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Topological properties of products of ordinals

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Subspaces of regular (T_2) topological spaces are also regular (T_2), moreover product spaces of arbitrary many regular (T_2) spaces are also regular (T_2). This says that the properties of "regular" and "T_2" are stable. However, the product space $\omega_1 \times (\omega_1 + 1)$ of the ordinals ω_1 and $\omega_1 + 1$ is a non-normal subspace of the compact space $(\omega_1 + 1)^2$, where a space X is *normal* if every disjoint pair F and H of closed sets are separated by disjoint open sets U and V, that is, U and V are disjoint open sets, $F \subset U$ and $H \subset V$. Since ordinals as well as compact spaces are normal, this says that the property "normal" is not stable. Spaces are assumed to be regular T_1.

We have seen that products of ordinals provide a fairly comprehensive store of basic counterexamples deliminating normality, countable paracompactness and closely related properties. In this note, we discuss about the following properties of products of ordinals presenting some open problems.

Definition 1 *A space X is countably paracompact (countably metacompact) if for every decreasing sequence $\{F_n : n \in \omega\}$ of closed sets with $\bigcap_{n\in\omega} F_n = \emptyset$, there exists a sequence $\{U_n : n \in \omega\}$ of open sets with $F_n \subset U_n$ for each $n \in \omega$ such that $\bigcap_{n\in\omega} \mathrm{Cl}U_n = \emptyset$ ($\bigcap_{n\in\omega} U_n = \emptyset$).*

A space X is subnormal if every disjoint pair F and H of closed sets are separated by disjoint G_δ-sets U and V (i.e., $U = \bigcap_{n\in\omega} U_n$ and $V = \bigcap_{n\in\omega} V_n$ for some open sets U_n's and V_n's).

A space X is κ-normal if every disjoint pair F and H of regular closed sets (i.e., $F = \mathrm{Cl}(\mathrm{Int}F)$ and $H = \mathrm{Cl}(\mathrm{Int}H)$) are separated by disjoint open sets U and V.

A space X is strongly zero-dimensional if every disjoint pair F and H of zero-sets (i.e., $F = f^{-1}[\{0\}]$ and $H = h^{-1}[\{0\}]$ for some real valued continuous maps f and h on X) are separated by disjoint clopen (= closed and open) sets U and V. Note that disjoint zero-sets are necessarily separated by disjoint open sets.

$\alpha, \beta, \gamma, \ldots$ stand for ordinals with the usual order topology. For simlicity, we mainly focus on subspaces of ω_1, ω_1^2, ω_1^3..... Some of results listed below can (but some of them cannot) be generalized for larger ordinals, details are shown in papers listed in the references.

A subset of ω_1 is *stationary* if it intersects all closed unbounded (club) subsets of ω_1. We frequently use the Pressing Down Lemma:

The Pressing Down Lemma (PDL) *Let X be a stationary subset of ω_1 and $f : X \to \omega_1$ a regressive function, that is, $f(\alpha) < \alpha$ for each $\alpha \in X$. Then there are a stationary subset $X' \subset X$ and a $\gamma < \omega_1$ such that $f(\alpha) = \gamma$ for each $\alpha \in X'$.*

It is well-known that ω_1^2 is normal. This fact is proved by using the PDL. First we conjectured that $A_0 \times A_1$ is normal whenever A_0 and A_1 are subspaces of ω_1. However this conjecture was false:

Theorem 2 [10] *For subspaces A_0 and A_1 of ω_1, $A_0 \times A_1$ is normal iff $A_0 \times A_1$ is countably paracompact iff A_0 or A_1 is non-stationary, or $A_0 \cap A_1$ is stationary.*

Since there are disjont stationary subsets A_0 and A_1 of ω_1, the product $A_0 \times A_1$ of such subsets is neither normal nor countably paracompact.

Countable metacompactness is known as to very closedly related notion of countable paracompactness. Obviously, countably paracompact spaces are countably metacompact and it is well-known that in normal spaces, countable metacompactness is equivalent to countable paracompactness. In this line, we had a big difference between countable metacompactness and countable paracompactness :

Theorem 3 [9] *All subspaces of ω_1^2 are countably metacompact, therefore all normal subspaces of ω_1^2 are countably paracompact.*

It is natural to ask:

Problem A *Are all countably paracompact subspaces of ω_1^2 normal?*

An partial positive answer was given:

Theorem 4 [8] *Assuming $V = L$, all countably paracompact subspaces of ω_1^2 are normal.*

However, Problem A still remains open.

Obviously normality implies subnormality and κ-normality, also it is known that strong zero-dimensionality and normality are not compatible but are very closedly related properties. So it is natural to ask whether normality can be replaced by these properties in Theorem 2. However we had unexpected results:

Theorem 5

1. *All subspaces of ω_1^2 are subnormal* [7].
2. *$A_0 \times A_1$ is κ-normal whenever A_0 and A_1 are subspaces of ω_1* [4].
3. *$A_0 \times A_1$ is strongly zero-dimensional whenever A_0 and A_1 are subspaces of ω_1* [unpublished work with Terasawa].

Then it is also natural to ask whether above results are extended for finite products, that is,

1. Are all subspaces of ω_1^n subnormal for every $n \in \omega$?
2. Is $\prod_{k<n} A_k$ κ-normal whenever A_k is a subspace of ω_1 for each $k < n$ with $n \in \omega$?
3. Is $\prod_{k<n} A_k$ strongly zero-dimensional whenever A_k is a subspace of ω_1 for each $k < n$ with $n \in \omega$?

We got an expected positive answer for 3:

Theorem 6 [1] *$\prod_{k<n} A_k$ is strongly zero-dimensional whenever A_k is a subspace of ω_1 for each $k < n$ with $n \in \omega$.*

However for 1, We got an unexpected answer:

Theorem 7 [7] *There is a subspace X of ω_1^3 which is not subnormal.*

Indeed, let

$$X = \{\langle \alpha, \beta, \gamma \rangle \in \omega_1^3 : \alpha \leq \beta \leq \gamma\} \setminus \{\langle \alpha, \alpha, \alpha \rangle : \alpha < \omega_1\}.$$

Then

$$F = \{\langle \alpha, \beta, \gamma \rangle \in \omega_1^3 : \alpha = \beta < \gamma\}$$

and

$$H = \{\langle \alpha, \beta, \gamma \rangle \in \omega_1^3 : \alpha < \beta = \gamma\}$$

are disjoint closed sets which cannot be separated by disjoint G_δ sets.

In some special cases, κ-normality and strong zero-dimensionality hold for infinite products:

Theorem 8 *If α_λ is an ordinal for each $\lambda \in \Lambda$, then $\prod_{\lambda \in \Lambda} \alpha_\lambda$ is κ-normal* [5] *and strongly zero-dimensional* [6].

In [5], Kalantan and Szeptycki used elementary submodel techniques to prove κ-normality of $\prod_{\lambda \in \Lambda} \alpha_\lambda$. In [6], analogeous proofs of both κ-normality and strong zero-dimensionality of $\prod_{\lambda \in \Lambda} \alpha_\lambda$ without using elementary submodel techniques were given. Since 3 is true, so naturally I conjectured that the answer of 2 is also true, however we had an unexpected result:

Theorem 9 [2] *There are subspaces A_0, A_1 and A_2 of ω_1 such that $A_0 \times A_1 \times A_2$ is not κ-normal.*

Indeed, let A_0, A_1 and A_2 be stationary subspaces of ω_1 having pairwise non-stationary intersection. Then in $X = A_0 \times A_1 \times A_2$,

$$F = \mathrm{Cl}\{\langle \alpha, \beta, \gamma \rangle \in X : \alpha > \beta, \beta < \gamma\}$$

and

$$H = \mathrm{Cl}\{\langle \alpha, \beta, \gamma \rangle \in X : \alpha < \beta, \beta > \gamma\}$$

are disjoint regular closed sets which cannot be separated by disjoint open sets. The details are very similar to those of Theorem 7.

The following still remains open:

Problem B *Is $\prod_{k\in\omega} A_k$ strongly zero-dimensional whenever A_k is a subspace of ω_1 for each $k \in \omega$?*

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