数理解析研究所講究録
1529 巻 2007 年 1-18

THE NAVIER-STOKES FLOW WITH LIPSCHITZ DATA

OKIHIRO SAWADA
Department of Mathematical Science, Waseda University
Okubo 3-4-1, 169-8555 Shinjuku, Japan

澤田 宙広
早稲田大学理工学術院数理科学（学振 PD）
〒 169-8555 新宿区大久保 3-4-1
e-mail: sawada@gm.math.waseda.ac.jp

ABSTRACT. Time-local existence and uniqueness of mild solutions to the non-stationary incompressible Navier-Stokes equations is established around a steady flow. The initial velocity U_0 is given by $U_0(x) := -f(x) + u_0(x)$, where $-f$ is a stationary solution and a globally Lipschitz continuous function, and a perturbation $u_0 \in L^p_\sigma(\mathbb{R}^n)$ for $p \geq n$. The key is to use the Ornstein-Uhlenbeck semigroup theory, since it is difficult to regard the drift terms (unbounded coefficients in front of first derivatives) as a minor perturbation of Laplacian. Our mild solution satisfies the Navier-Stokes equations in the classical sense when $f(x) = Mx$ with some matrix M and the pressure term is suitably chosen. Moreover, if M is skew-symmetric, then the solution is analytic in spatial variables.

1. INTRODUCTION.

This is a survey note of the author's recent papers [20] joint work with Matthias Hieber in Darmstadt University of Technology, and [19] joint work with Matthias Hieber and Abdelaziz Rhandi in University of Marrakesh.

Key words and phrases. Navier-Stokes equations, unbounded initial data, globally Lipschitz continuous, spatial analyticity.

1.1. **Problem and Known Results.** Consider the flow of an incompressible, viscous ideal fluid in the whole space. That is mathematically described by the Cauchy problem for the system of the Navier-Stokes equations in $\mathbb{R}^n$ for dimension $n \geq 2$:

$$
\left\{
\begin{array}{rcll}
U_t - \Delta U + (U, \nabla)U + \nabla P &=& F, & \text{in } \mathbb{R}^n \times (0,T), \\
\nabla \cdot U &=& 0 & \text{in } \mathbb{R}^n \times (0,T), \\
U|_{t=0} &=& U_0 & \text{in } \mathbb{R}^n.
\end{array}
\right.
\tag{1.1}
$$

Here, $U = (U^1, \dots, U^n)$ and P represent the unknown velocity and the unknown pressure of the fluid; U_0 is a given initial velocity, and F is a given external force term, for example, the acceleration of gravity.

There are many contributional works of studying (1.1), see e.g. [1, 7, 11, 26, 29]. In all these results the initial data are assumed that $U_0(x) \to 0$ as $|x| \to \infty$. In particular, when $F = 0$, (1.1) admits a time-local smooth solution provided the initial velocity U_0 belongs to $L^p_\sigma(\mathbb{R}^n)$ for $p \geq n$; see [15, 26]. Here $L^p = L^p(\mathbb{R}^n)$ denotes the standard Lebesgue space in $\mathbb{R}^n$ for $p \in [1, \infty]$, and its solenoidal subspace is denoted by $L^p_\sigma(\mathbb{R}^n)$. Throughout of this note we sometimes suppress the notation of domain ($\mathbb{R}^n$), and we do not distinguish functions of vector valued and scalar as well as function spaces, if no confusion occurs likely.

We are now strongly forced to study the solution to (1.1) around the stationary flow. For this purpose we consider the initial velocity of the form

$$
U_0(x) = -f(x) + u_0(x), \quad x \in \mathbb{R}^n. \tag{1.2}
$$

Here $u_0 \in L^p(\mathbb{R}^n)$ satisfies $\nabla \cdot u_0 = 0$ and f is a globally Lipschitz continuous function fulfilling the following two conditions:

$$
\begin{array}{ll}
(H1) & \nabla \cdot f = 0, \\
(H2) & \exists \text{ scalar function } \Pi \text{ s.t. } \quad \tilde{F} \in C(0,T; L^p_\sigma(\mathbb{R}^n)),
\end{array}
$$

where

$$
\tilde{F} := F - \Delta f - (f, \nabla)f - \nabla \Pi. \tag{1.3}
$$

In what follows, one may essentially take $(-f, \Pi)$ as the stationary solution to (1.1) into account; in this case $\tilde{F} = 0$ automatically.

Due to the results of Seregin and Šverák [43], it is known that a bounded function $-f$ satisfying stationary (NS) with $F = 0$ in the classical sense is automatically a constant. However, there are many non-trivial stationary solutions if $-f$ is allowed to take an unbounded function, even if $F = 0$. In fact, the pair $-f(x) = Mx + V$ and $\Pi = \frac{1}{2}(M^2x, x) + (V, M^Tx)$ is a stationary solution to (1.1) with $F = 0$. Here $M = (m_{ij})_{1 \le i,j \le n}$ is an $n \times n$ real-valued constant matrix enjoying $\operatorname{tr} M = 0$ and M^2 is symmetric, and V is a real-valued constant vector; M^T denotes the transposed matrix of M.

It is also known that (1.1) admits many exact solutions, which are studied in e.g. [9, 31, 36]. The reason why (1.1) with (1.2) is considered is to understand the classificatation of stationary solutions or exact solutions, for example, their uniqueness of solutions around these, stability or instability, asymptotic behavior, and so on.

We shall list up the results on the time-local existence and uniqueness of smooth solutions to (1.1) related to our situation. In [5, 6, 12, 28, 37], the non-decaying initial velocity is also treated, for example, $U_0 \in L^\infty$, BUC, $B^{-\varepsilon}_{\infty,\infty}$ for $0 \le \varepsilon < 1$, $\dot{B}^{-\varepsilon}_{\infty,\infty}$ for $0 < \varepsilon < 1$ or $BMO^{-1} = \dot{F}^{-1}_{\infty,2}$. Here BUC is the space of all bounded uniformly continuous functions, $B^s_{p,q}$ denotes the inhomogeneous Besov space as well as its homogeneous version $\dot{B}^s_{p,q}$, and $\dot{F}^s_{p,q}$ stands for the homogeneous Triebel-Lizorkin space; see e.g. [44, 45]. Such Besov or Triebel-Lizorkin spaces are strictly wider than L^∞, however, the obtained solutions $U(t)$ belong to L^∞ for any small $t > 0$.

Considering (1.2) with $f(x) = Rx$ where R is a skew-symmetric matrix, there are some results. In this situation we can employ the rotating coordinate to deduce the Navier-Stokes equations with the Coriolis terms. Since the Coriolis terms are linear perturbations, we may regard those as minor perturbation of Laplacian. So, it can be also shown easily that (1.1) admits a time-local unique smooth solution when the initial velocity is given by (1.2) with $f(x) = Rx$. In fact, in [38] the author proved the existence of a time-local smooth solution of (1.1) with (1.2) and $f(x) = Rx$, provided u_0 belongs to $\dot{B}^0_{\infty,1}$. Although

$\dot{B}^0_{\infty,1}$ is strictly smaller than L^∞, this space still contains the non-decaying function.

Moreover, since $U = Rx$ describes pure rotating fluid, it is also interesting to observe this. In particular, Constantin and Feffermann [8] showed the existence of time-global smooth solution in this situation, provided that the rotating speed is fast enough. This fact is called the global regularity. Babin, Mahalov and Nicolaenko [2, 3] also proved the global regularity for the less smooth initial data than that of [8].

Dealing with the problem of the rotating obstacle in the viscous fluid, one reads the similar equations to (1.4) below. On this problem Hishida [21, 22, 23] established the contraction semigroup theory in $L^2_\sigma(\Omega)$ where $\Omega \subset \mathbb{R}^3$ is an smooth exterior domain, and constructed time-local solutions provided initial disturbance belongs to a certain fractional power Sobolev space. Geissert, Heck and Hieber [10] established semigroup theory in L^p for general $p \in (1,\infty)$, and obtained the time-local solvability of rotating obstacle problem in L^p for $p \geq 3$. Recently, Hishida and Shibata [24] obtained a time-global smooth solution for small initial disturbance in L^3. In the recent works of the author [20, 19] he was inspired by Hishida's articles.

Further, the case $f(x) = Jx = (ax_1, ax_2, -2ax_3)$ with some constant $a \in \mathbb{R}$, was investigated by Giga and Kambe [13]. They studied the axisymmetric irrotational flow and the stability of the vortex.

Okamoto [35] obtained the uniqueness of classical solutions to (1.1), when U may grow linearly as $|x| \to \infty$; see also [27]. One of our purpose in this note is to construct the solutions which belong to the framework of Okamoto's uniqueness theorem. However, since Okamoto's uniqueness theorem requires the decay on the pressure at $|x| \to \infty$, we are not able to obtain such a solution so far.

1.2. **Main Results.** Before stating our main results, we consider simple substitutions $u := U + f$ and $\tilde{P} := P - \Pi$. Then the pair (U, P) satisfies (1.1) in the classical sense, if and only if $(u, \tilde{P})$ satisfies

$$(1.4) \qquad \begin{cases} u_t + Au + (u,\nabla)u - 2(u,\nabla)f + \nabla\tilde{P} &= \tilde{F}, \\ \nabla \cdot u &= 0, \\ u(0) &= u_0. \end{cases}$$

Recall $\tilde{F}$ is defined by (1.3), and we denote the matrix operator $A := -\Delta-(f,\nabla)+(\nabla f)$. Applying the Helmholtz projection $\mathbb{P}$ onto solenoidal subspace, we rewrite the first equations of (1.4) as an abstract equation

$$u' + Au + \mathbb{P}(u,\nabla)u - 2\mathbb{P}(u,\nabla)f = \tilde{F}. \tag{1.5}$$

Notice that $\mathbb{P}$ can be expressed explicitly by $\mathbb{P} := (\delta_{ij} + R_i R_j)_{i,j}$, where δ_{ij} stands for Kronecker's delta, and R_i is the Riesz transform defined by $R_i := \partial_i(-\Delta)^{-1/2}$ for $i = 1, \dots, n$.

Considering the realization of A (also denoted by A), A is an operator in $L^p_\sigma(\mathbb{R}^n)$ defined by

$$\begin{aligned} Au &:= -\Delta u - (f,\nabla)u + (u,\nabla)f \\ D(A) &:= \{u \in W^{2,p}(\mathbb{R}^n) \cap L^p_\sigma(\mathbb{R}^n); (f,\nabla)u \in L^p(\mathbb{R}^n)\}. \end{aligned}$$

Observe that A and $\mathbb{P}$ commute, since $\nabla \cdot Au = 0$ if $\nabla \cdot u = 0$. Since u, F and f are divergence-free, $\mathbb{P}u = u$ as well as $\mathbb{P}\tilde{F} = \tilde{F}$.

It is known that $-A$ generates a (non-analytic) C_0-semigroup $\{e^{-tA}\}_{t\ge0}$ on L^p_σ for $1 < p < \infty$. This semigroup theory follows from the results in [30, 33, 34] and a standard perturbation theory. This $\{e^{-tA}\}_{t\ge0}$ is often called the Ornstein-Uhlenbeck semigroup, we use this terminology. In general, there is not explicit representation formula of e^{-tA}. However, if $f(x) = Mx$, then

$$e^{-tA}\varphi(x) = \frac{e^{-tM}}{(4\pi)^{n/2}(detQ_t)^{1/2}} \int_{\mathbb{R}^n} \varphi(e^{tM}x - y)e^{-\frac{1}{4}(Q_t^{-1}y,y)}dy \tag{1.6}$$

for $x \in \mathbb{R}^n$ and $t > 0$, where Q_t is given by $Q_t := \int_0^t e^{sM}e^{sM^T}ds$.

It thus is straightforward to derive the integral equation by Duhamel's principle:

$$\begin{aligned} u(t) = e^{-tA}u_0 &- \int_0^t e^{-(t-s)A}\mathbb{P}(u(s),\nabla)u(s)ds \\ &+ 2\int_0^t e^{-(t-s)A}\mathbb{P}(u(s),\nabla)f ds + \int_0^t e^{-(t-s)A}\tilde{F}(s)ds \end{aligned} \tag{1.7}$$

for $t \in (0,T)$ and $u(0) = u_0$. We call a function $u \in C([0,T); L^p_\sigma(\mathbb{R}^n))$ a mild solution if u satisfies (1.7). Formally, (1.7) is equivalent to (1.4). In fact, under some condition a mild solution u and the suitable choice of $\tilde{P}$ satisfy (1.4) in the classical sense; see Theorem 1.1-(ii) below. In what follows, we rather discuss the mild solutions.

We now state the our existence and uniqueness results for mild solutions in L^p spaces.

1.1. Theorem. (i) *Let $n \geq 2$ and $p \in [n, \infty)$. Let f be a globally Lipschitz continuous function satisfying $(H1)$ and $(H2)$ with suitable Π and F. Assume that $u_0 \in L^p_\sigma(\mathbb{R}^n)$. Then there exist $T_0 > 0$ and a unique mild solution u in the following class:*

$$[t \mapsto t^{\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}u(t)] \in C([0,T_0); L^q_\sigma(\mathbb{R}^n)) \tag{1.8}$$

$$[t \mapsto t^{\frac{n}{2}(\frac{1}{p}-\frac{1}{q})+\frac{1}{2}}\nabla u(t)] \in C([0,T_0); L^q(\mathbb{R}^n)) \tag{1.9}$$

for $q \in [p, \infty]$.
(ii) *In addition, let $f(x) = Mx$ where M is a matrix. Then*

$$u \in C^\infty(\mathbb{R}^n \times (0,T_0)). \tag{1.10}$$

Moreover, u satisfies (1.4) in the classical sense provided $\tilde{P}$ is taken as

$$\partial_k \tilde{P} = \partial_k \sum_{i,j=1}^{n} R_i R_j u^i u^j - 2 \sum_{i,j=1}^{n} R_i R_k u^j (\partial_j f^i). \tag{1.11}$$

(iii) *In addition to the hypothesis of (ii), let M be skew-symmetric. Let $\Pi = \frac{1}{2}(M^2 x, x)$, and let F be analytic in x. Then $u(t)$ is analytic in x on $t \in (0, T_0)$.*

1.2. Remark. (a) *Because the Ornstein-Uhlenbeck semigroup $\{e^{-tA}\}_{t\geq 0}$ is not analytic, we cannot apply the usual argument to show our mild solution satisfies (1.4) like the Stokes case, for general Lipschitz function f. This means that we cannot control the time derivative of u with valued in L^p, although by Serrin's interior regularity theorem it seems true that*

$$u(t) \in C^\infty(\mathbb{R}^n) \quad \text{almost every} \quad t \in (0, T_0). \tag{1.12}$$

Unfortunately, (1.12) does not imply that u is a classical solution.
(b) *For neither $u_0 \in L^\infty_\sigma$ nor $u_0 \in BUC_\sigma$, it is not easy to get the mild solutions, since $\mathbb{P}$ is not bounded in such spaces as well as the Riesz transform. For dealing with non-decaying data we introduce the homogeneous Besov space $\dot{B}^0_{\infty,1} \subset L^\infty$, since $\mathbb{P}$ is bounded in the homogeneous Besov spaces; other properties of $\dot{B}^0_{\infty,1}$ are found in [41]. In*

fact, we may obtain the time-local existence and uniqueness results of the mild solutions $u \in C([0, T_0); \dot{B}^0_{\infty,1})$ *provided that* $u_0 \in \dot{B}^0_{\infty,1}$ *and* $\nabla \cdot u_0 = 0$ *at least for the the case* $f(x) = Mx$*; the details discussed in* [42].

(c) *Thanks to (ii), if* $f(x) = Mx$ *and* $p = n = 2$*, then we obtain the time-global solution by the following a priori estimate: there exist positive constants* D_1 *and* D_2 *depending only on* $u_0 \in L^2_\sigma(\mathbb{R}^2)$*,* M *and* $\tilde{F} \in C(0, \infty; L^2_\sigma(\mathbb{R}^2))$ *such that*

$$\|u(t)\|_2^2 \leq D_1 e^{D_2 t}, \quad t \geq 0.$$

This comes from the Energy estimate, multiplying u *into the first equations of (1.4) and integrating in* $x \in \mathbb{R}^2$*.*

(d) *Obviously, the analyticity in* x *implies that the propagation speed of mild solution is infinity, that is, the support of* $u(t)$ *coincides* $\mathbb{R}^n$ *for any small* $t > 0$*, even if the support of* u_0 *is compact.*

The proof of Theorem 1.1-(i) is based on Kato's iteration procedure. The key is to derive appropriate $L^p - L^q$ smoothing estimates for the Ornstein-Uhlenbeck semigroup e^{-tA}, including the gradient. Uniqueness follows by Gronwall's inequality.

To prove Theorem 1.1-(ii) we use the explicit representation formula of the Ornstein-Uhlenbeck semigroup (1.6), when $f(x) = Mx$. Involving the k-th derivatives in x into the iteration, it is proved that $u \in C(0, T_0; C^k(\mathbb{R}^n))$ for all $k \in \mathbb{N}$. To control the time derivatives of u we introduce the notion of a weak solution. From (1.10) we may see that u satisfies (1.5), and that $(u, \tilde{P})$ satisfies (1.4) provided $\tilde{P}$ is given by (1.11).

An observation of analyticity goes back to work of Masuda [32] based on the implicit function theory. In this note we give another proof of the analyticity of u in x. We shall derive the higher order derivatives in x of u. More precisely, we establish the following estimate:

$$\|\partial_x^\beta u(t)\|_q \leq D_3 (D_4 m)^m t^{-\frac{m}{2} - \frac{n}{2}(\frac{1}{p} - \frac{1}{q})} \tag{1.13}$$

with some positive constants D_3 and D_4 for all $t \in (0, T_0)$, $q \in [p, \infty]$ and $\beta \in \mathbb{N}_0^n$ with $m = |\beta|$. Here we use the conventional notation $\partial_x^\beta := \partial_1^{\beta_1} \cdots \partial_n^{\beta_n}$ for multi-index $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}_0^n$. Clearly, (1.13) yields the analyticity of u in x by Stirling's formula and Cauchy's criterion. Indeed, there exists a constant $C > 0$ such that the size of radius of the convergence of Taylor's expansion ($=: \rho(t)$) is estimated from below by

$$\rho(t) = \limsup_{m\to\infty} \left(\frac{\|\partial_i^m u(t)\|_\infty}{m!} \right)^{-1/m} \geq C\sqrt{t}$$

for each $i = 1, \ldots, n$. The main idea to derive (1.13) is dividing the time-interval of (1.7) into $(0, (1-\varepsilon)t)$ and $((1-\varepsilon)t, t)$, and taking $\varepsilon = 1/|\beta|$. This technique was developed by Giga and the author [17, 39] to show (1.13) when $f = 0$.

This note is organized as follows. In Section 2 we recall the Ornstein-Uhlenbeck semigroup theory, and also we prepare the estimates used in the proof of Theorem 1.1. In Section 3 we give propositions and their idea of proofs briefly.

(Acknowledgment). The author would like to thank Professor Giorgio Metafune and Professor Enrico Priola for giving him many advice on the Ornstein-Uhlenbeck semigroup theory. The author would also like to thank Professor Kenji Nakanishi for letting him know how to prove that the mild solution is a classical solution via weak solutions. The work of the author is partly supported by the Japan Society for the Promotion of Science.

2. Semigroup Theory.

We prepare the linear estimates used for the proof of Theorem 1.1. In this section let f be a vector-valued globally Lipschitz function satisfying $\nabla \cdot f = 0$.

We define the operator A by

$$\begin{aligned} Au &:= -\Delta u - (f, \nabla)\nabla u + (u, \nabla) f, \\ D(A) &:= \{u \in W^{2,p} \cap L^p_\sigma; (f, \nabla) u \in L^p\}. \end{aligned}$$

Thanks to add the lower order terms, we see $\nabla \cdot \{(f, \nabla)u - (u, \nabla)f\} = 0$ provided $\nabla \cdot u = 0$ and $\nabla \cdot f = 0$. Therefore, A and $\mathbb{P}$ commute.

It is known that $-A$ generates a non-analytic C_0-semigroup in $L^p_\sigma(\mathbb{R}^n)$ for $1 < p < \infty$. The family $\{e^{-tA}\}_{t\geq 0}$ is also a C_0-semigroup on L^1, and a semigroup in L^∞ (lack of strong continuity), however, it is difficult to make sense the Helmholtz decomposition in such spaces.

We are now state $L^p - L^q$ smoothing properties for the semigroup e^{-tA} as well as gradient estimates up to second derivatives. Note that due to the non-analyticity of Ornstein-Uhlenbeck semigroup, gradient estimates do not follow from the general theory of analytic semigroup.

2.1. **Lemma.** *Let $n \geq 2$, $1 \leq p \leq \infty$ and $p \leq q \leq \infty$. Then there exist constants $C > 0$ and $\omega \in \mathbb{R}$ such that*

$$\|\nabla^k e^{-tA}\varphi\|_q \leq Ce^{\omega t}t^{-\frac{k}{2}-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\|\varphi\|_p \tag{2.1}$$

for all $\varphi \in L^p(\mathbb{R}^n)$ and $k = 0, 1, 2$. Moreover, let either $1 \leq p \leq q \leq \infty$ and $k = 1, 2$ or $1 \leq p < q \leq \infty$ and $k = 0, 1, 2$. Then for $\varphi \in L^p(\mathbb{R}^n)$

$$t^{\frac{k}{2}+\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\|\nabla^k e^{-tA}\varphi\|_p \to 0 \qquad \text{as} \quad t \to 0. \tag{2.2}$$

The proof of (2.1) is given by [30, Proposition 5.4], [4, Theorem 4.7 and Corollary 4.8]. For more details see [18, Corollary 5.2 and Theorem 5.3]. To get (2.2) we use the triangle inequality, and the fact that C_0^∞ is a densely subset of L^p for $p < \infty$.

In the case where $f(x) = Mx$, the Ornstein-Uhlenbeck semigroup $\{e^{-tA}\}_{t\geq 0}$ has an explicit representation (1.6). Thanks to (1.6), we may derive the higher order derivatives.

2.2. **Lemma.** *Let $n \geq 2$, $1 \leq p \leq q \leq \infty$, $f(x) = Mx$ with some matrix M. Then there exist constants $C_1, C_2, C_3 > 0$ and $\omega_1, \omega_2, \omega_3, \omega_4 \in \mathbb{R}$ (depending only on n, p, q and M) such that*

$$\|\nabla^m e^{-tA}\varphi\|_q \leq C_1 e^{(\omega_1+\omega_2 m)t}t^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\|\nabla^m\varphi\|_p \tag{2.3}$$

for all $t > 0$, $m \in \mathbb{N}$ and $\varphi \in W^{m,p}(\mathbb{R}^n)$, and

$$\|\nabla^m e^{-tA}\varphi\|_q \leq C_2(C_3 m)^{m/2}e^{(\omega_3+\omega_4 m)t}t^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})-\frac{m}{2}}\|\varphi\|_p \tag{2.4}$$

for all $t > 0$, $m \in \mathbb{N}$ and $\varphi \in L^p(\mathbb{R}^n)$.

2.3. **Remark.** *If M is skew-symmetric, then $\omega_2 = 0$ in* (2.3).

The proof of above lemma was shown in [20]. Thanks to (1.6), we see that

$$\nabla e^{-tA} = e^{tM}e^{-tA}\nabla. \tag{2.5}$$

If M is skew-symmetric, then e^{tM} is unitary, so Remark 2.3 holds true.

3. Proof of Theorem 1.1.

For a given globally Lipschitz continuous function f satisfying $(H1)$ and $(H2)$ with suitable Π and F, we consider the substitution $u(x,t) := U(x,t) + f(x)$ and $\tilde{P}(x,t) := P(x,t) - \Pi(x)$. If (U, P) is a solution of (1.1) in the classical sense, then $(u, \tilde{P})$ satisfies (1.4). In what follows, we mainly deal with the mild solutions.

We only show the proof of Theorem 1.1 for the case $p = n$; because, in the case $p > n$ the proof is essentially similar and easier then that of $p = n$. Firstly, we state the proposition which yields Theorem 1.1-(i).

Proof of Theorem 1.1-(i). Let $n \geq 2$, $T > 0$ and $u_0 \in L^n_\sigma(\mathbb{R}^n)$. Assume that $\tilde{F} \in C(0,T; L^n_\sigma(\mathbb{R}^n))$. Recall that $\tilde{F} = F - \Delta f - (f, \nabla)f - \Pi$ with suitable scalar function Π, and that $\nabla \cdot f = 0$. For $j \in \mathbb{N}$ and $t \in (0,T)$ we define functions u_j successively by

$$u_1(t) := e^{-tA}u_0 + \int_0^t e^{-(t-s)A}\tilde{F}(s)ds,$$

$$u_{j+1}(t) := u_1(t) - \int_0^t e^{-(t-s)A}\mathbb{P}\{(u_j(s), \nabla)u_j(s) - 2(u_j(s), \nabla)f\}\, ds.$$

Since $\{e^{-tA}\}_{t\geq 0}$ acts on $L^p_\sigma(\mathbb{R}^n)$ for $p \in (1, \infty)$, it follows from the definition of the Helmholtz projection that the functions u_j are divergence-free for all $t > 0$ and all $j \in \mathbb{N}$.

As usual, using (2.1) and (2.2), we derive a priori estimates. In fact, for $\delta \in (0,1)$ we may obtain bounds for

$$sup_{0<t<T_0} t^{\frac{1-\delta}{2}} \|u_j(t)\|_{n/\delta} \quad \text{and} \quad sup_{0<t<T_0} t^{\frac{1}{2}} \|\nabla u_j(t)\|_n$$

for any $T \leq T_0$ uniformly in j provided that T_0 is small enough. These uniform bounds imply that $t^{\frac{1}{2}-\frac{n}{2q}}\|u_j(t)\|_q$ as well as $t^{1-\frac{n}{2q}}\|\nabla u_j(t)\|_q$ are bounded for $q \in [n, \infty)$, $t \leq T_0$ and all $j \in \mathbb{N}$. The continuity of these functions follows from similar calculations.

It can be also shown that these sequences are Cauchy sequences, once we choose T_0 small enough if necessary. We thus conclude that there are unique limit functions

$$[t \mapsto t^{\frac{1}{2}-\frac{n}{2q}}u(t)] \in C([0,T_0]; L^q_\sigma), \qquad [t \mapsto t^{1-\frac{n}{2q}}v(t)] \in C([0,T_0]; L^q)$$

of the sequences $\{t^{\frac{1}{2}-\frac{n}{2q}}u_j(t)\}_{j\geq 1}$ and $\{t^{1-\frac{n}{2q}}\nabla u_j(t)\}_{j\geq 1}$. Finally, note that $v(t) = t^{1/2}\nabla u(t)$ and that u is a mild solution on $[0, T_0]$. Uniqueness of mild solutions follows from standard Gronwall's inequality; see e.g. [14]. This completes the proof of Theorem 1.1-(i). □

Next, we show the idea of the proof of Theorem 1.1-(ii). Smoothness of mild solution is also obtained by a modification of the proof above.

Proof of Theorem 1.1-(ii). Consider the case when $f(x) = Mx$. To show (1.10) we establish the smoothing estimates with higher order differentiations; see Lemma 2.2. In order to get the up to ℓ-th derivative in x for $m \in \mathbb{N}$, we involve

$$\sup_{0<t<T} t^{\frac{\ell}{2}}\|\nabla^\ell u_j(t)\|_n \tag{3.1}$$

for all $\ell \leq m$ into the iteration scheme. To derive a priori estimates, we divide the time-interval $(0,t)$ of integrals of (1.7) into two parts $(0,t/2)$ and $(t/2,t)$ to distribute the singularities.

Similarly as the proof of Theorem 1.1-(i), we choose $T_m > 0$ small enough so that the quantities (3.1) are uniformly bounded. This implies

$$u \in C(0, T_m; C^m(\mathbb{R}^n)). \tag{3.2}$$

We see $T_m \sim m^{-m}$, in general. (It is possible to take T_ℓ independent of m, if we divide the time-interval more cleverly, and if M is skew-symmetric; see the proof of Theorem 1.1-(iii).)

We may extend the time-interval $(0, T_m)$ up to T_0, since mild solution exists uniquely (no blow-up) at least until T_0. We see (3.2) for all $m \in \mathbb{N}$, this yields $u \in C(0, T_0; C^\infty(\mathbb{R}^n))$.

For establishing the estimates for time derivatives, we will use the notion of a weak solution. Here the weak solution is a function satisfying (1.4) in distribution sense. Notice that our mild solution is a weak solution. We now take test-function $\varphi \in C_0^\infty(\mathbb{R}^n)$, and $h \in C^1(0,T)$ satisfying $h(0) = h(T) = 0$ for simplicity. Let

$$<\psi, \varphi>:= \int_{\mathbb{R}^n} \psi(x)\varphi(x)dx,$$

and A^* denotes the dual of A, i.e., $< A\psi, \varphi >=< \psi, A^*\varphi >$. Assume that u is the mild solution obtained in Theorem 1.1-(i). Multiplying φ

and h' into (1.7), and integrating over $(0,T)\times\mathbb{R}^n$, we get

$$\begin{aligned}
&\int_0^T <u(t),\varphi> h'(t)dt \\
&= \int_0^T <e^{-tA}u_0,\varphi> h'(t)dt - \int_0^T < \int_0^t e^{-(t-s)A}\mathbb{P}\tilde{F}(s)ds,\varphi> h'(t)dt \\
&\qquad + \int_0^T < \int_0^t e^{-(t-s)A}\mathbb{P}\{2(u(s),\nabla)f - (u(s),\nabla)u(s)\}ds,\varphi> h'(t)dt \\
&= \int_0^T <u_0, A^*e^{-tA^*}\varphi> h(t)dt - \int_0^T\int_0^t <\tilde{F}(s), A^*e^{-(t-s)A^*}\mathbb{P}\varphi> dsh(t)dt \\
&\qquad + \int_0^T\int_0^t <\mathbb{P}\{2(u(s),\nabla)f - (u(s),\nabla)u(s)\}, A^*e^{-(t-s)A^*}\varphi> dsh(t)dt \\
&= \int_0^T <Au(t) - \tilde{F}(t) + \mathbb{P}(u(t),\nabla)u(t) - 2\mathbb{P}(u(t),\nabla)f,\varphi> h(t)dt.
\end{aligned}$$

Note that $\varphi \in C_0^\infty \subset D(A)$. Since $u \in C((0,T_0];C^2(\mathbb{R}^n))$, we can make sense $Au(x,t)$ pointwisely. Moreover, the right-hand-side is well-defined at any $t \in (0,T_0]$ as well as these integrations are continuous in time. Hence, we can verify that $<u(\cdot),\varphi>\in C^1(0,T)$. We conclude that for all $t\in(0,T)$

$$<u_t(t) + Au(t) - \tilde{F}(t) + \mathbb{P}(u(t),\nabla)u(t) - 2\mathbb{P}(u(t),\nabla)f,\varphi>= 0.$$

Let $\tilde{P}$ be given by (1.11), from above we have

$$<u_t - \Delta u - (f,\nabla)u + (u,\nabla)u - (u,\nabla)f + \nabla\tilde{P} - \tilde{F},\varphi>= 0.$$

This holds true for all $\varphi \in C_0^\infty(\mathbb{R}^n)$. Therefore, $(u,\tilde{P})$ satisfies (1.4) in the classical sense at any $t\in(0,T_0)$ and $x\in\mathbb{R}^n$. Furthermore, higher order derivatives of u in time can be calculated, analogously. This implies that (1.10). □

Finally, we show the proof of Theorem 1.1-(iii). It is sufficient to establish the estimates for higher order derivatives of u in x, which is formally equivalent to (1.13). Again, we only discuss the case $p=n$ in what follows.

3.1. **Proposition.** *Let $n\geq 2$, $u_0\in L^n_\sigma(\mathbb{R}^n)$ and $f(x)=Mx$, where M is skew-symmetric. Let $\Pi=\frac{1}{2}(M^2x,x)$, and let the external force*

$F \in C(0,T; L^n_\sigma \cap C^\infty)$ with some $T > 0$. Let $\delta \in (1/2, 1]$. Suppose that there exist positive constants L_1 and L_2 such that

$$\|\partial_x^\beta F(t)\|_q \le L_1 (L_2 |\beta|)^{|\beta| - \delta} t^{-\frac{|\beta|}{2} - \frac{n}{2}(\frac{1}{n} - \frac{1}{q})} \tag{3.3}$$

hold for $t \in (0,T)$ and $q \in [n, \infty]$. Assume that u is a mild solution in the class

$$u \in C([0,T); L^n_\sigma) \cap C(0,T; L^r_\sigma)$$

for some $r > n$. Suppose that there exist positive constants M_1 and M_2 such that

$$M_1 \ge \sup_{0 \le t < T} \|u(t)\|_n, \quad M_2 \ge \sup_{0 < t < T} t^{\frac{n}{2}(\frac{1}{n} - \frac{1}{r})} \|u(t)\|_r.$$

Then there exist positive constants D_5 and D_6 depending only on n, r, M, L_1, L_2, M_1, M_2, T and δ such that

$$\|\partial_x^\beta u(t)\|_q \le D_5 (D_6 |\beta|)^{|\beta| - \delta} t^{-\frac{|\beta|}{2} - \frac{n}{2}(\frac{1}{n} - \frac{1}{q})} \tag{3.4}$$

for all $q \in [n, \infty]$, $t \in (0,T]$ and multi-index $\beta \in \mathbb{N}_0^n$.

Obviously, (3.4) implies (1.13). Notice that (3.3) holds true if $F(t)$ is analytic in x. Also, $\tilde{F} = F$, since $\Delta M x = 0$ and $(Mx, \nabla) Mx + \nabla \Pi = 0$.

Proof of Proposition 3.1. We use an induction with respect to $m = |\beta|$. Let $m_0 \ge 2$ (determined later). From above arguments we see

$$\|\partial_x^\beta u(t)\|_q \le D_5 t^{-\frac{m}{2} - \frac{n}{2}(\frac{1}{n} - \frac{1}{q})} \tag{3.5}$$

hold true for all $t \in (0,T)$ and $m = |\beta| \le m_0$, provided D_5 is chosen large enough.

Hence, we assume that $m \ge m_0$. We suppose by assumption of induction that (3.4) holds for all $q \in [n, \infty]$ and all $|\beta| \le m - 1$. We claim that (3.4) holds for $|\beta| = m$. For simplicity, we first prove the assertion under the additional assumptions that $T \le 1$, $n \ge 3$ and $q < \infty$. The claim then follows by minor modifications of the proof given below.

Let $q \in [n,\infty)$, and let $\varepsilon \in (0,1)$. We have

$$\begin{aligned}\|\partial_x^\beta u(t)\|_q &\le \|\partial_x^\beta u_1\|_q + \left(\int_0^{(1-\varepsilon)t} + \int_{(1-\varepsilon)t}^t\right)\|\partial_x^\beta e^{-(t-s)A}\mathbb{P}(u(s),\nabla)u(s)\|_q ds\\ &\quad + 2\left(\int_0^{(1-\varepsilon)t} + \int_{(1-\varepsilon)t}^t\right)\|\partial_x^\beta e^{-(t-s)A}\mathbb{P}(u(s),\nabla)f\|_q ds\\ &=: B_1 + B_2 + B_3 + B_4 + B_5.\end{aligned}$$

We shall estimate each the above terms $B_1 - B_5$ separately.

To this end, taking into account $\varepsilon = 1/m$, the estimates for B_1, B_2 and B_4 are derived from (2.4) as follows:

$$B_1 + B_2 + B_4 \le C_4(C_5 m)^{m-\delta} t^{-\frac{n}{2}(\frac{1}{n}-\frac{1}{q})-\frac{m}{2}}, \qquad t \in (0,T)$$

for constants C_4 and C_5 independent of t and β.

Main difficulties arise from B_3.

$$B_3 \le C\int_{(1-\varepsilon)t}^t (t-s)^{-\frac{1}{2}}\|\partial_x^\beta(u(s)\otimes u(s))\|_q ds$$

with some $C := C(n,M)$. Here we have used [20, Lemma 3.7], that is, $\|\nabla e^{-tA}\mathbb{P}\|_{\mathcal{L}(L^q)} \le Ct^{-1/2}e^{\omega t}$ for some $C>0$ and some $\omega \in \mathbb{R}$ for all $t>0$ and $q \in [1,\infty]$. We now calculate $\partial_x^\beta(u\otimes u)$ by Leibniz's rule. We divide the sum into two parts:

$$\begin{aligned}B_3 &\le 2C\int_{(1-\varepsilon)t}^t (t-s)^{-\frac{1}{2}}\|\partial_x^\beta u(s)\|_q\|u(s)\|_\infty ds\\ &\quad + C\int_{(1-\varepsilon)t}^t (t-s)^{-\frac{1}{2}}\sum_{0<\gamma<\beta}\binom{\beta}{\gamma}\|\partial_x^\gamma u(s)\|_q\|\partial_x^{\beta-\gamma}u(s)\|_\infty ds\\ &=: B_{3a} + B_{3b}.\end{aligned}$$

Here, $\gamma < \beta$ denotes $\gamma_i \le \beta_i$ for all i and $|\gamma| < |\beta|$ for multi-indices β and γ; $\binom{\beta}{\gamma} := \prod_{i=1}^n \frac{\beta_i!}{\gamma_i!(\beta_i-\gamma_i)!}$ is the binomial coefficient.

Recall that $\|u(s)\|_\infty \le Cs^{-1/2}$ for some C. So, we have

$$B_{3a} + B_5 \le C_6\int_{(1-\varepsilon)t}^t (t-s)^{-\frac{1}{2}}s^{-\frac{1}{2}}\|\partial_x^\beta u(s)\|_q ds$$

for some C_6, since we derive the estimate for B_5 as well.

Estimating B_{3b}, by assumption of induction we obtain

$$\begin{aligned} B_{3b} &\le C\int_{(1-\varepsilon)t}^{t}(t-s)^{-\frac{1}{2}}\sum_{0<\gamma<\beta}\binom{\beta}{\gamma}D_5(D_6|\gamma|)^{|\gamma|-\delta}s^{-\frac{n}{2}(\frac{1}{n}-\frac{1}{q})-\frac{|\gamma|}{2}} \\ &\quad\times D_5(D_6|\beta-\gamma|)^{|\beta-\gamma|-\delta}s^{-\frac{n}{2}(\frac{1}{n}-\frac{1}{q}-\frac{|\beta-\gamma|}{2})}ds \\ &\le CD_5^2D_6^{m-2\delta}J_\varepsilon\sum_{0<\gamma<\beta}\binom{\beta}{\gamma}|\gamma|^{|\gamma|-\delta}|\beta-\gamma|^{|\beta-\gamma|-\delta}t^{-\frac{n}{2}(\frac{1}{n}-\frac{1}{q})-\frac{m}{2}}. \end{aligned}$$

Here

$$J_\varepsilon := \int_{1-\varepsilon}^{1}(1-\tau)^{-\frac{1}{2}}\tau^{-\frac{n}{2}(\frac{1}{n}-\frac{1}{q})-\frac{m}{2}-\frac{1}{2}}d\tau.$$

Note that $J_{1/m}\le 1/(2\tilde{C}_3+2)$ and $\lim_{m\to\infty}J_{1/m}\to 0$, since $r>2$. For the multiplication of multi-sequences we apply Kahane's lemma [25, Lemma 2.1] to obtain

$$B_{3b}\le C_7D_5^2D_6^{m-2\delta}m^{m-\delta}t^{-\frac{n}{2}(\frac{1}{n}-\frac{1}{q})-\frac{m}{2}},$$

where C_7 depends also on δ; indeed, $C_7\sim\sum_{j=1}^{\infty}j^{-1/2-\delta/2}$.

Combining the estimates for B_1-B_5, and applying a Gronwall's type inequality [17, Lemma 2.4], there exists $\varepsilon_m\in(0,1)$ such that

$$\|\partial_x^\beta u(t)\|_q\le 2b_{\varepsilon_m}t^{-\frac{n}{2}(\frac{1}{n}-\frac{1}{q})-\frac{m}{2}},\qquad t\in(0,T).$$

We have taken $\varepsilon_m := 1/m$, we fix $m_0\in\mathbb{N}$ which is the smallest number satisfying $J_{1/m}\le\frac{1}{2C_6}$.

Finally, we verify (3.4) for all m under suitable choices of D_5 and D_6. To get (3.4) for $|\beta|=m\le m_0$, it is sufficient to choose D_5 large enough such that (3.5) holds, where m_0 is given above. Also, it is sufficient to take $D_6\ge(2C_7D_5)^{1/\delta}$, then (3.4) holds for all $m\ge m_0$. The proof is complete. □

If M is skew-symmetric, then $\|e^{tM}\|\le 1$. It is not enough to assume that $\|e^{tM}\|\le C$ for some $C>1$, at least for the author.

One can get the similar results on the Keller-Segel equations, Fujita equation (semilinear heat equation) of algebraic nonlinearity, Allen-Cahn equation, and other equations of parabolic type. See the details in [40].

At the end of this note we show a modification of iteration arguments. Recall that the mild solution u is a unique limit of successive approximation u_j. Take $\beta \in \mathbb{N}_0^n$ arbitrary. We now define for $j \in \mathbb{N}$

$$\psi_j(t) := \|\partial_x^\beta u_j(t)\|_q,$$

and argue in the similar way in the proof of Proposition 3.1 to get $\partial_x^\beta u \in C(0, T_0; L^q)$ by applying the sequence version of Gronwall's inequalities.

References

[1] H. Amann, *On the strong solvability of the Navier-Stokes equations*, J. Math. Fluid Mech., **2** (2000), 16-98.
[2] A. Banin, A. Mahalov and B. Nicolaenko, *Global regularity of 3D rotating Navier-Stokes equations for resonant domains*, Indiana Univ. Math. J. **48** (1999), 1133-1176.
[3] A. Banin, A. Mahalov and B. Nicolaenko, *3D Navier-Stokes and Euler equations with initial data characterized by uniformly large vorticity*, Indiana Univ. Math. J. **50** (2001), 1-35.
[4] M. Bertoldi, L. Lorenzi, *Estimates of the derivatives for parabolic operators with unbounded coefficients*, Trans. Amer. Math. **357** (2005), 2627-2664.
[5] J. R. Cannon and G. H. Knightly, *A note on the Cauchy problem for the Navier-Stokes equations*, SIAM J. Appl. Math., **18** (1970), 641-644.
[6] M. Cannone, *Ondelettes, Paraproduits et Navier-Stokes, Diderot Editeur*, Arts et Sciences Paris-New York-Amsterdam, (1995).
[7] M. Cannone and Y. Meyer, *Littlewood-Paley decomposition and Navier-Stokes equations*, Methods and Applications in Analysis, **2** (1995), 307-319.
[8] P. Constantin and C. Fefferman, *Direction of vorticity and the problem of global regularity for the Navier-Stokes equations*, Indiana Univ. Math. J., **42** (1993), 775-789.
[9] W.I. Fushchich, W.M. Shtelen and N.I. Serov, *Symmetry analysis and exact solutions of equations of biblinear mathematical physics*, Mathematics and its Applications, **246**, Kluwer Academic Publishers Group, Dordrecht, 1993.
[10] M. Geissert, H. Heck and M. Hieber *L^p-theory for the Navier-Stokes flow past moving or rotating obstacles*, J. Reine Angew. Math., **596** (2006), 45-62.
[11] Y. Giga, *Solutions for semilinear parabolic equations in L^p and regularity of weak solutions of the Navier-Stokes system*, J. Differential Equations, **62** (1986), 186-212.
[12] Y. Giga, K. Inui and S. Matsui, *On the Cauchy problem for the Navier-Stokes equations with nondecaying initial data*, Quaderni di Matematica, **4** (1999), 28-68.
[13] Y. Giga and T. Kambe, *Large time behaviour of the vorticity of two-dimensional viscous flow and its application to vortex formation*, Commun. Math. Phys., **117** (1988), 549-568.
[14] Y. Giga, S. Matsui and O. Sawada, *Global existence of two-dimensional Navier-Stokes flow with nondecaying initial velocity*, J. Math. Fluid Mech., **3** (2001), 302-315.
[15] Y. Giga and T. Miyakawa, *Solutions in L^r of the Navier-Stokes initial value problem*, Arch. Ration. Mech. Anal., **89** (1985), 267-281.

[16] Y. Giga, T. Miyakawa and H. Osada, *Two-dimensional Navier-Stokes flow with measures as initial vorticity*, Arch. Ration. Mech. Anal., **104** (1988), 223-250.

[17] Y. Giga and O. Sawada, *On regularizing-decay rate estimates for solutions to the Navier-Stokes initial value problem*, Nonlinear analysis and applications: to V. Lakshmikantham on his 80th birthday, 1 (2003), 549-562.

[18] M. Hieber, L. Lorenzi and A. Rhandi, *Uniform and L^p estimates for parabolic equations with unbounded coefficients in exterior domains*, Preprint 2006.

[19] M. Hieber, A. Rhandi and O. Sawada, *The Navier-Stokes flow for globally Lipschitz continuous initial data*, submitted.

[20] M. Hieber and O. Sawada, *The Navier-Stokes equations in $\mathbb{R}^n$ with Linearly growing initial data*, Arch. Ration. Mech. Anal., **175** (2005), 269-285.

[21] T. Hishida, *An existence theorem for the Navier-Stokes flow in the exterior of a rotating obstacle*, Arch. Ration. Mech. Anal., **150** (1999), 307-348.

[22] T. Hishida, *The Stokes operator with rotation effect in exterior domains*, Analysis, **19** (1999), 51-67.

[23] T. Hishida, *L^2 theory for the operator $\Delta + (k \times x) \cdot \nabla$ in exterior domains*, Nihonkai Math J., **11** (2000), 103-135.

[24] T. Hishida and Y. Shibata, *$L^p - L^q$-estimates of the Stokes operator and Navier-Stokes flows in the exterior of a rotating obstacle*, preprint 2006.

[25] C. Kahane, *On the spatial analyticity of solutions of the Navier-Stokes equations*, Arch. Ration. Mech. Anal., **33** (1969), 386-405.

[26] T. Kato, *Strong L^p-solutions of Navier-Stokes equations in $\mathbb{R}^n$ with applications to weak solutions*, Math. Z., **187** (1984), 471-480.

[27] N. Kim and D. Chae, *On the uniqueness of the unbounded classical solutions of the Navier-Stokes and associated equations*, J. Math. Anal. Appl., **186** (1994), 91-96.

[28] H. Koch and D. Tataru, *Well-posedness for the Navier-Stokes equations*, Adv. Math., **157**, 22-35 (2001).

[29] H. Kozono and M. Yamazaki, *Semilenear heat equations and the Navier-Stokes equation with disributions in new function spaces as initial data*, Comm. Partial Differential Equations, **19** (1994), 959-1014.

[30] A. Lunardi and G. Metafune, *On the domains of elliptic operators in L^1*, Differential Integral Equations, **17** (2004), 73-97.

[31] A. Majda, *Vorticity and the mathematical theory of incompressible fluid flow*, Comm. Pure Appl. Math., **34** (1986), 187-220.

[32] K. Masuda, *Weak solutions of Navier-Stokes equations*, Tohoku Math. J., **36** (1984), 623-646.

[33] G. Metafune, D. Pallara, E. Priola, *Spectrum of Ornstein-Uhlenbeck operators in L^p spaces with respect to invariant measures*, J. Funct. Anal., **196** (2002), 40-60.

[34] G. Metafune, J. Prüss, A. Rhandi, R. Schnaubelt, *The domain of the Ornstein-Uhlenbeck operator on an L^p-space with invariant measure*, Ann. Sc. Norm. Super. Pisa Cl. Sci., **1** (2002), 471-485.

[35] H. Okamoto, *A uniqueness theorem for the unbounded classical solution of the nonstationary Navier-Stokes equations in $\mathbb{R}^3$*, J. Math. Anal. Appl., **181** (1994), 191-218.

[36] H. Okamoto, *Exact solutions of the Navier-Stokes equations via Leray's scheme*, Japan J. Indust. Appl. Math., **14** (1997), 169-197.

[37] O. Sawada, *On time-local solvability of the Navier-Stokes equations in Besov spaces*, Adv. Differential Equations, **8** (2003), 385-412.

[38] O. Sawada, *The Navier-Stokes flow with linearly growing initial velocity in the whole space*, Bol. Soc. Paran. Mat., **22** (2004), 75-96.

[39] O. Sawada, *On analyticity rate of the solutions to the Navier-Stokes equations in Bessel-potential spaces*, J. Math. Anal. Appl., **312** (2005), 1-13.

[40] O. Sawada, *On the spatial analyticity of solutions to the Keller-Segel equations*, (submitted).

[41] O. Sawada and Y. Taniuchi, *On the Boussinesq flow with nondecaying initial data*, Funkcial. Ekvac., **47** (2004), 225-250.

[42] O. Sawada and T. Usui, *Uniqueness of the solutions to the nonstationary Navier-Stokes equations with linearly growing initial data*, (preprint).

[43] G. Seregin and V. Šverák, *The Navier-Stokes equations and backwards uniqueness*, Nonlinear problems in mathematical physiscs and related topics, Kluwer, New York, 2002.

[44] H. Triebel, *Theory of Function Spaces*, Birkhäuser, Basel-Boston-Stuttgart, (1983).

[45] H. Triebel, *Theory of Function Spaces II*, Birkhäuser, Basel-Boston-Stuttgart, (1992).

Department of Mathematical Science, School of Science and Engineering, Waseda University, Okubo 3-4-1, 169-8555 Shinjuku, Japan
E-mail address: `sawada@gm.math.waseda.ac.jp`