数理解析研究所講究録
1530 巻 2007 年 5-16

ALMOST DISJOINT FAMILIES ON LARGE UNDERLYING SETS

Sakaé Fuchino, Stefan Geschke and Lajos Soukup

January 13, 2007

Abstract

We show that, for any poset $\mathbb{P}$, the existence of a $\mathbb{P}$-indestructible mad family $\mathcal{F} \subseteq [\omega]^{\aleph_0}$ is equivalent to the existence of such a family over $\aleph_n$ for some/all $n \in \omega$. Under the very weak square principle $\square^{***}_{\omega_1,\mu}$ of Fuchino and Soukup [7] and $\mathrm{cf}([\mu]^{\aleph_0}, \subseteq) = \mu^+$ for all limit cardinals μ of cofinality ω, the equivalence for any proper poset $\mathbb{P}$ transfers to all cardinals. That is, under these assumptions, if $\mathbb{P}$ is a proper poset, then there is a $\mathbb{P}$-indestructible mad family on ω if and only if there is a $\mathbb{P}$-indestructible mad family on some/all infinite cardinals κ.

1 Introduction

For $\mathcal{X} \subseteq [S]^{\aleph_0}$ we say that an infinite family $\mathcal{F} \subseteq \mathcal{X}$ is *pairwise almost disjoint* (abbreviation: *ad*) if $x \cap y$ is finite for all distinct $x, y \in \mathcal{F}$. $\mathcal{F}$ is *maximal almost*

[0] The first author is partially supported by Chubu University grant 16IS55A. The second author is supported by Center of Excellence grant from the European Union. The third author is supported by Bolyai Grant and OTKA grant no. 61600.

The research of this paper was began when the first and second authors visited Alfréd Rény Institute in Budapest in 2002. Then it was resumed when the first author visited Centre de Recerca Matemàtica in Barcelona and the third Barcelona University in about the same period in 2006. The first and third authors would like to thank Professors Joan Bagaria and Juan-Carlos Martínez for the arrangement of the visit and their hospitality during the stay in Barcelona.

The authors also thank I. Juhász and P. Nykos for informing them about Lemma 12.

disjoint (*mad*, for short) in $\mathcal{X}$ if it is pairwise almost disjoint and maximal among such subsets of $\mathcal{X}$ (with respect to $\subseteq$). If $\mathcal{F}$ is mad in $[S]^{\aleph_0}$ we shall also say that $\mathcal{F}$ is *mad* on S or $\mathcal{F}$ is a *mad family* on S.

For $S \subseteq \mathrm{On}$ such that $otp(\sup S) = \omega$, let

$$(S)^\omega = \{x \in [S]^{\aleph_0} : \sup x = \sup S,\ otp(x) = \omega\}.$$

If $\mathcal{F}$ is mad in $(S)^\omega$ we shall also say that $\mathcal{F}$ is *cof-mad on* S or $\mathcal{F}$ is a *cof-mad family on* S.

For a poset $\mathbb{P}$, a mad (cof-mad) family $\mathcal{F}$ on $S \subseteq \mathrm{On}$ is said to be $\mathbb{P}$-indestructible if $\Vdash_{\mathbb{P}}$ "$\mathcal{F}$ is mad on S" ($\Vdash_{\mathbb{P}}$ "$\mathcal{F}$ is cof-mad on S"). We shall call $\mathcal{F}$ a *$\mathbb{P}$-indestructible mad family on S in a broad sense* if $\mathcal{F}$ is either a $\mathbb{P}$-indestructible mad family on S or $\mathcal{F}$ is finite partition of S modulo finite (i.e $\mathcal{F}$ is finite, $\mathcal{F}$ is ad and $(\bigcup\mathcal{F})\triangle S$ is finite).

$\mathbb{P}$-indestructible mad families on ω for various posets $\mathbb{P}$ are studied extensively in recent papers, e.g., Hrušák [8], Hrušák and Ferreira[9], Brendle and Yatabe [5], and authors' [6].

The present note shows that results on $\mathbb{P}$-indestructibility of mad families on ω can be transfered to corresponding results on mad families on an uncountable support.

For a poset $\mathbb{P}$ and a set $S \subseteq \mathrm{On}$, let $\mathbb{A}^{\mathbb{P}}(S)$ and $\mathbb{A}^{\mathbb{P}}_{cof}(S)$ be the following assertions:

$$\begin{aligned}
\mathbb{A}^{\mathbb{P}}(S) &\iff \text{there exists a } \mathbb{P}\text{-indestructible mad family on } S;\\
\mathbb{A}^{\mathbb{P}}_{cof}(S) &\iff \text{there exists a } \mathbb{P}\text{-indestructible cof-mad family on } S.
\end{aligned}$$

2 $\mathbb{P}$-indestructible mad families on sets of ordinals

In this section, let $\mathbb{P}$ be an arbitrary poset.

Lemma 1. *If $\mathbb{A}^{\mathbb{P}}(\omega)$ then $\mathbb{A}^{\mathbb{P}}_{cof}(\alpha)$ for all limit $\alpha < \omega_1$.*

Proof. Let $\mathcal{C}$ be a $\mathbb{P}$-indestructible mad family on ω. Without loss of generality we may assume that $\mathcal{C} = \{c_\xi : \xi < \eta\}$ and ω is the disjoint union of c_n, $n \in \omega$.

If $\alpha = \beta + \omega$, let $f : \omega \to \alpha$; $n \mapsto \beta + n$.

Then $\tilde{\mathcal{C}} = \{f''c_\xi : \xi < \eta\}$ is a $\mathbb{P}$-indestructible cof-mad family on α: Clearly $\tilde{\mathcal{C}} \subseteq (\alpha)^\omega$ and $\tilde{\mathcal{C}}$ is ad. Suppose that x is an element of $(\alpha)^\omega$ in a $\mathbb{P}$-generic extension such that $x \notin \tilde{\mathcal{C}}$. By $\mathbb{P}$-indestructibility of $\mathcal{C}$, there is $\xi < \eta$ such that $|\, c_\xi \cap f^{-1}{}''x \,| = |\, f''c_\xi \cap x \,| = \aleph_0$. Hence $\tilde{\mathcal{C}} \cup \{x\}$ is not ad. This shows that $\tilde{\mathcal{C}}$ is a $\mathbb{P}$-indestructible cof-mad family on α.

If $\alpha < \omega_1$ is a limit of limits, then let $\langle \alpha_n : n \in \omega \rangle$ be a strictly increasing sequence of ordinals such that $\alpha = \sup_{n<\omega} \alpha_n$, $\alpha_0 = 0$ and $\alpha_{n+1} \setminus \alpha_n$ is infinite for all $n < \omega$.

Let $f : \omega \to \alpha$ be such that $f''c_n = \alpha_{n+1} \setminus \alpha_n$ for all $n \in \omega$. Let $\mathcal{D} = \{f''c_\xi : \xi \in \eta \setminus \omega\}$. Then, similarly to the previous case, $\mathcal{D}$ is a $\mathbb{P}$-indestructible mad family on $(\alpha)^\omega$. □ (Lemma 1)

For $\mathcal{F} \subseteq [A]^{\aleph_0}$ and $A' \subseteq A$, let

$$\mathcal{F} \restriction A' = \{a \cap A' : a \in \mathcal{F}\} \cap [A]^{\aleph_0}.$$

Lemma 2. *Suppose that $\mathcal{F}$ is a $\mathbb{P}$-indestructible mad family on A.*

(1) *If $A' \subseteq A$ is uncountable, then $\mathcal{F} \restriction A'$ is a $\mathbb{P}$-indestructible mad family on A'.*

(2) *If $A' \subseteq A$ is countable and $\mathcal{F} \restriction A'$ is infinite, then $\mathcal{F} \restriction A'$ is a $\mathbb{P}$-indestructible mad family on A'.*

Proof. (1): Clearly $\mathcal{F} \restriction A'$ is an ad family. If A' is infinite then $\mathcal{F} \restriction A'$ is also infinite since otherwise $A' \setminus \bigcup(\mathcal{F} \restriction A')$ would be infinite so any countable subset of this set would be almost disjoint to $\mathcal{F}$.

If $\mathcal{F} \restriction A'$ were not a $\mathbb{P}$-indestructible mad family, then there would be an element x of $[A']^{\aleph_0}$ in a $\mathbb{P}$-extension such that x is almost disjoint to every element of $\mathcal{F} \restriction A'$. But then x would be also almost disjoint to every element of $\mathcal{F}$. A contradiction to $\mathbb{P}$-indestructibility of $\mathcal{F}$.

(2): Similarly. □ (Lemma 2)

Lemma 3. *For any ordinal α, if $\mathbb{A}^{\mathbb{P}}(\alpha)$ then $\mathbb{A}^{\mathbb{P}}(\beta)$ for all $\omega \leq \beta \leq \alpha$.*

Proof. We prove this for $|\,\beta\,| = \omega$. The case for $|\,\beta\,| > \omega$ follows from Lemma 2,(1).

Let $\mathcal{F} \subseteq [\alpha]^{\aleph_0}$ be a $\mathbb{P}$-indestructible mad family. For ω distinct elements a_i, $i < \omega$ of $\mathcal{F}$, let $s = \bigcup_{i<\omega} a_i$. Then s is countable and $\mathcal{F} \restriction s$ is infinite. By Lemma 2,(2), it follows that $\mathcal{F} \restriction s$ is a $\mathbb{P}$-indestructible mad family on s.

Let $\varphi : s \to \beta$ be a bijection. Then $\mathcal{F}^* = \{\varphi''a \,:\, a \in \mathcal{F} \restriction s\}$ is a $\mathbb{P}$-indestructible mad family on β. □ (Lemma 3)

Lemma 4. *For any ordinal α with* $\mathrm{cf}(\alpha) = \omega$, *if* $\mathbb{A}^{\mathbb{P}}(\alpha)$ *then* $\mathbb{A}^{\mathbb{P}}_{cof}(\alpha)$.

Proof. By Lemma 3, we have $\mathbb{A}^{\mathbb{P}}(\omega)$. Hence, by Lemma 1, there is a $\mathbb{P}$-indestructible cof-mad family $\mathcal{C}_\beta$ on each $\beta \in Lim(\omega_1)$.

Let $\mathcal{F}$ be a $\mathbb{P}$-indestructible mad family on α.

Without loss of generality we may assume that sup a is a limit for each $a \in \mathcal{F}$. For $a \in \mathcal{F}$, let $f_a : otp(a) \to a$ be the order isomorphism. Let

$$\mathcal{D} = \{f_a''b \,:\, a \in \mathcal{F},\ \sup a = \alpha,\ b \in \mathcal{C}_{otp(a)}\}.$$

Then $\mathcal{D}$ is a $\mathbb{P}$-indestructible cof-mad family on α: Clearly $\mathcal{D} \subseteq (\alpha)^\omega$ and $\mathcal{D}$ is ad. Suppose that x is an element of $(\alpha)^\omega$ in a $\mathbb{P}$-generic extension such that $x \notin \mathcal{D}$. By $\mathbb{P}$-indestructibility of $\mathcal{F}$, there is $a \in \mathcal{F}$ such that $|\, a \cap x \,| = \aleph_0$. Let $x' = a \cap x$. Then we have $\sup a = \sup x' = \alpha$. Hence $x' \in (a)^\omega$. By $\mathbb{P}$-indestructibility of the cof-mad family $\{f_a''b \,:\, b \in \mathcal{C}_{otp(a)}\}$ on a, there is $b \in \mathcal{C}_{otp(a)}$ such that $|\, f_a''b \cap x' \,| = \aleph_0$. Thus $\mathcal{D} \cap \{x\}$ is not ad. □ (Lemma 4)

Theorem 5. (1) *For any cardinal κ,* $\mathbb{A}^{\mathbb{P}}(\kappa)$ *implies* $\mathbb{A}^{\mathbb{P}}(\kappa^+)$.

(2) *If* $\mathrm{cf}(\kappa) > \omega$ *and* $\mathbb{A}^{\mathbb{P}}(\lambda)$ *for all* $\lambda < \kappa$ *then* $\mathbb{A}^{\mathbb{P}}(\kappa)$.

Proof. (1): By Lemma 4, there is a $\mathbb{P}$-indestructible cof-mad family $\mathcal{C}_\alpha$ on α for each $\alpha \in E^\omega_{\kappa^+} = \{\alpha < \kappa^+ \,:\, \mathrm{cf}(\alpha) = \omega\}$. Let

$$\mathcal{F} = \bigcup_{\alpha \in E^\omega_{\kappa^+}} \mathcal{C}_\alpha.$$

Then $\mathcal{F}$ is a $\mathbb{P}$-indestructible mad family on κ^+: Clearly $\mathcal{F}$ is ad. Suppose that x is an element of $[\kappa^+]^{\aleph_0}$ in a $\mathbb{P}$-generic extension such that $x \notin \mathcal{F}$. By cutting off a finite end segment of x and thinning it out, if necessary, we may assume that $x \in (\alpha)^\omega$ for some limit $\alpha < \kappa^+$ of cofinality ω. By $\mathbb{P}$-indestructibility of $\mathcal{C}_\alpha$, there is $a \in \mathcal{C}_\alpha$ such that $|\, x \cap a \,| = \aleph_0$. Thus $\mathcal{F} \cup \{x\}$ is not ad.

(2): Similarly to (1). □ (Theorem 5)

Corollary 6. *For any poset* $\mathbb{P}$ *and* $n < \omega$, $\mathbb{A}^{\mathbb{P}}(\omega)$ *if and only if* $\mathbb{A}^{\mathbb{P}}(\omega_n)$.

Proof. By Lemma 3 and Theorem 5, (1). □ (Corollary 6)

The following lemma is used as a building block for Theorem 10:

Lemma 7. *For an uncountable* κ, *suppose that* $\mathbb{A}^{\mathbb{P}}(\kappa)$ *and* $\kappa = \bigcup\{A_n : n < \omega\}$. *For an ad family* $\mathcal{F} \subseteq [\kappa]^{\aleph_0}$, *if* $\mathcal{F} \restriction A_n$ *is a* $\mathbb{P}$*-indestructible mad family in a broad sense for all* $n < \omega$ *and* $\mathcal{F} = \bigcup_{n<\omega}\left(\mathcal{F} \cap [A_n]^{\aleph_0}\right)$, *then there is a* $\mathbb{P}$*-indestructible mad family* $\mathcal{B}$ *on* κ *with* $\mathcal{B} \supseteq \mathcal{F}$.

Proof. Let $A'_n = A_n \setminus \bigcup\{A_m : m < n\}$ for $n < \omega$ and

$$\delta = \textstyle\sum_{n<\omega} otp(A'_n).$$

That is, δ is the order type of the linear order obtained by concatenating A'_n, $n < \omega$ one after another. Note that we have $\mathrm{cf}(\delta) = \omega$. Let $f : \kappa \to \delta$ be a bijection such that

$$f''A'_n = \left[\textstyle\sum_{k<n} otp(A'_k),\ \sum_{k<n+1} otp(A'_k)\right)$$

for all $n < \omega$. By Lemma 4, there is a $\mathbb{P}$-indestructible cof-mad family $\mathcal{B}'$ on δ.

Claim 7.1. $\mathcal{B} = \mathcal{F} \cup \{f^{-1}{}''b : b \in \mathcal{B}'\}$ *is a* $\mathbb{P}$*-indestructible mad family on* κ.

$\vdash$ To show that $\mathcal{B}$ is ad, it is enough to show that for all $a \in \mathcal{F}$ and $b \in \mathcal{B}'$ we have $|\,a \cap f^{-1}{}''b\,| < \aleph_0$. By assumption there is $n < \omega$ such that $a \subseteq [A_n]^{\aleph_0}$. Thus a corresponds to a bounded subset of δ. Hence $|\,a \cap f^{-1}{}''b\,| = |\,f''a \cap b\,| < \aleph_0$.

To show that $\mathcal{B}$ is $\mathbb{P}$-indestructibly mad, suppose that x is an element of $[\kappa]^{\aleph_0}$ in a $\mathbb{P}$-generic extension such that $x \notin \mathcal{B}$. If $f''x$ is cofinal in δ, then by $\mathbb{P}$-indestructibility of $\mathcal{B}'$, there is $b \in \mathcal{B}'$ such that $|\,x \cap b\,| = \aleph_0$. If $f''x$ is not cofinal in δ, there is $n < \omega$ such that $x \in [A_n]^{\aleph_0}$. Then by $\mathbb{P}$-indestructibility of $\mathcal{F} \restriction A_n$, there is $a \in \mathcal{F}$ such that $|\,x \cap a\,| = \aleph_0$. $\dashv$ (Claim 7.1) □ (Lemma 7)

3 Very weak weak square principle

In this section we review some results from [7] and prove a consequence of them (Theorem 9) which will be used in the proof of our main theorem (Theorem 10).

For a regular cardinal κ and $\mu > \kappa$, let $\square^{***}_{\kappa,\mu}$ be the following assertion: there exists a sequence $\langle C_\alpha \rangle_{\alpha<\mu^+}$ and a club set $D \subseteq \mu^+$ such that for all $\alpha \in D$ with $\mathrm{cf}(\alpha) \geq \kappa$

(3.1) $C_\alpha \subseteq \alpha$, C_α is unbounded in α;

(3.2) $[\alpha]^{<\kappa} \cap \{C_{\alpha'} : \alpha' < \alpha\}$ dominates $[C_\alpha]^{<\kappa}$ (with respect to $\subseteq$).

Note that, in (3.2), we also consider $\alpha' < \alpha$ of cofinality $< \kappa$.

Since (3.2) remains valid when C_α's for $\alpha \in D$ are slimed down, we may replace (3.1) by

(3.1)′ $C_\alpha \subseteq \alpha$, C_α is unbounded in α and $otp(C_\alpha) = \mathrm{cf}(\alpha)$.

Suppose now that κ is a regular cardinal and $\mu > \kappa$ is such that $\mathrm{cf}(\mu) < \kappa$. Let $\mu^* = \mathrm{cf}(\mu)$ — the case we later consider is when $\kappa = \omega_1$ (and $\mathrm{cf}(\mu) = \omega$). For a sufficiently large regular χ and $x \in \mathcal{H}(\chi)$, let us call a sequence $\langle M_{\alpha,\beta} \rangle_{\alpha<\mu^+,\beta<\mu^*}$ a (κ,μ)-*dominating matrix over* x, or just *dominating matrix over* x if it is clear from the context which κ and μ are meant — if the following conditions hold:

(3.3) $M_{\alpha,\beta} \prec \mathcal{H}(\chi)$, $x \in M_{\alpha,\beta}$, $\kappa + 1 \subseteq M_{\alpha,\beta}$ and $|M_{\alpha,\beta}| < \mu$ for all $\alpha < \mu^+$ and $\beta < \mu^*$;

(3.4) $\langle M_{\alpha,\beta} \rangle_{\beta<\mu^*}$ is an increasing sequence for each $\alpha < \mu^+$;

(3.5) if $\alpha < \mu^+$ is such that $\mathrm{cf}(\alpha) \geq \kappa$, then there is $\beta^* < \mu^*$ such that, for every $\beta^* \leq \beta < \mu^*$, $[M_{\alpha,\beta}]^{<\kappa} \cap M_{\alpha,\beta}$ is cofinal in $([M_{\alpha,\beta}]^{<\kappa}, \subseteq)$;

For $\alpha < \mu^+$, let $M_\alpha = \bigcup_{\beta<\mu^*} M_{\alpha,\beta}$. By (3.3) and (3.4), we have $M_\alpha \prec \mathcal{H}(\chi)$.

(3.6) $\langle M_\alpha \rangle_{\alpha<\mu^+}$ is continuously increasing and $\mu^+ \subseteq \bigcup_{\alpha<\mu^+} M_\alpha$.

Theorem 8. (Theorem 7 in [7], see also the remark after Theorem 7 in [7]) *Suppose that κ is a regular cardinal and $\mu > \kappa$ is such that $\mu^* = \mathrm{cf}(\mu) < \kappa$. If we have $\mathrm{cf}([\lambda]^{<\kappa}, \subseteq) = \lambda$ for cofinally many $\lambda < \mu$ and $\square^{***}_{\kappa,\mu}$ holds, then, for any sufficiently large χ and $x \in \mathcal{H}(\chi)$, there is a (κ,μ)-dominating matrix over x such that*

(3.7) *for $\alpha < \alpha' < \mu^+$ and $\beta < \mu^*$, there is $\beta' < \mu^*$ such that $M_{\alpha,\beta} \subseteq M_{\alpha',\beta'}$.* □

In [7] it is shown that, for any singular cardinal μ and regular cardinal $\kappa < \mu$ such that $\mathrm{cf}([\lambda]^{<\kappa}) \leq \mu$ for all $\lambda < \mu$, Jensen's weak square principle $\square^*_\mu$ implies

$\square^{***}_{\kappa,\mu}$ (Lemma 4 in[7]) and $\square^{***}_{\omega_1,\aleph_\omega}$ does not hold in a model of GCH + Chang's Conjecture for $\aleph_\omega$, i.e. $(\aleph_{\omega+1,}, \aleph_\omega) \twoheadrightarrow (\aleph_1, \aleph_0)$ (Theorem 12 in [7]).

The following is the consequence of Theorem 8 we need in the proof of Theorem 10:

Theorem 9. *Suppose that* $\omega < \mu$, $\mathrm{cf}(\mu) = \omega$, $\mathrm{cf}([\mu]^{\aleph_0}, \subseteq) = \mu^+$ *and* $\mathrm{cf}([\lambda]^{\aleph_0}, \subseteq) = \lambda$ *for cofinally many* $\lambda < \mu$. *If* $\square^{***}_{\aleph_1,\mu}$ *holds then there is a matrix* $\langle A_{\alpha,k}\rangle_{\alpha<\mu^+,k<\omega}$, *such that*

(3.8) $A_{\alpha,k} \in [\mu]^{<\mu}$ *for all* $\alpha < \mu^+$ *and* $k < \omega$;

(3.9) $\langle A_{\alpha,k}\rangle_{k<\omega}$ *is an increasing sequence (with respect to* $\subseteq$*) for all* $\alpha < \mu^+$;

(3.10) $\langle \bigcup_{k<\omega} A_{\alpha,k} : \alpha < \mu^+\rangle$ *is a continuously increasing sequence.*

(3.11) $\bigcup_{\alpha<\mu^+} \bigcup_{k<\omega} [A_{\alpha,k}]^{\aleph_0} = [\mu]^{\aleph_0}$;

(3.12) *for* $\alpha < \mu^+$, *if* $\mathrm{cf}(\alpha) > \omega$, *then* $\bigcup_{k<\omega}[A_{\alpha,k}]^{\aleph_0} = \bigcup_{\gamma<\alpha} \bigcup_{\ell<\omega}[A_{\gamma,\ell}]^{\aleph_0}$.

Proof. Let $\langle c_\alpha : \alpha < \mu^+\rangle$ be such that $x = \{c_\alpha : \alpha < \mu^+\}$ is a cofinal subset of $[\lambda]^{\aleph_0}$. By Theorem 8, there is a (ω_1, μ)-dominating matrix $\langle M_{\alpha,n}\rangle_{\alpha<\mu^+,n<\omega}$ over x with (3.7). Let

$$A_{\alpha,k} = \mu \cap M_{\alpha,k}$$

for $\alpha < \mu^+$ and $k < \omega$. We claim that these $A_{\alpha,k}$'s satisfy the conditions (3.8) to (3.12).

(3.8) follows from (3.3); (3.9) from (3.4). (3.10) follows from (3.6).

To show (3.11), suppose that $a \in [\mu]^{\aleph_0}$. Let $\beta < \mu^+$ be such that $a \subseteq c_\beta$ and let $\alpha^* < \mu^+$, and $k^* < \omega$ be such that $\beta \in M_{\alpha^*,k^*}$ — we can find such α^* and k^*, by (3.6). Since $\langle c_\alpha : \alpha < \mu^+\rangle \in M_{\alpha^*,k^*}$, we have $c_\beta \in M_{\alpha^*,k^*}$ and thus $a \subseteq c_\beta \subseteq M_{\alpha^*,k^*}$. It follows that $a \in [A_{\alpha^*,k^*}]^{\aleph_0}$.

To show (3.12), assume that $\alpha < \mu^+$ and $\mathrm{cf}(\alpha) > \omega$. First, suppose that $a \in \bigcup_{k<\omega}[A_{\alpha,k}]^{\aleph_0}$. By (3.4) and (3.5), there is $k^* < \omega$ and $c \in [M_{\alpha,k^*}]^{\aleph_0} \cap M_{\alpha,k^*}$ such that $a \subseteq c$. By the first part of (3.6), there is $\alpha^* < \alpha$ such that $c \in M_{\alpha^*}$. Let $k^{**} < \omega$ be such that $c \in M_{\alpha^*,k^{**}}$. Then $a \subseteq c \subseteq M_{\alpha^*,k^{**}}$. Hence $a \in [M_{\alpha^*,k^{**}}]^{\aleph_0}$.

Now, suppose $a \in \bigcup_{\gamma<\alpha} \bigcup_{\ell<\omega}[A_{\gamma,\ell}]^{\aleph_0}$, say $a \in [A_{\gamma^*,\ell^*}]^{\aleph_0}$ for some $\gamma^* < \alpha$ and $\ell^* < \omega$. Then by (3.7) there is $m^* < \omega$ such that $M_{\gamma^*,\ell^*} \subseteq M_{\alpha,m^*}$. It follows that $a \in [A_{\alpha,m^*}]^{\aleph_0}$. $\square$ (Theorem 9)

Since $\langle A_{\alpha,k}\rangle_{k<\omega}$ for each $\alpha < \mu^+$ may be replaced by its subsequence, we may assume that $\langle A_{\alpha,k}\rangle_{\alpha<\mu^+,k<\omega}$ satisfies the following strengthening of (3.9):

(3.9)' $\langle A_{\alpha,k}\rangle_{k<\omega}$ is an increasing sequence, and $A_{\alpha,0}$ as well as $A_{\alpha,k+1} \setminus A_{\alpha,k}$ for all $k < \omega$ are uncountable for all $\alpha < \mu^+$.

4 $\mathbb{A}^{\mathbb{P}}(\mu)$ at limit cardinals of countable cofinality under the very weak weak square principle

Theorem 10. *Suppose that $\omega < \mu$, $\mathrm{cf}(\mu) = \omega$, $\mathrm{cf}([\mu]^{\aleph_0}, \subseteq) = \mu^+$, $\mathrm{cf}([\lambda]^{\aleph_0}, \subseteq) = \lambda$ for cofinally many $\lambda < \mu$ and $\square^{***}_{\aleph_1,\mu}$. If $\mathbb{P}$ is a proper poset and $\mathbb{A}^{\mathbb{P}}(\kappa)$ holds for all $\kappa < \mu$ then we have $\mathbb{A}^{\mathbb{P}}(\mu)$.*

Proof. Let $\langle A_{\alpha,k}\rangle_{\alpha<\mu^+,k<\omega}$ be as in Theorem 9 with (3.9) replaced by (3.9)'. By induction on $\alpha < \mu^+$, we define $\mathcal{F}_\alpha \subseteq [\mu]^{\aleph_0}$ such that, for all $\alpha < \mu^+$

(4.1) $\mathcal{F}_\alpha$ is ad;

(4.2) $\mathcal{F}_\beta \subseteq \mathcal{F}_\alpha$ for all $\beta < \alpha$;

(4.3) $\mathcal{F}_\alpha \subseteq \bigcup_{k<\omega}[A_{\alpha,k}]^{\aleph_0}$;

(4.4) For all $k < \omega$, $\mathcal{F}_\alpha$ induces a $\mathbb{P}$-indestructible mad family over $A_{\alpha,k}$; I.e.

$$\mathcal{F}_\alpha \restriction A_{\alpha,k} = \{a \cap A_{\alpha,k} \,:\, a \in \mathcal{F}_\alpha\} \cap [A_{\alpha,k}]^{\aleph_0}$$

is a $\mathbb{P}$-indestructible mad family over $A_{\alpha,k}$.

Suppose that $\mathcal{F}_\alpha$, $\alpha < \mu^+$ as above have been constructed. Let $\mathcal{F} = \bigcup_{\alpha<\mu^+} \mathcal{F}_\alpha$.

Claim 10.1. *$\mathcal{F}$ is a $\mathbb{P}$-indestructible mad family over μ.*

$\vdash$ $\mathcal{F}$ is ad by (4.1) and (4.2). Suppose that x is an element of $[\mu]^{\aleph_0}$ in a $\mathbb{P}$-generic extension such that $x \not\in \mathcal{F}$. Since $\mathbb{P}$ is proper, there is $c \in [\mu]^{\aleph_0}$ in the ground model such that $x \subseteq c$. By (3.11), there is $\alpha < \mu^+$ and $k < \omega$ such that $c \in [A_{\alpha,k}]^{\aleph_0}$. So $x \in [A_{\alpha,k}]^{\aleph_0}$ in the $\mathbb{P}$ extension. By (4.4) there is $a \in \mathcal{F}_\alpha$ such that $|\, a \cap A_{\alpha,k} \cap x \,| = \aleph_0$. Thus $\mathcal{F} \cup \{x\}$ is not an ad family. $\dashv$ (Claim 10.1)

For inductive construction of $\mathcal{F}_\alpha$, $\alpha < \mu^+$, let $A_{\alpha,-1} = \emptyset$ for each $\alpha < \mu^+$. Suppose that $\mathcal{F}_\beta$, $\beta < \alpha$ have been constructed in accordance with (4.1) $\sim$ (4.4). We define $\mathcal{F}_\alpha$ as follows:

Case 0: $\alpha = 0$. For each $k \in \omega$, let $\mathcal{F}_{0,k}$ be a $\mathbb{P}$-indestructible mad family on $A_{0,k} \setminus A_{0,k-1}$. This is possible by the assumption of the theorem and since $|A_{0,k}| < \mu$ by (3.8). Let

$$\mathcal{F}_0 = \bigcup_{k<\omega} \mathcal{F}_{0,k}.$$

Then $\mathcal{F}_0$ satisfies (4.3) and (4.4).

Case 1: $\mathrm{cf}(\alpha) > \omega$. Let $\mathcal{F}_\alpha = \bigcup_{\beta<\alpha} \mathcal{F}_\beta$. Then $\mathcal{F}_\alpha$ satisfies (4.2). By (3.12) we also have

$$\mathcal{F}_\alpha \subseteq \bigcup_{\beta<\alpha} \bigcup_{k<\omega} [A_{\beta,k}]^{\aleph_0} = \bigcup_{k<\omega} [A_{\alpha,k}]^{\aleph_0}.$$

Thus $\mathcal{F}_\alpha$ satisfies (4.3). It also satisfies (iii):

Claim 10.2. *$\mathcal{F}_\alpha$ is a $\mathbb{P}$-indestructible mad family over $A_{\alpha,k}$ for all $k < \omega$.*

$\vdash$ Similarly to the proof of Theorem 5. $\dashv$ (Claim 10.2)

Case 2: $\alpha = \beta + 1$. For each $k, \ell < \omega$, let

$$B_{k,\ell} = \left(A_{\alpha,k} \setminus A_{\alpha,k-1}\right) \cap A_{\beta,\ell}$$

and

$$\mathcal{B}_k = \bigcup \{\mathcal{F}_\beta \restriction B_{k,\ell} \,:\, \ell < \omega\}.$$

Each $\mathcal{F}_\beta \restriction B_{k,\ell}$ is a $\mathbb{P}$-indestructible mad family on $B_{k,\ell}$ by (4.4) for $\mathcal{F}_\beta$ and Lemma 2, (1). Hence by Lemma 7, there is $\mathcal{F}_{\alpha,k} \subseteq [A_{\alpha,k}]^{\aleph_0}$ for each $k < \omega$ such that $\mathcal{F}_{\alpha,k} \cup \mathcal{B}_k$ is a $\mathbb{P}$-indestructible mad family on $A_{\alpha,k}$. Let

$$\mathcal{F}_\alpha = \mathcal{F}_\beta \cup \bigcup_{k<\omega} \mathcal{F}_{\alpha,k}.$$

Case 3: $\mathrm{cf}(\alpha) = \omega$. Let $\langle \alpha_n \,:\, n \in \omega \rangle$ be an increasing sequence of ordinals below α such that $\lim_{n<\omega} \alpha_n = \alpha$. Let $\{C_\ell \,:\, \ell < \omega\}$ be an enumeration of $\{A_{\alpha_n,k} \,:\, n, k < \omega\}$ and let

$$\mathcal{F}_\alpha^- = \bigcup_{\beta<\alpha} \mathcal{F}_\beta = \bigcup_{n<\omega} \mathcal{F}_{\alpha_n}.$$

For each $k, \ell < \omega$, let $B_{k,\ell} = A_{\alpha,k} \setminus A_{\alpha,k-1} \cap C_\ell$. By (4.4) for $\mathcal{F}_{\alpha_n}$, $n < \omega$ and Lemma 2, $\mathcal{F}_\alpha^- \restriction B_{k,\ell}$ is a $\mathbb{P}$-indestructible mad family. Hence by Lemma 7, there is a $\mathcal{B}_k$, such that

$\mathcal{F}_{\alpha,k} = \mathcal{B}_k \cup \bigcup_{\ell<\omega} \mathcal{F}_\alpha^- \restriction B_{k,\ell}$

is a $\mathbb{P}$-indestructible mad family. Then

$\mathcal{F}_\alpha = \bigcup_{k<\omega} \mathcal{F}_{\alpha,k}$

is as desired. □ (Theorem 10)

It is easy to see that

(4.5) $\mathrm{cf}([\kappa^+]^{\aleph_0}, \subseteq) = \max\{\kappa^+, \mathrm{cf}([\kappa]^{\aleph_0}, \subseteq)\}$ for all cardinal κ and

(4.6) $\mathrm{cf}([\kappa]^{\aleph_0}, \subseteq) = \sup(\{\kappa\} \cup \{\mathrm{cf}([\lambda]^{\aleph_0}, \subseteq) : \lambda < \kappa\})$ for all limit cardinal of cofinality $> \omega$.

Hence if $\mathrm{cf}([\mu]^{\aleph_0}, \subseteq) = \mu^+$ for all limit cardinal of cofinality ω, we have $\mathrm{cf}([\kappa]^{\aleph_0}, \subseteq) = \kappa$ for all cardinal of cofinality $> \omega$.

Corollary 11. *Assume that* $\mathrm{cf}([\mu]^{\aleph_0}, \subseteq) = \mu^+$ *and* $\square^{***}_{\aleph_1,\mu}$ *holds for all limit cardinals* $> \omega$ *of countable cofinality. If* $\mathbb{P}$ *is a proper poset, then* $\mathbb{A}^{\mathbb{P}}(\omega)$ *holds if and only if* $\mathbb{A}^{\mathbb{P}}(\kappa)$ *holds for some/any cardinal* κ.

Proof. By Lemma 2, Theorem 5 and Theorem 10. Note that Theorem 10 is applicable here by the remark above. □ (Corollary 11)

5 Almost disjoint number for uncountable supports

The construction of mad families on large underlying sets given in the proofs of the previous sections are also quite optimal concerning the possible minimal size of the mad families.

Note first that an ad family $\mathcal{F}$ is mad if and only if it is hitting where a family $\mathcal{F} \subseteq [S]^{\aleph_0}$ is *hitting* if, for any $x \in [S]^{\aleph_0}$, there is an $a \in \mathcal{F}$ such that $|\, x \cap a\,| = \aleph_0$.

The authors were informed about the following lemma by I. Juhász who learned it from P. Nykos.

Lemma 12. (Baumgartner) *For any cardinal* κ *the minimal possible size of a hitting family* $\subseteq [\kappa]^{\aleph_0}$ *is equal to* $\mathrm{cf}([\kappa]^{\aleph_0}, \subseteq)$.

Proof. Let λ be the minimal possible cardinality of a hitting family in $[\kappa]^{\aleph_0}$. The inequality $\lambda \leq \mathrm{cf}([\kappa]^{\aleph_0}, \subseteq)$ is clear. To prove $\lambda \geq \mathrm{cf}([\kappa]^{\aleph_0}, \subseteq)$ consider $T = {}^{\omega>}\kappa$ as a κ-ary tree of height ω. We have $|T| = \kappa$. So let $\mathcal{F} \subseteq [T]^{\aleph_0}$ be a hitting family of cardinality λ.

For $x \in [\kappa]^{\aleph_0}$, let $f : \omega \to x$ be an onto mapping. Let $B(f) = \{f \restriction n : n \in \omega\} \in [T]^{\aleph_0}$. Since $\mathcal{F}$ is hitting there is some $a \in \mathcal{F}$ such that $B(f) \cap a$ is infinite. It follows that $\bigcup\{\mathrm{rng}(t) : t \in a\} \supseteq x$. This shows that $\{\bigcup\{\mathrm{rng}(t) : t \in a\} : a \in \mathcal{F}\}$ is cofinal in $[\kappa]^{\aleph_0}$. Thus we have

$$|\mathcal{F}| \geq |\bigcup\{\{\mathrm{rng}(t) : t \in a\} : a \in \mathcal{F}\}| \geq \mathrm{cf}([\kappa]^{\aleph_0}, \subseteq).$$

$\square$ (Lemma 12)

For a family $\mathcal{S}$ of countable sets, let $\mathfrak{a}(\mathcal{S})$ be the minimal size of a maximal pairwise almost disjoint family $\subseteq \mathcal{S}$. Thus the usual almost disjoint number $\mathfrak{a}$ can be written as $\mathfrak{a} = \mathfrak{a}([\omega]^{\aleph_0})$.

Theorem 13. (1) *For all $n < \omega$, we have $\mathfrak{a}([\aleph_n]^{\aleph_0}) = \max\{\mathfrak{a}, \aleph_n\}$.*

(2) *Assume that $\mathrm{cf}([\mu]^{\aleph_0}, \subseteq) = \mu^+$ and $\square^{***}_{\aleph_1,\mu}$ holds for all limit cardinals $> \omega$ of countable cofinality. Then $\mathfrak{a}([\kappa]^{\aleph_0}) = \max\{\mathfrak{a}, \kappa\}$ for any cardinal κ of cofinality $> \omega$ and $\mathfrak{a}([\mu]^{\aleph_0}) = \max\{\mathfrak{a}, \mu^+\}$ for any cardinal μ of countable cofinality.*

Proof. Starting from a mad family $\mathcal{F}$ of size $\mathfrak{a}$, we can construct $\mathbb{P}$-indestructible mad families on all $\kappa > \omega$ using the constructions of Theorem 5 and Theorem 10, e.g. for the trivial poset $\mathbb{P} = \{\mathbb{1}_{\mathbb{P}}\}$. We can check easily that the mad families obtained thus have the needed minimal cardinality. $\square$ (Theorem 13)

References

[1] T. Bartoszyński and H. Judah, *Set Theory: on the structure of the real line*, A K Peters, (1995).

[2] A. Blass, Combinatorial cardinal characteristics of the continuum, in: Handbook of Set Theory (A. Kanamori et al., eds.), to appear.

[3] J. Brendle, Around splitting and reaping, Commentationes Mathematicae Universitatis Carolinae 39 (1998), 269-279.

[4] J. Brendle, Mob families and mad families, Arch. Math. Logic 37 (1998), 183–197.

[5] J. Brendle and S. Yatabe, Forcing indestructibility of MAD families, Annals of Pure and Applied Logic 132 (2005) 271–312.

[6] S. Fuchino, S. Geschke and L. Soukup, How to drive our families mad, submitted.

[7] S. Fuchino and L. Soukup, More set-theory around the weak Freese-Nation property, Fundamenta Matematicae 154 (1997), 159–176.

[8] M. Hrušák, MAD families and the rationals, Comment. Math. Univ. Carolinae 42,2 (2001), 343–350.

[9] M. Hrušák and S. G. Ferreira, Ordering MAD families a la Katětov, The Journal of Symbolic Logic, Volume 68, Number 4, Dec. 2003.

[10] T. Jech, *Set theory, The Third Millennium Edition*, Springer-Verlag (2002).

[11] K. Kunen, *Set Theory*, North-Holland (1980).

[12] T. Leathrum, A special class of almost disjoint families, J. Symb. Logic 60 (1995), 879–891.

[13] J. Steprāns, Combinatorial consequences of adding Cohen reals, in: H. Judah, ed., Set theory of the reals, Israel Math. Conf. Proc., vol. 6, (1993), 583–617.

Sakaé Fuchino

Dept. of Natural Science and Mathematics
College of Engineering, Chubu University, Kasugai Aichi 487-8501 Japan.
`fuchino@isc.chubu.ac.jp`

Stefan Geschke

Department of Mathematics, Boise State University,
Boise, Idaho 83725-1555 U.S.A.
`geschke@math.boisestate.edu`

Lajos Soukup

Alfréd Rényi Institute of Mathematics, Budapest, Hungary.
`soukup@renyi.hu`