数理解析研究所講究録
1529 巻 2007 年 42-63

Optimal Control Problem Associated with Jump-Diffusion Processes and Optimal Stopping

Yasushi Ishikawa, Dept. Mathematics, Ehime University

石川保志 (愛媛大学理)

1 Introduction

In this note we study optimal consumption problem and optimal stopping problem both associated with (1-dimensional) jump-diffusion. Methods employed are stochastic calculus of jump type, Hamilton-Jacobi inequality, Bellman principle, the notion of viscosity solution and some classical calculus associated with positive maxmal principle.

In part I a topic in optimal consumption problem will be presented, and in part II an optimal stopping problem associated with jump-diffusion process. Maretials in Part I is based on [14] and those in part II is based on [15]. Many interpretations have been added.

The process appearing in Part I is 2-dimensional, whereas that appears in Part II is 1-dimensional. However, formulation of the problem and proofs proceed in a similar way. We shall describe mainly for Part II.

2 Part I - Optimal consumption

Let $\tilde{N}(dtdz)=N(dtdz)-\mu(dz)dt$ be a compensated Poisson random measure on $[0,T]\times\mathbf{R}$, whose mean measure (Lévy measure) satisfies $\int_{\mathbf{R}\setminus\{0\}}\min(z^2,1)\mu(dz)<+\infty$. We admit μ to be a fairely discrete measure satisfying this condition, i.e., sum of point masses on $\mathbf{R}$.

Let Z_t be a Lévy process given by

(1) $$Z_t=rt+\int_0^t\int_{|z|<1}z\tilde{N}(dsdz)+\int_0^t\int_{|z|\geq 1}zN(dsdz).$$

Here we do not admit Gaussian part, and trajectories are chosen from the right continuous version. We put $S_t=S_0e^{Z_t}$ with $S_0>0$ being a constant. The process (S_t) is called a *geometric Lévy process.*

Then S_t satisfies, by Itô formula, the SDE

$$dS_t=rS_tdt+S_t\int_{|z|<1}(e^z-1-z)\mu(dz)dt$$

$$(2) \qquad +S_{t-}\Big(\int_{|z|<1}(e^z-1)\tilde{N}(dtdz)+\int_{|z|\geq 1}(e^z-1)N(dtdz)\Big).$$

We assume

$$(3) \qquad \int_{|z|\geq 1}(e^z-1)\mu(dz)<\infty.$$

Then (2) can be rewritten as

$$dS_t = rS_t dt + S_t\int_{\mathbf{R}\setminus\{0\}}(e^z-1-z1_{\{|z|<1\}})\mu(dz)dt + S_{t-}\int_{\mathbf{R}\setminus\{0\}}(e^z-1)\tilde{N}(dtdz).$$

We put

$$\tilde{r} = r + \int_{\mathbf{R}\setminus\{0\}}(e^z-1-z1_{\{|z|<1\}})\mu(dz),$$

which is finite due to (3). Then

$$dS_t = \tilde{r}S_t dt + S_{t-}\int_{\mathbf{R}\setminus\{0\}}(e^z-1)\tilde{N}(dtdz).$$

Let $\mathcal{S}$ be

$$\mathcal{S} = \{(x,y); y>0, y+\beta x>0\}.$$

Here $\beta>0$ is a weight factor which describes the dumping rate of the average past consumption (e.g., buying durable goods).

Based on the driving processes $(Z_t), (S_t)$, we shall construct the processes $X = X_t^x, Y = Y_t^y$ depending on the parameter process (π_t, C_t, L_t) by

$$(4) \quad X_t = x - C_t + \int_0^t (r_0+(\tilde{r}-r_0)\pi_s)X_s ds + L_t + \int_0^t \pi_{s-}X_{s-}\int_{\mathbf{R}\setminus\{0\}}(e^z-1)\tilde{N}(dsdz), X_0 = x,$$

$$Y_t = ye^{-\beta t} + \beta\int_0^t e^{-\beta(t-s)}dC_s, Y_0 = y.$$

The background of defining X_t is the self-financing investment policy accordthe portfolio π_t :

$$\frac{dX_t}{X_{t-}} = (1-\pi_t)\frac{dB_t}{B_t} + \pi_t\frac{dS_t}{S_{t-}},$$

where B_t denotes the riskless bond given by $dB_t = r_0 B_t dt$. The second equation in (4) means $dY_t = -\beta Y_t dt + \beta dC_t$.

Here (π_t, C_t, L_t) denotes a control which satisfies the following conditions:

(i) $C_t = \int_0^t c_s ds$, and $t \mapsto c_t$ is a non-decreasing adapted cádlàg process of finite variation such that $0 \leq c_t \leq M_1$ for all $t \geq 0$ for some $M_1 > 0$, and that $c_t > 0$ only for such t that $X_t \geq 0$.

(ii) L_t is a non-decreasing adapted càdlàg process such that $L_{0-} = 0, L_t \geq 0$ a.s., $E[L_t] < \infty$ for all $t \geq 0$, $\Delta L_t > 0$ only for such t that $X_{t-} \in \mathcal{S}$ and $X_{t-} + \Delta X_t \notin \mathcal{S}$, and $L_t^c > 0$ only for such t that $X_t \leq 0$. Here L_t^c denotes the continuous part of L_t.

(iii) π_t is an adapted càdlàg process with values in $[0, 1]$.

(iv) π_t, C_t, L_t are processes such that

$$(*) \qquad \text{if } (x,y) \in \mathcal{S} \quad \text{then} \quad (X_t, Y_t) \in \bar{\mathcal{S}} \ a.s.$$

holds for $t \geq 0$.

Those controls (π_t, c_t, L_t) which satisfy **(i)** - **(iv)** will be called *admissible*, and the set of admissible controls for (X_t, Y_t) starting from (x, y) will be denoted by $\mathcal{A}_{(x,y)}$ which may often be written simply be $\mathcal{A}$.

Viewing $(\pi_., c_., L_.)$ as a fixed parameter, we put $v^{(\pi_.,c_.,L_.)}$ by

$$v^{(\pi_.,c_.,L_.)}(t; x, y) = E^{(X_{t\wedge.}^{(\pi_.,c_.,L_.)}, Y_{t\wedge.}^{(\pi_.,c_.,L_.)})}[\int_0^t e^{-\alpha s} U(c_s) ds],$$

where $X_t^{(\pi_.,c_.,L_.)}, Y_t^{(\pi_.,c_.,L_.)}$ are processes X_t, Y_t given $(\pi_., c_., L_.)$. Also we put the value functions

$$(5) \qquad v(t; x, y) = \sup_{(\pi,c,L)\in\mathcal{A}} E^{(X_{t\wedge.}^{(\pi_.,c_.,L_.)}, Y_{t\wedge.}^{(\pi_.,c_.,L_.)})}[\int_0^t e^{-\alpha s} U(c_s) ds]$$

$$(6) \qquad v(x, y) = \sup_{(\pi,c,L)\in\mathcal{A}} E^{(X^{(\pi_.,c_.,L_.)}, Y^{(\pi_.,c_.,L_.)})}[\int_0^\infty e^{-\alpha s} U(c_s) ds],$$

where $\alpha > 0$ is the dumping rate of the utility, and the supremum is taken over admissible controls $(\pi_., c_., L_.)$, and the expectation is taken with respect to the law of (X_t, Y_t) due to $N(dtdz)$.

It is more realistic to consider the case

$$\mathcal{S} = \{(x, y); y > 0, y + \beta x > 0, x^2 + y^2 < R\}$$

for some $R > 0$. However, if we consider the case that small jumps are dominant, it is expected that it takes long time before the process (X_t, Y_t) crosses the boundary of $\mathcal{S}$ at the magnitute R. Then due to the time dumping factor $e^{-\alpha s}$ in $v(x, y)$, the effect of (X_t, Y_t) near the boundary decrease to small.

Our goal is to characterize v as a viscosity solution to the HJB equation stated below.

The Hamilton-Jacobi equation (HJB equation) associated with (X_t, Y_t) is given by as follows.

(7) $$\max\{Nv, \sup_{\pi,c}\{Av\}, Mv\} = 0 \quad \text{in} \quad \mathcal{S}.$$

$$v = 0 \qquad \text{outside of} \quad \mathcal{S}.$$

Here

(8) $$Av(x,y) = -\alpha v - \beta y v_y$$
$$+\{(r+\pi(\hat{b}-r))xv_x + \int (v(x+\pi x(e^z-1),y) - v(x,y) - \pi x v_x(e^z-1))\mu(dz)\}$$
$$+U(c) - c(v_x - \beta v_y), \quad \pi \in [0,1], c \in [0, M_1],$$

and

$$Nv = v_x \cdot 1_{\{x\le 0\}}$$

(9) $$Mv = (\beta v_y - v_x)\cdot 1_{\{x\ge 0\}}.$$

The principal part $A_0 = \{\cdots\}$ of A is an operator which satisfies the *positive maximum principle* :

$$\text{if } u(x_0,y_0) = \sup_{(x,y)\in\mathcal{S}} u(x,y) \ge 0, \text{then } Au(x_0,y_0) \le 0.$$

Hence $A|_{C_0^\infty}$ becomes a pseudo-differential operator having certain symbol $a(x,y;\xi,\eta)$ which is negative definite (cf. [7], [17]).

In general, if

$$Lf(x) = b(x,\pi)f_x(x) + \int\{f(x+\gamma(x,u,z)) - f(x) - \gamma(x,u,z)f_x(x)\}\mu(dz),$$

where $\gamma(x,u,z) = xu(e^z-1)$ and $u=\pi$, denotes the infinitesimal generator of the process X_t satisfying the positive maximal principle, and if

$$J^x(s,u) = E[\int_0^T e^{-\alpha(s+t)}h(t,X_t,u_t)dt + g(X_T)]$$

denotes the performance criterion for a control u with respect to some function h, we can say the following.

We assume there exists $u^* \in \mathcal{A}$ such that $J(s,u^*) = \sup_{u\in\mathcal{A}} J^x(s,u)$. Then we write $\Phi(s,x) = J(s,u^*)$. Viewing L above as a Lagrangean, we shall perform a canonical transformation from L to the Hamiltonian H.

$$H(t,x,u,p,r) = h(t,x,u) + b(x,u)p + \int \gamma(x,u,z)r(t,z)\mu(dz).$$

We consider the following Hamilton-Jacobi (stochastic) equation

$$dp(t) = -\frac{\partial}{\partial x} H(t, X_t, u_t, p(t), r(t,\cdot))dt + \int r(t,z)\tilde{N}(dtdz), \quad t < T$$

$$p(T) = \frac{\partial}{\partial x} g(X_T).$$

It is shown

Theorem ([12]) *Assume* $\Phi(s,x) \in C^{1,3}(\mathbf{R}_+ \times \mathbf{R})$. *Define*

$$p(t) = \frac{\partial \Phi}{\partial x}(t, X_t^*),$$

$$r(t,z) = \frac{\partial \Phi}{\partial x}(t, X_t^* + \gamma(X_t^*, u_t^*, z)) - \frac{\partial \Phi}{\partial x}(t, X_t^*).$$

Here X_t^* *denotes* $X_t^{u^*}$ *the process associated with* u^*.

Then $p(t), r(t,z)$ *solve the Hamilton-Jacobi equation.*

This verifies the validity of the method.

We next introduce the notion of viscosity solutions.

We write

$$B^\pi((x,y),v) = \int (v(x + \pi x(e^z - 1), y) - v(x,y) - \pi x v_x(e^z - 1))\mu(dz),$$

and for $\delta > 0, p \in \mathbf{R}$,

$$B^{\pi,\delta}((x,y),\phi,p) = \int_{|z|>\delta} (\phi(x + \pi x(e^z - 1), y) - \phi(x,y) - \pi x p(e^z - 1))\mu(dz),$$

$$B_\delta^\pi((x,y),\phi,p) = \int_{|z|\le\delta} (\phi(x + \pi x(e^z - 1), y) - \phi(x,y) - \pi x p(e^z - 1))\mu(dz);$$

so that

$$B^\pi((x,y),v) = B^{\pi,\delta}((x,y),v,v_x) + B_\delta^\pi((x,y),v,v_x), \quad \delta > 0.$$

Further we use the notation $F = F^{\delta,c}$ given by

(10)
$$F((x,y),w,s,t;\phi,p,\psi,q) = -\alpha w - \beta y t + \max_{0\le\pi\le1}\{(r + \pi(\hat{b} - r))xs$$
$$+ B^{\pi,\delta}((x,y),\phi,p) + B_\delta^\pi((x,y),\psi,q)\} + U(c) - c(s - \beta t)$$

when it is necessary. Here s, t, p, q are scalars. We note that

$$Av(x,y) = F((x,y), v, v_x, v_y; v, v_x, v, v_x).$$

To introduce the notion of the viscosity solutions, we put

$$C_l(\bar{S}) = \{\phi \in C(\bar{S});\ \sup_{(x,y)\in\bar{S}} |\frac{\phi(x,y)}{(1+|x|+|y|)^l}| < \infty\} \tag{1.11}$$

for $l \geq 0$. This is a space of functions having the constraint on the asymptotic order at infinity.

Definition 2.1 (cf. [3], [4])

Let $E \subset \bar{S}$. (1) Any $v \in C(\bar{S})$ is a viscosity subsolution (resp. supersolution) of (7) in E iff for all $(x,y) \in E$ all $\delta > 0$ and all $\phi \in C^2(\bar{S}) \cap C_1(\bar{S})$ such that (x,y) is a global maximizer (resp. minimizer) of $v - \phi$ relative to E, it holds that

$$\max(N\phi, \sup_c(F(.,v,\phi_x,\phi_y;\phi,\phi_x,\phi,\phi_x)), M\phi)(x,y) \geq 0. \tag{11}$$

$$(resp.\ \ \max(N\phi, \sup_c(F(.,v,\phi_x,\phi_y;\phi,\phi_x,\phi,\phi_x)), M\phi)(x,y) \leq 0.)$$

(2) $v \in C(\bar{S})$ is a constrained viscosity solution of (7) iff v is a viscosity subsolution of (7) in $\bar{S}$ and a supersolution of (7) in S.

We have now our first main result.

Theorem 2.2 *The value function $v(x,y)$ is well defined, and it is a constrained viscosity solution of (7).*

Lemma 2.3 (Bellman Principle) *For any stopping time τ and any $t \geq 0$,*

$$v(x,y) = \sup_{(\pi,c,L)\in\mathcal{A}} E[\int_0^{\tau\wedge t} e^{-\alpha s}U(c_s)ds + e^{-\alpha(\tau\wedge t)}v(X^x_{\tau\wedge t}, Y^y_{\tau\wedge t})],\ (x,y) \in S \tag{12}$$

where $(\pi_., c_., L_.)$ is taken over admissible controls.

The Bellman principle plays a role to show the semigroup property concerning the value function, which helps to verify the Theorem 2.2 above. Here we need this principle since we take supremum with respect to the control triplet $(\pi_., c_., L_.)$. In the case of optimal stopping problem, we have a similar statement for the value function. In this case, however, the strong Markov property of the basic process will suffice. See Theorem 4.4 in Part II.

With respect to the uniqueness of the viscosity solution, we have the following theorem.

Theorem 2.4 *For each $\gamma > 0$ choose $\alpha > 0$ so that $\alpha > k(\gamma)$. Assume $v_0 \in C_\gamma(\bar{\mathcal{S}})$ is a subsolution of (7) in $\bar{\mathcal{S}}$ and $\bar{v} \in C_{\bar{\gamma}}(\bar{\mathcal{S}})$ is a supersolution of (7) in $\mathcal{S}$. Then*

$$v_0 \leq \bar{v} \quad on \quad \bar{\mathcal{S}}.$$

Here $k(\gamma) = \max_\pi[\gamma(r + \pi(\hat{b} - r)) + \int_{\mathbf{R}\setminus\{0\}}((1 + \pi(e^z - 1))^\gamma - 1 - \gamma\pi(e^z - 1))\nu(dz)]$.

Consequently, the HJB equation admits at most one constrained viscosity solution in $C_{\bar{\gamma}}(\bar{\mathcal{S}})$.

This implies that the solution must coincide with the value function, since it is bounded and hence belongs to $C_{\bar{\gamma}}(\bar{\mathcal{S}})$ for all $\bar{\gamma} > 0$.

3 Part II - Optimal stopping

Consider the optimal stopping problem for the stock price in mathematical finance. Define the following quantities:

$$\begin{aligned}
X(t) &= \text{the stock price at time } t \\
r &= \text{expected return of the stock, } r > 0, \\
B(t) &= \text{1-dimensional standard Brownian motion} \\
Z(t) &= \text{1-dimensional Lévy process} \\
\sigma &= \text{the positive diffusion constant} \\
\tau &= \text{exercise time or stopping time} \\
g(x) &= \text{the reward function of the stock} \\
\mathcal{S} &= \text{the set of stopping times} \\
\mathcal{S}_b &= \text{the set of bounded stopping times.}
\end{aligned}$$

Here the Lévy process $Z(t)$ is given as in Part I.

We assume that the stock price $X = \{X(t)\}$ evolves according to the stochastic differential equation of jump-diffusion type

$$dX(t) = (r + \int_{|z|<1}(e^z - 1 - z)\mu(dz))X(t)dt + \sigma X(t)dB(t)$$

$$(1) \quad +X(t-)\int_{|z|<1}(e^z - 1)\tilde{N}(dtdz) + X(t-)\int_{|z|\geq1}(e^z - 1)N(dsdz), \quad X(0) = x > 0,$$

on a complete probability space $(\Omega, \mathcal{F}, P)$, carrying a standard Brownian motion $\{B(t)\}$ and a Poisson random measure $N(dtdz)$, endowed with the natural filtration $\mathcal{F}_t$ generated by $\sigma(B(s), s \leq t)$ and $\sigma(N(dsdz), s \leq t)$.

We assume μ is non-degenerate, and that

$$\int_{|z|\geq 1} (e^z - 1)\mu(dz) < +\infty$$

and put

$$\tilde{r} = r + \int (e^z - 1 - z \cdot 1_{|z|<1})\mu(dz)$$

as in Part I. Then $X(t)$ can be written

$$(1)' \qquad dX(t) = \tilde{r}X(t)dt + \sigma X(t)dB(t) + X(t-)\int (e^z - 1)\tilde{N}(dtdz), \quad X(0) = x > 0.$$

We assume here

$$\mu \ \text{ is symmetric.}$$

This together with the above imply that $\int_{\mathbf{R}\setminus\{0\}} (e^z - 1)\mu(dz) > 0$.

The reward function $g(x)$ is assumed to have the following property:

$$(2) \qquad g \geq 0, \quad g \in \mathcal{C},$$

where $\mathcal{C}$ denotes the Banach space $C_0([0,\infty))$ of all continuous functions on $[0,\infty)$ vanishing at infinity, with norm $\|h\| = \sup_{x\geq 0} |h(x)|$.

The objective is to find an optimal stopping time τ^* so as to maximize the expected reward function:

$$(3) \qquad J(\tau) = E[e^{-\tilde{r}\tau} g(X(\tau))]$$

over the class $\mathcal{S}$ of all stopping times τ, where $e^{-\tilde{r}\tau} g(X_\tau)$ at $\tau = \infty$ is interpreted as zero. Instead of HJB equations, we consider the variational inequality:

$$(4) \qquad \begin{cases} \max(Lv, g - v) = 0 \ \text{ in } \ (0,\infty), \\ v(0) = g(0). \end{cases}$$

Here

$$Lv = -\tilde{r}v + \frac{1}{2}\sigma^2 x^2 v'' + rxv' + \int \{v(x + \gamma(x,z)) - v(x) - v'(x) \cdot \gamma(x,z)\}\mu(dz)$$

where $\gamma(x,z) = x(e^z - 1)$. We write $Lv = -\tilde{r}v + L_0 v$ in the sequel.

Since L satisfies the positive maximum principle, L can be viewed as a pseudo-differential operator with the symbol $a(x,\xi)$ given by

$$a(x,\xi) = a_1(x,\xi) + a_2(x,\xi),$$

where

$$a_1(x,\xi) = -\tilde{r} - \frac{1}{2}\sigma^2 x^2 \xi^2 + irx\xi$$

$$a_2(x,\xi) = \int \{e^{i\xi\gamma(x,z)} - 1 - i\xi \cdot \gamma(x,z)\}\mu(dz).$$

The symbol of L_0 is given by $(a_1(x,\xi)+\tilde{r}) + a_2(x,\xi)$.

By the assumption that $\sigma > 0$, the symbol a_1 is elliptic. On the other hand, the symbol a_2 satisfies

$$a_2(x,\xi) \sim c(x)|\xi|^{\alpha(x)} \text{ for each } x \in (0,\infty).$$

Here $\alpha(x)$ is a measurable function taking values in $(0,2)$. Due to the initial assumption that $\sigma > 0$ we may assume the ssymbol a is elliptic.

To solve (4), we need to study the penality equation for $\varepsilon > 0$:

$$\begin{cases} \tilde{r}u = L_0 u + \frac{1}{\varepsilon}(u-g)^- \text{ in } (0,\infty), \\ u(0) = g(0), \end{cases} \tag{5}$$

origined by Bensoussan and Lions.

Remark 3.1 The condition (2) is fulfilled if the reward function is given by the bounded function

$$g(x) = (K-x)^+$$

for the strike price $K > 0$ of a put option.

Suppose that the variational inequality (4) admits a solution $v \in C^2((0,\infty))$. Then the optimal stopping time $\hat{\tau}$ is given by

$$\hat{\tau} = \inf\{t : v(X(t)) \le g(X(t))\}.$$

From (4) it follows that

$$Lv = 0 \text{ if } v > g.$$

Hence

$$Lv(X(t)) = 0 \text{ for } t < \hat{\tau}.$$

By Itô formula, under some additional assumptions on v, we obtain

$$\begin{aligned} E[e^{-\tilde{r}\hat{\tau}}v(X(\hat{\tau}))] &= v(x) + E[\int_0^{\hat{\tau}} e^{-\tilde{r}t}Lv(X(t))dt] + E[\int_0^{\hat{\tau}} e^{-\tilde{r}t}v'(X(t))\sigma X(t)dB(t) \\ &+ \int_0^{\hat{\tau}}\int e^{-\tilde{r}t}(v(X(t)+\gamma(X(t),z)) - v(X(t)))\tilde{N}(dsdz)] \\ &= v(x). \end{aligned}$$

Thus

$$E[e^{-\tilde{r}\hat{\tau}}g(X(\hat{\tau}))] \ge E[e^{-\tilde{r}\hat{\tau}}v(X(\hat{\tau}))] = v(x).$$

On the other hand, since

$$Lv \le 0,$$

Itô formula gives

$$E[e^{-\bar{r}\tau}v(X(\tau))] \le v(x), \quad \tau \in \mathcal{S}.$$

We assume v is bounded, as in Remark above, and let $\tau \to \hat{\tau}$. Therefore we seem to obtain the optimality of $\hat{\tau}$, and we have $\Phi(x) = v(x)$, where $\Phi(x) = \sup_\tau J^x(\tau)$. However, we remark that $v \in C^2$ may be violated, because v is connected to g at some point x which is only continuous.

4 Penalized Problem

In this section, we show the existence of a unique solution u of the penalty equation (5).

We begin with a probabilistic penalty equation

$$u(x) = E[\int_0^\infty e^{-(\bar{r}+\frac{1}{\varepsilon})t}\frac{1}{\varepsilon}(u \vee g)(X(t))dt], \tag{6}$$

for $x \ge 0$.

Theorem 4.1 We assume (2). Then, for each $\epsilon > 0$, there exists a unique nonnegative solution $u = u_\epsilon \in \mathcal{C}$ of (6).

Proof. Define

$$\mathcal{T}h(x) = E[\int_0^\infty e^{-(\bar{r}+\frac{1}{\varepsilon})t}\frac{1}{\varepsilon}(h \vee g)(X(t))dt] \quad \text{for} \quad h \in \mathcal{C}_+, \tag{7}$$

where $\mathcal{C}_+ = \{h \in \mathcal{C} : h \ge 0\}$. Clearly, $\mathcal{C}_+$ is a closed subset of $\mathcal{C}$. By (7), we have

$$\begin{aligned} 0 \le \mathcal{T}h(x) &= E[\int_0^\infty e^{-(\bar{r}+\frac{1}{\varepsilon})t}\frac{1}{\varepsilon}(h \vee g)(X(t))dt] \\ &\le \|h \vee g\| \int_0^\infty e^{-(\bar{r}+\frac{1}{\varepsilon})t}\frac{1}{\varepsilon}dt \\ &= \frac{\|h \vee g\|}{\bar{r}\varepsilon + 1} \le \|h \vee g\|. \end{aligned}$$

Then, by the Gronwall inequality

$$\begin{aligned} |\mathcal{T}h(y) - \mathcal{T}h(x)| &\le E[\int_0^\infty e^{-(\bar{r}+\frac{1}{\varepsilon})t}\frac{1}{\varepsilon}\{|h(X(t)) - h(Y(t))|\}dt] \\ &\to 0 \quad \text{as} \quad y \to x, \end{aligned}$$

where $\{Y(t)\}$ be the solution of (1) with the initial condition $Y(0) = y > 0$.

Indeed, since

$$X(t)-Y(t)=(x-y)+\int_0^t r(X(s)-Y(s))ds$$

$$+\sigma\int_0^t (X(s)-Y(s))dB(s)+(X(t)-Y(t))\int_0^t\int(e^z-1)\tilde{N}(dsdz),$$

we have

$$E[\sup_{u\le t}|X(u)-Y(u)|^2]\le |x-y|^2+C\int_0^t E[\sup_{u\le s}|X(u)-Y(u)|^2]ds.$$

Hence we have the conclusion by the Gronwall inequality.

Moreover,

$$\mathcal{T}h(x)=E[\int_0^\infty e^{-(\bar{r}+\frac{1}{\varepsilon})t}\frac{1}{\varepsilon}(h\vee g)(X(t))dt]\to 0 \quad \text{as} \quad x\to\infty,$$

since $(P_h)_{h\in\mathcal{C}_+}, P_h=P^{h\circ X}$, is tight in the space $D=D([0,t]), t>0$.

Indeed, let f be any element in C^2 having bounded derivatives. Since $P_h(A)=P^{h\circ X}(A)=P^X(h^{-1}(A))$,

$$f(h(X(t))-f(h(X(0))-\int_0^t\{\frac{\partial}{\partial x}f(h(X(s)))+[(r+\int_{|z|<1}(e^z-1-z)\mu(dz))h(X(s-))]$$

$$+\frac{1}{2}\sigma^2h^2(X(s))\frac{\partial^2}{\partial x^2}f(h(X(s-)))$$

$$+\int\{f(h(X(s-))+h(X(s-))(e^z-1))-f(h(X(s-))-\frac{\partial}{\partial x}f(h(X(s-)))h(X(s-))(e^z-1)\}\mu(dz)\}ds$$

is a P_h-martingale.

Hence, since h is bounded,

$$|\int_0^t\{\cdots\}ds|\le C\int_0^t ds\|f''\|\int(e^z-1)\mu(dz)\le c_f t$$

for some constant c_f. Hence by Proposition 3.2 of [1], (P_h) is tight in $D([0,t]), t>0$. Thus $\mathcal{T}$ maps $\mathcal{C}_+$ into $\mathcal{C}_+$.

Now, by (7), we have

$$\begin{aligned}|\mathcal{T}h_1(x)-\mathcal{T}h_2(x)| &\le E[\int_0^\infty e^{-(\bar{r}+\frac{1}{\varepsilon})t}\frac{1}{\varepsilon}|h_1(X(t))-h_2(X(t))|dt]\\ &\le E[\int_0^\infty e^{-(\bar{r}+\frac{1}{\varepsilon})t}\frac{1}{\varepsilon}\|h_1-h_2\|dt]\\ &= \frac{1}{\bar{r}\varepsilon+1}\|h_1-h_2\|, \quad h_1,h_2\in\mathcal{C}_+.\end{aligned}$$

This yields that $\mathcal{T}$ is a contraction mapping. Thus $\mathcal{T}$ has a fixed point u, which solves (6). The proof is finished.

Consider the penalty equation for $u = u_\epsilon$:

$$\tilde{r}u = L_0 u + \frac{1}{\varepsilon}(u-g)^- \quad \text{in} \quad (0,\infty), \tag{8}$$

with boudary condition $u(0) = g(0)$. Since

$$u \vee g = u + (u-g)^-,$$

we rewrite (8) as

$$(\tilde{r} + \frac{1}{\varepsilon})u = L_0 u + \frac{1}{\varepsilon}(u \vee g) \quad \text{in} \quad (0,\infty). \tag{9}$$

We introduce here a notion of weak solution.

Definition 4.2 Let $w \in C([0,\infty))$ and $w(0) = g(0)$. Then w is called a viscosity sub- or super- solution of (8) as follows;

(a) w is a viscosity subsolution of (8), that is, for any $\phi \in C^2((0,\infty))$ and any local maximum point $z > 0$ of $w - \phi$,

$$\tilde{r}w(z) \le L_0\phi(z) + \frac{1}{\varepsilon}(w-g)^-(z),$$

and

(b) w is a viscosity supersolution of (8), that is, for any $\phi \in C^2((0,\infty))$ and any local minimum point $\bar{z} > 0$ of $w - \phi$,

$$\tilde{r}w(\bar{z}) \ge L_0\phi(\bar{z}) + \frac{1}{\varepsilon}(w-g)^-(\bar{z}).$$

Theorem 4.3 We make the assumption of Theorem 4.1. Then u in (6) is a viscosity solution of (8).

Proof. We see that $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t^X\}; X)$ is a strong Markov process, that is,

$$P_x(X(t+\tau) \in A | \mathcal{F}_\tau^X) = P_{X_\tau}(X(t) \in A), \quad P_x\text{-}a.s., \quad t \ge 0,$$

for any Borel set A of $\mathbf{R}$ and $\tau \in \mathcal{S}_b$, where P_x denotes the probability measure P with $X(0) = x$.

Let $x > 0$. By (6), we get

$$u(x) = E[\int_0^\infty e^{-(\tilde{r}+\frac{1}{\varepsilon})t}\frac{1}{\varepsilon}(u \vee g)(X(t))dt].$$

Hence

$$E[\int_\tau^\infty e^{-(\tilde r+\frac{1}{\varepsilon})t}\frac{1}{\varepsilon}(u\vee g)(X(t))dt|\mathcal{F}_\tau^X] = e^{-(\tilde r+\frac{1}{\varepsilon})\tau}E[\int_0^\infty e^{-(\tilde r+\frac{1}{\varepsilon})t}\frac{1}{\varepsilon}(u\vee g)(X(t+\tau))dt|\mathcal{F}_\tau^X]$$
$$= e^{-(\tilde r+\frac{1}{\varepsilon})\tau}u(X(\tau)),\quad a.s.$$

Thus for each $\theta > 0$

$$u(x) = E[\int_0^{\tau\wedge\theta} e^{-(\tilde r+\frac{1}{\varepsilon})t}\frac{1}{\varepsilon}(u\vee g)(X(t))dt + e^{-(\tilde r+\frac{1}{\varepsilon})\tau\wedge\theta}u(X(\tau\wedge\theta))].$$

This relation corresponds to the *dynamic programming principle (Bellman principle)* for u. By the same line as the proof of Theorem 1 in [14], we deduce that u is a viscosity solution to (9), and also to (8).

We study the smoothness of the solution u to (8). We fix $\varepsilon > 0$ temporarily.

Theorem 4.4 We make the assumption of Theorem 4.1. Then there exists a solution u of (8) which coincides with u in (6) in $C((0,\infty))$. The solution is unique in C_+. Further, for any $\tau \in \mathcal{S}$, we have

$$u(x) = E[\int_0^\tau e^{-\tilde r t}\frac{1}{\varepsilon}(u-g)^-(X(t))dt + e^{-\tilde r\tau}u(X(\tau))]. \tag{10}$$

In particular,

$$u(x) = E[\int_0^\infty e^{-\tilde r t}\frac{1}{\varepsilon}(u-g)^-(X(t))dt]. \tag{11}$$

Proof.

1. Let $[a,b] \subset (0,\infty)$ be an arbitrary finite interval and we consider the boundary value problem:

$$\begin{aligned} \tilde r\chi(x) &= L_0\chi(x) + \frac{1}{\varepsilon}(u-g)^- \text{ in } (a,b),\\ \chi(a) &= u(a),\quad \chi(b) = u(b). \end{aligned} \tag{12}$$

By the uniform ellipticity and linearlity, Theorem 2.5.4 in [17] yields that (12) has a smooth solution χ. In view of Theorem 4.3 above and Theorem 2 in [14], we can obtain the uniqueness of the viscosity solution of (12). Therefore we deduce that $u = \chi \in C((a,b))$, and hence $u \in C((0,\infty))$.

2. We set

$$\tau_R = \inf\{t \ge 0 : X(t) > R \text{ or } X(t) < 1/R\} \tag{13}$$

for $R > 1$ and $\varrho = \tau\wedge\tau_R$. By Itô formula and (8), we get, if $\frac{1}{R} < x < R$,

$$e^{-\tilde r(\varrho\wedge n)}u(X(\varrho\wedge n)) = u(x) + \int_0^{\varrho\wedge n} e^{-\tilde r t}(-\tilde r u(X(t)) + L_0u(X(t))dt$$

$$
\begin{aligned}
&+ \int_0^{\varrho\wedge n} e^{-\bar{r}t} u'(X(t))\sigma X(t)dB(t) \\
&+ \int_0^{\varrho\wedge n} e^{-\bar{r}t} X(t) \int \{u(X(t-)+\gamma(X(t-),z)) - u(X(t-)) \\
&- u'(X(t-))\cdot\gamma(X(t-),z)\}\tilde{N}(dtdz) \\
&= u(x) - \int_0^{\varrho\wedge n} e^{-\bar{r}t}\frac{1}{\varepsilon}(u-g)^-(X(t))dt \\
&+ \int_0^{\varrho\wedge n} e^{-\bar{r}t} u'(X(t))\sigma X(t)dB(t) \\
&+ \int_0^{\varrho\wedge n} e^{-\bar{r}t} X(t) \int \{u(X(t-)+\gamma(X(t-),z)) \\
&- u(X(t-)) - u'(X(t-))\cdot\gamma(X(t-),z)\}\tilde{N}(dtdz), \quad a.s., \quad \forall n\in \mathbf{N}.
\end{aligned}
$$

Since u' is bounded on $[1/R, R]$, we see that

$$
E[\int_0^{\varrho\wedge n} e^{-\bar{r}t}u'(X(t))\sigma X(t)dB(t)] = E[\int_0^n e^{-\bar{r}t}u'(X(t))\sigma X(t)1_{\{t\le\varrho\}}dB(t)] = 0,
$$

$$
\begin{aligned}
&E[\int_0^{\varrho\wedge n} e^{-\bar{r}t}X(t)\{u(X(t-)+\gamma(X(t-),z)) - u(X(t-)) - u'(X(t-))\cdot\gamma(X(t-),z)\}\tilde{N}(dtdz)] \\
&= E[\int_0^n e^{-\bar{r}t}X(t)\{u(X(t-)+\gamma(X(t-),z)) - u(X(t-)) - u'(X(t-))\cdot\gamma(X(t-),z)\}1_{\{t\le\varrho\}}\tilde{N}(dtdz)] = 0.
\end{aligned}
$$

Hence

$$
u(x) = E[\int_0^{\varrho\wedge n} e^{-\bar{r}t}\frac{1}{\varepsilon}(u-g)^-(X(t))dt + e^{-\bar{r}(\varrho\wedge n)}u(X(\varrho\wedge n))].
$$

Letting $n\to\infty$, by the dominated convergence theorem, we have

$$
u(x) = E[\int_0^{\tau\wedge\tau_R} e^{-\bar{r}t}\frac{1}{\varepsilon}(u-g)^-(X(t))dt + e^{-\bar{r}(\tau\wedge\tau_R)}u(X(\tau\wedge\tau_R))].
$$

Note that $\tau_R \nearrow \theta$ as $R\nearrow\infty$. Passing to the limit, we deduce (10). The statement (11) is immediate from (10) with $\tau=\infty$.

3. By the same line as (11), we have

$$
u(x) = E[\int_0^\infty e^{-(\bar{r}+\frac{1}{\varepsilon})t}\frac{1}{\varepsilon}(u\vee g)(X(t))dt].
$$

For two solutions u_1, u_2 of (8) in C_+, we get by (7)

$$
\|u_1-u_2\| \le \frac{1}{\bar{r}\varepsilon+1}\|u_1-u_2\|,
$$

which implies $u_1=u_2$.

5 Passaing to the limit as $\varepsilon \to 0$

We study the convergence of $u = u_\varepsilon \in \mathcal{C}_+$ as $\varepsilon \to 0$. Define the Green function

$$G_\beta h(x) = E[\int_0^\infty e^{-\beta t} h(X(t))dt], \quad \beta > 0,$$

and

$$\mathcal{G} = \{G_\beta(\beta h) : \ h \in \mathcal{C}, \beta > \tilde{r}\}.$$

Our objective is to prove the following.

Theorem 5.1

We assume (2). Let $\epsilon_n > 0$ be any sequence such that $\epsilon_n \to 0$ and that $\sum_{n=1}^\infty \epsilon_n < +\infty$. Then we have

(14) $$u_{\varepsilon_n} \quad \to \quad v \in \mathcal{C}.$$

For the proof of this theorem, we prepapre the following three lemmas, whose proofs we shall omit. See [15].

Lemma 5.2 The class $\mathcal{G}$ is dense in $\mathcal{C}$.

Lemma 5.3 Let $\tilde{u} \in \mathcal{C}_+$ be the solution of (8) with $\tilde{g} \in \mathcal{C}_+$ replacing g. Then we have

(15) $$\|u - \tilde{u}\| \le \|g - \tilde{g}\|.$$

Lemma 5.4 Under (2), we have

(16) $$u_\varepsilon(x) = \sup_{\tau \in \mathcal{S}} E[e^{-\tilde{r}\tau}\{g - (u_\varepsilon - g)^-\}(X(\tau))].$$

Proof of Theorem 5.1

1. We claim that

(17) $$(u_\varepsilon - g)^- \le \varepsilon \|\beta h + (\tilde{r} - \beta) g\|,$$

if $g = G_\beta(\beta h) \in \mathcal{G}$ for some $h \in \mathcal{C}$.

Indeed, by the same line as the proof of Theorem 4.3, we observe that g is a unique viscosity solution of

$$\begin{cases} \beta g = L_0 g + \beta h & \text{in} \quad (0, \infty), \\ g(0) = h(0), \end{cases}$$

or equivalently,

$$\begin{cases} (\tilde{r} + \frac{1}{\varepsilon}) g = L_0 g + \hat{h} + \frac{1}{\varepsilon} g & \text{in} \quad (0, \infty), \\ g(0) = \frac{\varepsilon}{\tilde{r}\varepsilon + 1}\{\hat{h}(0) + \frac{1}{\varepsilon} g(0)\}, \end{cases}$$

where $\hat{h} = \beta h + (\tilde{r} - \beta)g$ (see the proof of Theorem 6.3 below for the uniqueness). Hence we have $g = G_{\tilde{r}+\frac{1}{\varepsilon}}(\hat{h} + \frac{1}{\varepsilon}g)$. Therefore, by (6)

$$\begin{aligned} u_\varepsilon(x) - g(x) &= E[\int_0^\infty e^{-(\tilde{r}+\frac{1}{\varepsilon})t}\{\frac{1}{\varepsilon}(u_\varepsilon \vee g)(X(t)) - (\hat{h}(X(t)) + \frac{1}{\varepsilon}g(X(t)))\}dt] \\ &\geq -E[\int_0^\infty e^{-(\tilde{r}+\frac{1}{\varepsilon})t}\ \hat{h}(X(t))dt] \\ &\geq -\varepsilon\|\hat{h}\|, \quad x > 0, \end{aligned}$$

which implies (17).

2. Let $g = G_\beta(\beta h) \in \mathcal{G}$. Applying (17) to $u_{\varepsilon_{n+1}}(x)$ and $u_{\varepsilon_n}(x)$, by Lemma 5.4, we have

$$\begin{aligned} |u_{\varepsilon_{n+1}}(x) - u_{\varepsilon_n}(x)| &\leq \sup_{\tau\in\mathcal{S}} E[e^{-\tilde{r}\tau}|(u_{\varepsilon_{n+1}} - g)^- - (u_{\varepsilon_n} - g)^-|(X(\tau))] \\ &\leq (\varepsilon_{n+1} + \varepsilon_n)\|\beta h + (\tilde{r} - \beta)g\|. \end{aligned}$$

Thus

$$\sum_{n=1}^{\infty} \|u_{\varepsilon_{n+1}} - u_{\varepsilon_n}\| \leq \sum_{n=1}^{\infty} (\varepsilon_{n+1} + \varepsilon_n)\|\beta h + (\tilde{r} - \beta)g\| < \infty.$$

This implies that $\{u_{\varepsilon_n}\}$ is a Cauchy sequence in $\mathcal{C}$, and we get (14).

3. Let g satisfy (2). By Lemma 5.2, there exists a sequence $\{g_m\} \subset \mathcal{G}$ such that $g_m \to g$ in $\mathcal{C}$. Let u_ε^m be the solution of (8) corresponding to g_m. By **2**, we see that

(18) $$u_{\varepsilon_n}^m \quad \to \quad v^m \in \mathcal{C} \quad \text{as} \quad n \to \infty.$$

By Lemma 5.3,

$$\|u_{\varepsilon_n}^m - u_{\varepsilon_n}^{m'}\| \leq \|g_m - g_{m'}\|.$$

Letting $n \to \infty$, we have

$$\|v^m - v^{m'}\| \leq \|g_m - g_{m'}\|.$$

Hence $\{v^m\}$ is a Cauchy sequence, and

(19) $$v^m \quad \to \quad v \in \mathcal{C}.$$

Thus

$$\begin{aligned} \|u_{\varepsilon_n} - v\| &\leq \|u_{\varepsilon_n} - u_{\varepsilon_n}^m\| + \|u_{\varepsilon_n}^m - v^m\| + \|v^m - v\| \\ &\leq \|g - g_m\| + \|u_{\varepsilon_n}^m - v^m\| + \|v^m - v\|. \end{aligned}$$

Letting $n \to \infty$ and then $m \to \infty$, we obtain (14). The limit does not depend on the choice of (ϵ_n) and $\{g_m\}$ as long as (ϵ_n) satisfy the condition in the Theorem.

6 Viscosity Solutions of Variational Inequalities

In this section, we study the viscosity solution of the variational inequality:

$$(20) \qquad \begin{cases} \max(Lv, g - v) \leq 0, \\ v(0) = g(0). \end{cases}$$

Definition 6.1 Let $v \in C([0, \infty))$. Then v is called a viscosity solution of (20), if the following assertions are satisfied:

(*a*) For any $\phi \in C^2$ and for any local minimum point $\bar{z} > 0$ of $v - \phi$,

$$-\tilde{r}v(\bar{z}) + L_0\phi(\bar{z}) \leq 0,$$

(*b*) $v(x) \geq g(x)$ for all $x \geq 0$,

(*c*) For any $\phi \in C^2$ and for any local maximum point $z > 0$ of $v - \phi$,

$$(-\tilde{r}v + L_0\phi)(v - g)^+ \mid_{x=z} \geq 0.$$

Theorem 6.2 We assume (2). Then the limit v in Theorem 5.1 is a viscosity solution of (20).

Proof. Let $\phi \in C^2$ and let $z > 0$ be a local maximum point of $v - \phi$ such that

$$v(z) - \phi(z) > v(x) - \phi(x), \quad x \in \bar{B}_\delta(z), \quad z \neq x$$

for some $\delta > 0$.

By the uniform convergence in Theorem 5.1, the function $u_{\varepsilon_n} - \phi$ attains a local maximum at $x_n \in \bar{B}_\delta(z)$.

We deduce

$$x_n \rightarrow z \quad \text{as} \quad n \rightarrow \infty.$$

Indeed, for $0 < \delta < \delta_0$, it is easy to check

$$u(x) - \phi(x) < u(z) - \phi(z), \quad \text{for} \quad x \in \bar{B}_\delta(z), z \neq x.$$

For the sequence of local maximum points (x_n) in $\bar{B}_\delta(z)$ of $(u_{\varepsilon_n} - \phi)$, choose a subsequence (x_{n_k}) of (x_n) so that for some $z' \in \bar{B}_\delta(z)$

$$x_{n_k} \rightarrow z'.$$

By Theorem 5.1

$$(u_{\varepsilon_{n_k}} - \phi)(x_{n_k}) \rightarrow (v - \phi)(z')$$

and

$$\max_{x\in \bar{B}_\delta(z)}(u_{\varepsilon_{n_k}}(x)-\phi(x)) \to \max_{x\in \bar{B}_\delta(z)}(v(x)-\phi(x)).$$

Hence $(v-\phi)(z') \geq (v-\phi)(x), x \in \bar{B}_\delta(z)$, and hence $(v-\phi)(z') \geq (v-\phi)(z)$. Hence we have $z' = z$.

Now, by Theorem 4.3, we have

$$-\tilde{r}u_{\varepsilon_n}(x) + L_0\phi(x) + \frac{1}{\varepsilon_n}(u_{\varepsilon_n} - g)^-(x)|_{x=x_n} \geq 0.$$

Multiply both sides by $(u_{\varepsilon_n} - g)^+$ to obtain

$$(-\tilde{r}u_{\varepsilon_n}(x_n) + L_0\phi(x_n))(u_{\varepsilon_n} - g)^+(x_n) \geq 0.$$

Letting $n \to \infty$, we get

$$(-\tilde{r}v(z) + L_0\phi(z))(v-g)^+(z) \geq 0.$$

Next, by (17), we have

$$(u^m_{\varepsilon_n} - g_m)^- \leq \varepsilon_n \|\beta h_m + (\tilde{r} - \beta)g_m\|,$$

where $g_m = G_\beta(\beta h_m)$ for some $h_m \in \mathcal{C}$ and $u^m_{\varepsilon_n}$ is as in the proof of Theorem 5.1. Letting $n \to \infty$, by (18), we have

$$v^m(x) \geq g_m(x), \quad x \geq 0,$$

and then, by (19)

$$v(x) \geq g(x) \quad \text{for all} \quad x \geq 0.$$

Finally, let $\bar{z}$ be the minimizer of $v-\phi$, and $\bar{x}_n$ be the sequence of the local minimizers of $u_{\varepsilon_n} - \phi$ such that $\bar{x}_n \to \bar{z}$. Then, by Theorem 4.3

$$-\tilde{r}u_{\varepsilon_n}(x) + L_0\phi(x) + \frac{1}{\varepsilon_n}(u_{\varepsilon_n} - g)^-(x)|_{x=\bar{x}_n} \leq 0,$$

from which

$$-\tilde{r}u_{\varepsilon_n}(\bar{x}_n) + L_0\phi(\bar{x}_n) \leq 0.$$

Letting $n \to \infty$, we deduce

$$-\tilde{r}v(\bar{z}) + L_0\phi(\bar{z}) \leq 0.$$

Thus we get the assertion of the theorem.

Theorem 6.3 We make the assumption of Theorem 6.2. Let $v_i \in \mathcal{C}, i = 1, 2$, be two viscosity solutions of (20). Then we have

$$v_1 = v_2.$$

We omit the proof of this theorem since it is too long. See [15].

Theorem 6.4 We make the assumption of Theorem 6.2. Then we have

$$v(x) = \sup_{\tau \in \mathcal{S}} E[e^{-\tilde{r}\tau} g(X(\tau))].$$

Proof.

1. Let $x > 0$ and $\tau \in \mathcal{S}$. By (10), we get

$$\begin{aligned} u_{\varepsilon_n}(x) &= E[\int_0^{\tau} e^{-\tilde{r}t} \frac{1}{\varepsilon}(u_{\varepsilon_n} - g)^-(X(t))dt + e^{-\tilde{r}\tau} u_{\varepsilon_n}(X(\tau))] \\ &\geq E[e^{-\tilde{r}\tau} u_{\varepsilon_n}(X(\tau))]. \end{aligned}$$

Letting $n \to \infty$, by Theorems 5.1 and 6.2, we have

$$v(x) \geq E[e^{-\tilde{r}\tau} v(X(\tau))] \geq E[e^{-\tilde{r}\tau} g(X(\tau))].$$

2. For any $m \in \mathbf{N}$, we set

$$\varrho_m = \inf\{t \geq 0 : \ v(X(t)) - \frac{1}{m} \leq g(X(t))\}. \tag{22}$$

Since

$$v(X(t)) - \frac{1}{m} > g(X(t)) \quad \text{on} \quad \{t < \varrho_m\},$$

we have

$$\begin{aligned} E[\int_0^{\varrho_m} e^{-\tilde{r}t}(u_{\varepsilon_n} - g)^-(X(t))dt] &\leq E[\int_0^{\varrho_m} e^{-\tilde{r}t}(u_{\varepsilon_n} - (v - \frac{1}{m}))^-(X(t))dt] \\ &\leq E[\int_0^{\varrho_m} e^{-\tilde{r}t}(\frac{1}{m} - \|u_{\varepsilon_n} - v\|)^-(X(t))dt] \\ &= 0 \end{aligned} \tag{23}$$

for sufficiently large n. Hence, by (10)

$$\begin{aligned} u_{\varepsilon_n}(x) &= E[\int_0^{\varrho_m} e^{-\tilde{r}t} \frac{1}{\varepsilon_n}(u_{\varepsilon_n} - g)^-(X(t))dt + e^{-\tilde{r}\varrho_m} u_{\varepsilon_n}(X(\varrho_m))] \\ &= E[e^{-\tilde{r}\varrho_m} u_{\varepsilon_n}(X(\varrho_m))]. \end{aligned}$$

Letting $n \to \infty$, by (23), we get

$$\begin{aligned} v(x) &= E[e^{-\tilde{r}\varrho_m} v(X(\varrho_m))] \leq E[e^{-\tilde{r}\varrho_m}\{g(X(\varrho_m)) + \frac{1}{m}\}] \\ &\leq \sup_{\tau \in \mathcal{S}} E[e^{-\tilde{r}\tau} g(X(\tau))] + \frac{1}{m}. \end{aligned}$$

Passing to the limit, we deduce

$$v(x) \leq \sup_{\tau \in \mathcal{S}} E[e^{-\tilde{r}\tau} g(X(\tau))].$$

7 Solution of the Optimal Stopping Problem

In this section, we give a synthesis of the optimal stopping time.

Theorem 7.1

We assume (2). Then the optimal stopping time τ^* is given by

$$\tau^* = \inf\{t \geq 0 : v(X(t)) \leq g(X(t))\}$$

for $x > 0$.

Proof.

1. For any $\tau \in \mathcal{S}$ and τ_R of (13), we set $\varrho = \tau \wedge \tau_R$. By Itô's formula, we have

$$E[e^{-\bar{r}\varrho}u_{\varepsilon_n}(X(\varrho))] = u_{\varepsilon_n}(x) + E[\int_0^{\varrho} e^{-\bar{r}t}\{-\bar{r}u_{\varepsilon_n} + \frac{1}{2}\sigma^2 x^2 u''_{\varepsilon_n} + rxu'_{\varepsilon_n}$$
$$+ \int \{u_{\varepsilon_n}(x+\gamma(x,z)) - u_{\varepsilon_n}(x) - u'_{\varepsilon_n}(x)\cdot\gamma(x,z)\}\mu(dz)\}|_{x=X(t)}dt]$$
$$= u_{\varepsilon_n}(x) - E[\int_0^{\varrho} e^{-\bar{r}t}\frac{1}{\varepsilon}(u_{\varepsilon_n} - g)^-(X(t))dt] \leq u_{\varepsilon_n}(x).$$

Letting $R \to \infty$ and then $\varepsilon_n \to 0$, by the dominated convergence theorem, we deduce

$$E[e^{-\bar{r}\tau}g(X(\tau))] \leq E[e^{-\bar{r}\tau}v(X(\tau))] \leq v(x).$$

2. We set $\bar{\tau} = \tau_R \wedge \varrho_m$ for ϱ_m of (22). By (23), it is clear that

$$E[\int_0^{\bar{\tau}} e^{-\bar{r}t}(u_{\varepsilon_n} - g)^-(X(t))dt] = 0$$

for sufficiently large n. Hence, applying Itô's formula, we have

$$E[e^{-\bar{r}\bar{\tau}}u_{\varepsilon_n}(X(\bar{\tau}))] = u_{\varepsilon_n}(x) + E[\int_0^{\bar{\tau}} e^{-\bar{r}t}\{-\bar{r}u_{\varepsilon_n} + \frac{1}{2}\sigma^2 x^2 u''_{\varepsilon_n} + rxu'_{\varepsilon_n}$$
$$+ \int \{u_{\varepsilon_n}(x+\gamma(x,z)) - u_{\varepsilon_n}(x) - u'_{\varepsilon_n}(x)\cdot\gamma(x,z)\}\mu(dz)\}|_{x=X(t)}dt]$$
$$= u_{\varepsilon_n}(x) - E[\int_0^{\bar{\tau}} e^{-\bar{r}t}\frac{1}{\varepsilon}(u_{\varepsilon_n} - g)^-(X(t))dt] = u_{\varepsilon_n}(x).$$

Letting $n \to \infty$ and then $R \to \infty$, we get

$$E[e^{-\bar{r}\varrho_m}v(X(\varrho_m))] = v(x).$$

Note that $\varrho_m \nearrow \tau^*$ as $m \nearrow \infty$. Passing to the limit, we deduce

$$E[e^{-\bar{r}\tau^*}g(X(\tau^*))] = E[e^{-\bar{r}\tau^*}v(X(\tau^*))] = v(x).$$

Thus, we obtain the assertion.

References

[1] R.F. Bass, Uniqueness in law for pure jump Markov processes, Probab. Th. Rel. Fields 79 (1988), 271–287.

[2] N. Bellamy, Wealth optimization in an incomplete market driven by a jump-diffusion process, J. Math. Econom. 35 (2001), 259–287.

[3] F. Benth, K. Karlsen and K. Reikvam, Optimal portfolio selection with consumption and nonlinear integro-differential equations with gradient constraint: a viscosity solution approach, Finance Stoch. 5 (2001), 275–303.

[4] F. Benth, K. Karlsen and K. Reikvam, Optimal portfolio management rules in a non-Gaussian market with durability and intertemporal substitution. Finance Stoch. 5 (2001), 447–467.

[5] F. Benth, K. Karlsen and K. Reikvam, On the existence of optimal controls for a singular stochastic control problem in finance, Mathematical Finance (Konstanz, 2000), M. Kohlmann and S. Tang eds., 79–88, Birkhauser, 2001.

[6] F. Benth, K. Karlsen and K. Reikvam, Portfolio optimization in a Lévy market with international substitution and transaction costs, Stoch. Stoch. Reports 74 (2002), 517–569.

[7] C. Cancelier, Problmes aux limites pseudo-diffrentiels donnant lieu au principe du maximum, Comm. Partial Differential Equations 11 (1986), 1677–1726.

[8] M. Chesney and M. Jeanblanc, Pricing American currency options in an exponential Lévy model, Appl. Math. Finance 11 (2004), 207–225 .

[9] M. Crandall and P-L. Lions, Viscosity solutions of Hamilton-Jacobi equations, Trans. A. M. S. 277 (1983), 1–42.

[10] M. Crandall, H. Ishii and P-L. Lions, User's guide to viscosity solutions of second order partial differential equations, Bull. Amer. Math. Soc. (N.S.) 27 (1992), 1–67.

[11] N. Framstad, B. Oksendal and A. Sulem, Optimal consumption and portfolio in a jump diffusion market with proportional transaction costs, J. Math. Econom. 35 (2001), 233–257.

[12] N. Framstad, B. Oksendal and A. Sulem, Sufficient stochastic maximum principle for the optimal control of jump diffusion and applications to finance, JOTA 121 (2004), 77–98.

[13] A. Hindy and C. Huang, Optimal consumption and portfolio rules with durability and local substitutions, Econometrica 61 (1993), 85–121.

[14] Y. Ishikawa, Optimal control problem associated with jump processes, Appl. Math. Optim. 50 (2004), 21–65.

[15] Y. Ishikawa, Optimal stopping problem associated with jump-diffusion processes, Preprint 2006.

[16] N. V. Krylov, Controlled diffusion processes, Applications of Mathematics, 14, Springer-Verlag, New York-Berlin, 1980.

[17] H. Kumano-go, Pseudodifferential operators, MIT Press, Cambridge, 1981.

[18] R. C. Merton, Optimum consumption and portfolio rules in a continuous-time model. J. Econom. Theory 3, 373–413. Erratum, J. Econom. Theory 6 (1973), 213–214.

[19] E. Mordecki, Optimal stopping and perpetual options for Levy processes, Finance Stochast. 6 (2002), 473–493.

[20] B. Oksendal, Stochastic differential equations. An introduction with applications. Fourth edition. Universitext. Springer-Verlag, Berlin, 1995.

[21] Ph. Protter, Stochastic integration and differential equations. A new approach. Applications of Mathematics, 21. Springer-Verlag, Berlin, 1990.

Department of Mathematics

Faculty of Science

Ehime University

Matsuyama Ehime 7908577 Japan
E-mail: slishi@math.sci.ehime-u.ac.jp