y,s) := \chi(x_0 + ry, t_0 + r^2 s)$ the following holds:

$$\text{a)} \int_{Q_\rho(x_1,t_1)} |(\nabla \chi_r \cdot x \, + \, 2t\partial_t \chi_r) * \psi_\delta|$$

$$\leq \; C(\delta, Z, T, S, \rho) \left(\Psi_{(x_0,t_0)}(r\sqrt{\frac{-t_1 + \delta + \rho^2}{2}}) \; - \; \Psi_{(x_0,t_0)}(r\sqrt{\frac{-t_1 - \delta - \rho^2}{2}}) \right)$$

for $-S \leq t_1 \leq -T < 0$, $\delta + \rho^2 \leq \frac{T}{2}$, $|x_1| \leq Z$ and, in the case of $\tau = 0$, $t_0 - 2r^2(-t_1 + \rho^2 + \delta) > 0$.

$$\text{b)} \int_{Q_\rho(t_1,x_1)} |(\nabla \chi_r \cdot \xi) * \psi_\delta| \; \leq \; C(\delta) \int_{Q_{\sqrt{\delta}+\rho}(t_1,x_1)} |\nabla u_r \cdot \xi|$$

for $\xi \in \partial B_1(0)$, $t_1 < 0$ and, in the case of $\tau = 0$, $t_0 - r^2(-t_1 + (\rho + \sqrt{\delta})^2) > 0$.

$$\text{c)} \int_{t_1}^{t_2} \partial_t((|\nabla u_r|^2 + \chi_r) * \phi_\delta)(t, x_0) \; \leq \; \int_{t_1}^{t_2} \int_{\mathbf{R}} 2\partial_t u_r(t,z) \nabla u_r(t,z) \cdot \nabla \phi_\delta(x_0 - z) \, dz$$

for $-\infty < t_1 < t_2 < \infty$ and, in the case of $\tau = 0$, $t_0 + r^2 t_1 > 0$.

Remark 3.3. As the function χ is defined only almost everywhere, all pointwise equalities/inequalities involving χ should be understood as equalities/inequalities that hold almost everywhere with respect to the Lebesgue measure.

The reader may wonder whether a solution in the sense of distributions (possibly defined by the identity in [15, Lemma 11.3]) would not be good enough for the purposes of this paper. It turns however out that the information yielded by the order parameter χ in Definition 3.2 carries information that is essential in what follows. Incidentally, χ may be different from $\chi_{\{u>0\}}$ (see [15, Remark 4.1]).

4. Flatness Classes

Definition 4.1. Let $0 < \sigma_+, \sigma_- < 1$ and $\tau \geq 0$. We say that

$$u \in F(\sigma_+, \sigma_-, \tau) \quad \text{in } Q_\rho \quad \text{in direction } e_n$$

if

(1) (u, χ) is a solution in the sense of Definition 3.2 in a domain containing Q_ρ.
(2)

$$(0, \rho^2) \in \partial\{u > 0\},$$

$$u(x,t) = \chi(x,t) = 0 \text{ when } (x,t) \in Q_\rho \text{ and } x_n \geq \sigma_+ \rho,$$

$$\chi(x,t) = 1 \text{ and } u(x,t) \geq -(x_n + \sigma_- \rho) \text{ when } (x,t) \in Q_\rho \text{ and } x_n \leq -\sigma_- \rho\,.$$

(3)

$$|\nabla u| \leq 1 + \tau \text{ in } Q_\rho.$$

When the origin is replaced by (x_0, t_0) and the flatness direction e_n is replaced by ν then we define u to belong to the flatness class $F(\sigma_+, \sigma_-, \tau)$ in $Q_\rho(x_0, t_0)$ in direction ν.

5. Flatness on the side of $\{\chi = 0\}$ implies flatness on the side of $\{\chi > 0\}$

The aim of this and the following sections is to draw information from properties of an inhomogeneous blow-up limit. One of the central problems when using blow-up arguments is "*not-strong convergence*" or "*energy loss*" in the limit. Here we avoid those problems by working with *uniform convergence* (not some Sobolev norm). The approach is based on a powerful idea by H.W. Alt-L.A. Caffarelli who used "flatness on the side of $\{u = 0\}$ implies flatness on the side of $\{u > 0\}$" to prove uniform convergence to an inhomogeneous blow-up limit (cf [1, Section 7]). In this section we extend their result to a weaker class of solutions and to the parabolic case, using results in [15].
The following theorem extends [1, Lemma 7.2].

Theorem 5.1. *There exists a constant $C \in (0, +\infty)$ depending only on the space dimension n such that if $u \in F(\sigma, 1, \sigma)$ in Q_ρ then $u \in F(C\sigma, C\sigma, \sigma)$ in $Q_{\rho/2}(0, y_n, 0))$ for some $|y_n| \leq C\sigma$.*

The idea is to touch the boundary $\partial\{\chi = 0\}$ with the graph of a C^2-function, and to proceed then with a Harnack inequality argument.

6. Inhomogeneous Blow-up

In this section we consider inhomogeneous scaling of the solution and the free boundary. The following lemma is our version of [1, Lemma 7.3]

Lemma 6.1. *Suppose that $u_k \in F(\sigma_k, \sigma_k, \tau_k)$ in Q_{ρ_k}, that $\sigma_k \to 0$ and that $\tau_k/\sigma_k^2 \to 0$, and define*

$$f_k^+(x', t) := \sup\{h : \limsup_{r \to 0} r^{-n-2} \int_{Q_r(\rho_k x', \sigma_k \rho_k h, \rho_k^2 t)} \chi > 0\},$$

$$f_k^-(x', t) := \inf\{h : \limsup_{r \to 0} r^{-n-2} \int_{Q_r(\rho_k x', \sigma_k \rho_k h, \rho_k^2 t)} \chi > 0\}.$$

Then, as a subsequence $k \to \infty$, f_k^+ and f_k^- converge in $L^\infty_{\text{loc}}(Q_1')$ to some function f, and f is continuous in Q_1'.

The next Proposition follows the lines of [2, Lemma 5.7].

Proposition 6.2. *Suppose that the assumptions of Lemma 6.1 are satisfied and that k is the subsequence of Lemma 6.1. Then*

$$w_k(x', h, t) = \frac{u_k(\rho_k x', \rho_k h, \rho_k^2 t) + \rho_k h}{\sigma_k}$$

is for each $\delta \in (0,1)$ bounded in $Q_{1-\delta} \cap \{x_n < 0\}$ (by a constant depending only on δ and n) and converges on compact subsets of Q_1^- in C^2 to a caloric function w.
Moreover, $w(x', h, t)$ is non-decreasing in the h-variable in Q_1^- and

$$\lim_{Q_1^- \ni (y,s) \to (x',0,t) \in Q_1', k \to \infty} w_k(y, s) = f(x', t);$$

here f is the function defined in Lemma 6.1.

7. Scaling discrepancy and C^∞-regularity of blow-up limits

In order to obtain "better-than-Lipschitz"-regularity of the inhomogeneous blow-up limit f, H.W. Alt-L.A. Caffarelli used the non-positive mean curvature of $\partial\{u > 0\}$ at singularities. More precisely, for any smooth test set D each classical solution $\bar{u}$ of the stationary problem satisfies

$$0 = \int_{D \cap \{\bar{u}>0\}} \Delta \bar{u} = -\int_{D \cap \partial\{\bar{u}>0\}} 1 + \int_{\{\bar{u}>0\} \cap \partial D} \nabla \bar{u} \cdot \nu ,$$

implying by the fact that $|\nabla \bar{u}| \le 1 + C\text{dist}(x, \{\bar{u} = 0\})^\alpha$ that the perimeter of $\{\bar{u} > 0\}$ is less than the Hausdorff measure of $\{\bar{u} > 0\} \cap \partial D$ plus $o(1)$ and thereby "almost" non-positive mean curvature of $\partial\{\bar{u} > 0\}$.
The analogue of the non-positive mean curvature property can still be proved in the time-dependent case, however that path leads to problems in the sequel. Therefore we replace it by a scaling discrepancy argument which gives hope to be applicable in more general situations. We obtain C^∞-regularity of f.

Proposition 7.1. *Suppose that the assumptions of Lemma 6.1 are satisfied and that k is the subsequence of Lemma 6.1. Then $\partial_n w = 0$ on $Q'_{1/2}$ in the sense of distributions.*

Proof. The reason why the proposition holds is that the definition of w_k results in different terms scaling at different orders, i.e. a *scaling discrepancy.* The rigorous proof is however rather lengthy.

Corollary 7.2. *Suppose that the assumptions of Lemma 6.1 are satisfied and that k is the subsequence of Lemma 6.1. Then $f \in C^\infty(Q_{1/2})$; moreover,*

$$\left| \frac{\partial^{\alpha+k} f}{\partial x^\alpha \partial t^k} \right| \le C(n, |\alpha|, k)$$

in $Q_{1/4}$ for any $k \in \mathbf{N}$ and multi-index $\alpha \in \mathbf{N}^n$.

8. Flatness improvement and regularity

Concluding regularity is then a standard procedure. We obtain:

Theorem 8.1. *There exists a constant $\sigma_0 > 0$ such that if $u \in F(\sigma, 1, \tau)$ in $Q_\rho(t_0, x_0)$, $\sigma \leq \sigma_0$ and $\tau \leq \sigma_0\sigma^2$, then the topological free boundary $\partial\{u > 0\}$ is in $Q_{\rho/4}(t_0, x_0)$ the graph of a $C^{1+\alpha,\alpha}$-function; in particular the space normal is Hölder continuous in $Q_{\rho/4}(t_0, x_0)$.*

Corollary 8.2. *For each point (x_0, t_0) of the set R, the topological free boundary $\partial\{u > 0\}$ is in an open neighborhood of (x_0, t_0) the graph of a $C^{1+\alpha,\alpha}$-function; in particular, the space normal is Hölder continuous in an open space-time neighborhood of (x_0, t_0).*

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