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DOWKER 空間の二つの構成法
(RUDIN と BALOGH の DOWKER 空間)

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1. INTRODUCTION

In [2], Dowker proved that if a topological space $\mathfrak{X}$ is Hausdorff and normal, $\mathfrak{X}$ is countably paracompact iff $\mathfrak{X} \times [0,1]$ is normal. Moreover, he asked if a Hausdorff normal space is countably paracompact.

The first discovery of its counterexample is due to Rudin in [2]. She proved that if Suslin Hypothesis fails, then there exists a Hausdorff normal space which is not countably paracompact. A Hausdorff normal space which is not countably paracompact is called a *Dowker space*. Her Dowker space is first countable and of size $\aleph_1$. In [6], she asked questions as follows. (All of these questions are asked "*from only* ZFC*?*")

(1) Does there exist a Dowker space of size $\aleph_1$?
(2) Does there exist a first countable Dowker space?
(3) Does there exist a first countable Dowker space of size $\aleph_1$?

Three of them has been still unknown. The best known ZFC-example of a Dowker space is of size $\min\{2^{\aleph_0}, \aleph_{\omega+1}\}$ by combining of results due to Balogh [1] and Kojiman-Shelah [3]. (It should be note here that the first discovery of a ZFC-example of a Dowker space is also due to Rudin in [6].)

In this note, we summarize two constructions of a Dowker space: Rudin's one and Balogh's one. The following is the key theorem to introduce that our constructions are Dowker.

Theorem 1.1 (Dowker [2]). *Suppose that $\mathfrak{X}$ is a Hausdorff normal space. The following are equivalent.*

(D0): *$\mathfrak{X}$ is not countably paracompact.*
(D1): *There exists a sequence $\langle C_n; n \in \omega \rangle$ of closed subsets of $\mathfrak{X}$ such that*
- $C_{n+1} \subseteq C_n$ *for every* $n \in \omega$,
- $\bigcap_{n\in\omega} C_n = \emptyset$,
- *for every sequence $\langle U_n; n \in \omega \rangle$ of open subsets of $\mathfrak{X}$ such that $C_n \subseteq U_n$ for all $n \in \omega$, $\bigcup_{n\in\omega} U_n \neq \emptyset$.*

(D2): *There exists a sequence $\langle U_n; n \in \omega \rangle$ of open subsets of $\mathfrak{X}$ such that*
- $U_{n+1} \supseteq U_n$ *for every* $n \in \omega$,
- $\bigcup_{n\in\omega} U_n = \mathfrak{X}$,
- *for every sequence $\langle C_n; n \in \omega \rangle$ of closed subsets of $\mathfrak{X}$ such that $C_n \subseteq U_n$ for all $n \in \omega$, $\bigcup_{n\in\omega} C_n \neq \mathfrak{X}$.*

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2. Rudin's Dowker space

In this section, we summarize a construction of Rudin's Dowker space in [5]. It have to note that Suslin Hypothesis is independent from ZFC.

She constructed a Dowker space as follows. Suppose that a Suslin line exists. At first, a Suslin tree is constructed from its Suslin line by the standard method. Next, a topological space is defined using its Suslin tree and it is proved that it is Dowker. Here we will see her construction by using modern terminologies: the density of forcing notions and maximal antichains.

Theorem 2.1 (Rudin [5]). *If Suslin's Hypothesis fails, then there exists a first countable Dowker space of size* $\aleph_1$.

Proof. Suppose that T is a Suslin tree. For a countable ordinal α, let T_α be the set of nodes in T with level α, and for such an α and $t \in T$ with level larger than α, let $t \restriction \alpha$ be the nodes with α-th level below t in T. For each $t \in T$, we write $\mathsf{lv}(t)$ as the level of t.

To define our topological space, for each $\alpha \in \omega_1 \cap \mathsf{Lim}$, we fix a function $\pi_\alpha : T_\alpha \to [T_\alpha]^{\aleph_0}$ such that

- for any $t \in T_\alpha$ and $\beta \in \alpha$, the set $\{s \in \pi_\alpha(t); t\restriction\beta <_T s\}$ is infinite,
- for any distinct nodes t and t' in T_α, $\pi_\alpha(t) \cap \pi_\alpha(t') = \emptyset$.

Let $\mathfrak{X} := T \times \omega$. We define a neighborhood of the point $\langle t, n\rangle$ of $\mathfrak{X}$ by induction on n and $\mathsf{lv}(t)$ as follows.

(I): If $\mathsf{lv}(t) \notin \mathsf{Lim}$, then a neighborhood of $\langle t, n\rangle$ is $\langle\langle t, n\rangle\rangle$.

(II): If $\mathsf{lv}(t) \in \mathsf{Lim}$ and $n = 0$, then the neighborhood of $\langle t, n\rangle$ is the set

$$(\{s \in T; s <_T t \ \&\ \beta < \mathsf{lv}(s)\} \times \{0\}) \cup \{\langle t, 0\rangle\},$$

for some $\beta \in \mathsf{lv}(t)$.

(III): If $\mathsf{lv}(T) \in \mathsf{Lim}$ and $n > 0$, then a neighborhood of $\langle t, n\rangle$ is a union of
- neighborhood of points in the set $(\pi_\alpha(t) \setminus \sigma) \times \{n-1\}$,
- neighborhoods of points in the set $\{s \in T; s <_T t \ \&\ \beta < \mathsf{lv}(s)\} \times \{n\}$, and
- $\{\langle t, n\rangle\}$,

for some $\sigma \in [\pi_\alpha(t)]^{<\aleph_0}$ and $\beta \in \mathsf{lv}(t)$.

By the definition, $\mathfrak{X}$ is first countable and of size $\aleph_1$.

The next proposition lists types of open and closed subsets of $\mathfrak{X}$ we will use in the proof below. We omit the proof here.

Proposition 2.2. (1) *$\mathfrak{X}$ is T_1.*
(2) *The set $T \times n$ is open for each $n \in \omega$.*
(3) *The set $\bigcup_{\alpha \leq \delta} T_\alpha \times \omega$ is clopen for each $\delta \in \omega_1$.*
(4) *The set $\{s \in T; s <_T t \ \&\ \beta < \mathsf{lv}(s)\} \times \{n\}$ is closed for each $t \in T$ with $\mathsf{lv}(t) \in \mathsf{Lim}$, $\beta \in \mathsf{lv}(t)$ and $n \in \omega$.*
(5) *The set $(\pi_{\mathsf{lv}(t)}(t) \setminus \sigma) \times \{n\}$ is close for each $t \in T$ with $\mathsf{lv}(t) \in \mathsf{Lim}$, $\sigma \in [\pi_{\mathsf{lv}(t)}(t)]^{<\aleph_0}$ and $n \in \omega$.* $\dashv_{2.2}$

The next proposition can be shown from the definition of the topology. We omit the proof again.

Proposition 2.3. *For every $t \in T$ with a limit level, $\beta \in \mathsf{lv}(t)$, $n \in \omega \setminus \{0\}$ and $m \in n$, every neighborhood of the point $\langle t, n\rangle$ has a point $\langle s, m\rangle$ such that $t\restriction\beta <_T s$.* ⊣2.3

Lemma 2.4. *$\mathfrak{X}$ satisfies* **(D1)**.

Proof of Lemma 2.4. Let $C_n := T \times (\omega \setminus n)$ for each $n \in \omega$. Then $C_{n+1} \subseteq C_n$ for any $n \in \omega$ and $\bigcap_{n\in\omega} C_n = \emptyset$. We show that the sequence $\langle C_n; n \in \omega\rangle$ is a witness for **(D1)**.

Let $\langle U_n; n \in \omega\rangle$ be a sequence of open subsets of $\mathfrak{X}$ such that $C_n \subseteq U_n$.

Claim 2.5. *For every $n \in \omega$, the set*

$$\mathcal{D}_n := \{t \in T; \{s \in T; t <_T s\} \times \{0\} \subseteq U_n\}$$

is dense in T.

Proof of Lemma 2.5. Assume not, i.e. there exists $t \in T$ such that for any $s >_T t$, we can find $u >_T s$ such that $\langle u, 0\rangle \notin U_n$. Then there is a sequence $\langle \delta_i, A_i; i \in \omega\rangle$ such that

- δ_i is a countable ordinal and $\delta_i < \delta_{i+1}$ for every $i \in \omega$,
- A_i is a maximal antichain above t for every $i \in \omega$, and
- for any member s in A_i, $\delta_i \leq \mathsf{lv}(s) < \delta_{i+1}$ and $\langle s, 0\rangle \notin U_n$.

Let $\delta := \sup_{i\in\omega} \delta_i$. Since $C_n \subseteq U_n$, there exists $u \in T$ such that $\mathsf{lv}(u) = \delta$ and $\langle u, 0\rangle \in U_n$ by Proposition 2.3. However then we can show that $\langle u, 0\rangle$ is in the closure of $\mathfrak{X} \setminus U_n$, which is just $\mathfrak{X} \setminus U_n$. This is a contradiction.

For the proof that the point $\langle u, 0\rangle$ belongs to the closure of $\mathfrak{X} \setminus U_n$, let N be a neighborhood of $\langle u, 0\rangle$, say

$$N := (\{s \in T; s <_T t \ \&\ \beta < \mathsf{lv}(s)\} \times \{0\}) \cup \{\langle t, 0\rangle\}$$

for some $\beta \in \mathsf{lv}(u) = \delta$. Let $i \in \omega$ be such that $\beta \leq \delta_i$. Then there is $s \in A_i$ which is compatible with u in T, that is, $s <_T u$. Then the point $\langle s, 0\rangle$ is a common point of both N and $\mathfrak{X} \setminus U_n$, i.e. $N \cap (\mathfrak{X} \setminus U_n) \neq \emptyset$. ⊣2.5

For each $n \in \omega$, let $B_n \subseteq \mathcal{D}_n$ be a maximal antichain in T. Take $\gamma \in \omega_1 \cap \mathsf{Lim}$ such that for any $t \in \bigcup_{n\in\omega} B_n$, $\mathsf{lv}(t) < \gamma$. Then for each $n \in \omega$, $T_\gamma \times \{0\} \subseteq U_n$. Therefore $\bigcap_{n\in\omega} U_n \neq \emptyset$. ⊣2.4

Lemma 2.6. *$\mathfrak{X}$ is normal.*

Proof of Lemma 2.6. Suppose that H and K be disjoint closed subsets of $\mathfrak{X}$. For each $n \in \omega$, let

$$H_n := \{t \in T; \langle t, n\rangle \in H\}$$

and

$$K_n := \{t \in T; \langle t, n\rangle \in K\}.$$

Claim 2.7. *Let m and n be in ω. Then the set*

$$\mathcal{E}_{m,n} := \{t \in T; \{s \in T; t <_T s\} \textit{ is disjoint from } H_m \textit{ or } K_n\}$$

is dense in T.

Proof of Claim 2.7. Assume not, i.e. there exists $t \in T$ such that for any $s >_T t$, we can find $u >_T s$ such that $u \in H_m \cap K_n$. Then there is a sequence $\langle \delta_i, A_i; i \in \omega \rangle$ such that

- δ_i is a countable ordinal and $\delta_i < \delta_{i+1}$ for every $i \in \omega$,
- A_i is a maximal antichain above t for every $i \in \omega$, and
- for any member s in A_i, $\delta_i \le \mathsf{lv}(s) < \delta_{i+1}$ and $s \in H_m \cap K_n$.

Let $\delta := \sup_{i \in \omega} \delta_i$. Then we observe that $\{s \in T_\delta; t <_T s\} \subseteq H_m \cap K_n$ because both H and K are closed. Since H and K are disjoint, $m \neq n$.

Without loss of generality, we may assume that $m < n$. Let $s \in T_\delta$ such that $t <_T s$. Then $\langle s, n \rangle \in K$. By Proposition 2.3 and the above observation, $\langle s, n \rangle \in H$ which is a contradiction. ⊣2.7

Therefore for each $n \in \omega$, the set

$$\mathcal{E}'_n := \{t \in T; \{s \in T; t <_T s\} \times (n+1) \text{ is disjoint from } H \text{ or } K\}$$

is also dense in T. There exists $\delta \in \omega_1$ such that for every $n \in \omega$, $\mathcal{E}'_n$ has a maximal antichain contained in the set $\bigcup_{\alpha \le \delta} T_\alpha$. Let

$$H' := H \cap \left(\bigcup_{\alpha \le \delta} T_\alpha \times \omega \right)$$

and

$$K' := K \cap \left(\bigcup_{\alpha \le \delta} T_\alpha \times \omega \right).$$

Let $\{p_i; i \in \omega\}$ enumerate the set $\bigcup_{\alpha \le \delta} T_\alpha \times \omega$, and say $p_i := \langle t_i, n_i \rangle$.

Recursively choose closed subsets M_i and N_i of $\mathfrak{X}$, for each $i \in \omega$ as follows.

Case 1: Suppose that $p_i \notin K \cup \bigcup_{j \in i} N_j$.

(a): If $\mathsf{lv}(t_i) \notin \mathsf{Lim}$, then let $M_i := \{p_i\}$ and $N_i = \emptyset$.

(b): If $\mathsf{lv}(t_i) \in \mathsf{Lim}$ and $n_i = 0$, then since $K \cup \bigcup_{j \in i} N_j$ is closed, we can find $\beta_i \in \mathsf{lv}(t_i)$ such that

$$((\{s \in T; s <_T t_i \ \&\ \beta_i < \mathsf{lv}(s)\} \times \{0\}) \cup \{p_i\}) \cap \left(K \cup \bigcup_{j \in i} N_j \right) = \emptyset.$$

Then let

$$M_i := (\{s \in T; s <_T t_i \ \&\ \beta_i < \mathsf{lv}(s)\} \times \{0\}) \cup \{p_i\}$$

and $N_i = \emptyset$.

(c): If $\mathsf{lv}(t_i) \in \mathsf{Lim}$ and $n_i > 0$, then since $K \cup \bigcup_{j \in i} N_j$ is closed, we can find $\beta_i \in \mathsf{lv}(t_i)$ and $\sigma_i \in \left[\pi_{\mathsf{lv}(t_i)}(t_i)\right]^{<\aleph_0}$ such that there exists a neighborhood of p_i disjoint from $K \cup \bigcup_{j \in i} N_j$, which is a union of

- neighborhoods of points in the set $\left(\pi_{\mathsf{lv}(t_i)}(t_i) \setminus \sigma_i\right) \times \{n_i - 1\}$,
- neighborhoods of points in the set $\{s \in T; s <_T t_i \ \&\ \beta_i < \mathsf{lv}(s)\} \times \{n_i\}$ and
- $\{p_i\}$.

Then let

$$M_i := \left(\left(\pi_{\mathsf{lv}(t_i)}(t_i) \setminus \sigma_i\right) \times \{n_i - 1\}\right) \\ \cup \left(\{s \in T; s <_T t_i \ \& \ \beta_i < \mathsf{lv}(s)\} \times \{n_i\}\right) \cup \{p_i\}$$

and $N_i = \emptyset$.

Case 2: Otherwise. Then since H and K are disjoint, $p_i \notin H \cup \bigcup_{j \in i} M_i$. Then we perform as in the case 1 above replacing $K \cup \bigcup_{j\in i} N_i$ to $H \cup \bigcup_{j \in i} M_i$.

Let

$$U' := H \cup \bigcup_{i \in \omega} M_i$$

and

$$V' := K \cup \bigcup_{i \in \omega} N_i.$$

We note that $H' \subseteq U'$, $K' \subseteq V'$, $U' \cap V' = \emptyset$, and both $U'$and V' are open.

Let

$$U := U' \cup \bigcup \Big\{ \{s \in T; t <_T s\} \times (n+1); \\ t \in T_\delta \cap \mathcal{E}'_n \ \& \ (\{s \in T; t <_T s\} \times (n+1)) \cap H \neq \emptyset \Big\}$$

and

$$V := V' \cup \bigcup \Big\{ \{s \in T; t <_T s\} \times (n+1); \\ t \in T_\delta \cap \mathcal{E}'_n \ \& \ (\{s \in T; t <_T s\} \times (n+1)) \cap K \neq \emptyset \Big\}$$

Then $H \subseteq U$, $K \subseteq V$, $U \cap V = \emptyset$, and both U and V are open. ⊣2.6

Since $\mathfrak{X}$ is T_1 and normal, $\mathfrak{X}$ is Hausdorff, therefore $\mathfrak{X}$ is a Dowker space. □

Paul B. Larson asks whether we need the Suslinness of T to introduce it to be Dowker [4].

3. Balogh's Dowker space

In this section, we summarize Balogh's construction of a Dowker space in [1].

Theorem 3.1 (Balogh [1]). *There exists a Dowker space of size continuum.*

Summary of proof. For an infinite cardinal κ, let $\mathbf{B}(\kappa)$ be the statement that there exists a sequence $\langle \mathcal{F}_\alpha; \alpha \in \kappa \rangle$ of subsets of $\mathcal{P}(\kappa)$ such that

(i): each $\mathcal{F}_\alpha$ is closed under finite intersections,

(ii): $\bigcap \mathcal{F}_\alpha = \emptyset$ for all $\alpha \in \kappa$,

(iii): for any disjoint subsets $I$and J of κ, there exists a sequence $\langle F_\alpha; \alpha \in I \cup J\rangle$ such that $F_\alpha \in \mathcal{F}_\alpha$ for each $\alpha \in I \cup J$ and

$$\left(\bigcup_{\alpha \in I} F_\alpha\right) \cap \left(\bigcup_{\beta \in J} F_\beta\right) = \emptyset,$$

(iv): κ is not σ-decomposable, where $I \in \mathcal{P}(\kappa)$ is called σ-decomposable if there exists $f : I \to \omega$ such that for any sequence $\langle F_\alpha; \alpha \in I\rangle$ with $F_\alpha \in \mathcal{F}_\alpha$ and $\alpha \neq \beta$ in I, if $f(\alpha) = f(\beta)$, then $\alpha \notin F_\beta$ and $\beta \notin F_\alpha$.

Balogh proves in his paper that

(1) $\mathbf{B}(2^{\aleph_0})$ holds, and
(2) If $\mathbf{B}(\kappa)$ holds, then there exists a Dowker space of size κ (in fact, his Dowker space is σ-relatively discrete and hereditarily normal).

His construction is as follows. Suppose that $\mathbf{B}(\kappa)$ holds and we take a witness $\langle \mathcal{F}_\alpha; \alpha \in \kappa \rangle$ for $\mathbf{B}(\kappa)$. $\mathfrak{X} := \kappa \times \omega$, and for $\langle \alpha, n \rangle \in \mathfrak{X}$, we define an open neighborhood of $\langle \alpha, n \rangle$ by induction on n as follows. If $n = 0$, then a neighborhood of $\langle \alpha, n \rangle$ is $\{\langle \alpha, n \rangle\}$, and if $n > 0$, then a neighborhood of $\langle \alpha, n \rangle$ is a union of neighborhoods of points in the set $F \times \{n-1\}$ and the singleton $\{\langle \alpha, n \rangle\}$ for some $F \in \mathcal{F}_\alpha$. We can prove that it is a Dowker space. (The property **(i)** guarantees that $\mathfrak{X}$ is a topology (and hence it is σ-relatively discrete by the definition), **(ii)** guarantees that $\mathfrak{X}$ is T_1, **(iii)** guarantees the hereditary normality of $\mathfrak{X}$, and **(iv)** guarantees that $\mathfrak{X}$ satisfies **(D2)**.)

Show only that $\mathfrak{X}$ satisfies **(D2)**.

At first, we show that for each $n \in \omega$ and $I \in \mathcal{P}(\kappa)$ which is not σ-decomposable, the set

$$I^+ := \left\{ \alpha \in I; \langle \alpha, n+1 \rangle \in \overline{I \times \{n\}} \right\}$$

is not σ-decomposable. For such n and I, let $J := I \setminus I^+$. Then for each $\alpha \in J$, there exists $F_\alpha \in \mathcal{F}_\alpha$ such that $F_\alpha \cap I = \emptyset$. Then $\langle F_\alpha; \alpha \in J \rangle$ is a witness that J is σ-decomposable (in fact, 1-decomposable). So if I^+ is σ-decomposable, then $I = I^+ \cup J$ is also σ-decomposable, which is a contradiction.

For $n \in \omega$, let $U_n := \kappa \times (n+1)$, which is open in our topology. Show that the sequence $\langle U_n; n \in \omega \rangle$ is a witness for **(D2)**. Let $\langle C_n; n \in \omega \rangle$ be a sequence of closed subsets of $\mathfrak{X}$ such that $C_n \subseteq U_n$ for all $n \in \omega$ and $\bigcup_{n \in \omega} C_n = \mathfrak{X}$. Then we can find $m \in \omega$ such that the set

$$\{\alpha \in \kappa; \langle \alpha, 0 \rangle \in C_m\}$$

is not σ-decomposable by the property **(iv)**. Then we can conclude that $C_n \not\subseteq U_n$ by the above observation. □

The author would like to ask if $\mathbf{B}(\aleph_1)$ holds under ZFC, and what about a general $\mathbf{B}(\kappa)$.

In the last of the note, the author give one construction of a topological space of size $\aleph_1$, which is moreover first countable, under ZFC by modifying Balogh's Dowker space. Unfortunately, it will be observed that it is not a Dowker space.

Theorem 3.2. *There exists a first countable, σ-relatively discrete, Hausdorff space of size $\aleph_1$ such that for any closed subsets H and K, if H and K are disjoint, then either H or K is countable.*

Proof. Let $\langle S_n; n \in \omega \rangle$ be a sequence of disjoint stationary subsets of countable ordinals. Let

$$\mathfrak{X} := \bigcup_{n \in \omega} S_n \times \{n\},$$

and define that a subset U of $\mathfrak{X}$ is open iff for every point $\langle \alpha, n \rangle$ in U, if $n > 0$, then there exists $\beta \in \alpha$ such that the set

$$(S_n \cap (\beta, \alpha)) \times \{n-1\}$$

is contained in U. We will prove that this $\mathfrak{X}$ satisfies the statement of the theorem.

From the definition, $\mathfrak{X}$ is first countable, σ-relatively discrete, T_1. To show the rest, we see the property of the closed subset of $\mathfrak{X}$.

Claim. *Assume that H is a closed subset of $\mathfrak{X}$ and $n \in \omega$ satisfies that the set*

$$I_n^H := \{\alpha \in S_n; \langle \alpha, n \rangle \in H\}$$

is uncountable. Then the set I_{n+1}^H contains a club.

Proof of Claim. Suppose that the set $S_{n+1} \setminus I_{n+1}^H$ is stationary. Then for each $\alpha \in S_{n+1} \setminus I_{n+1}^H$, there exists $\beta_\alpha \in \alpha$ such that

$$((S_n \cap (\beta_\alpha, \alpha)) \times \{n\}) \cap H = \emptyset.$$

By Fodor's Theorem, there are a stationary subset S of $S_{n+1} \setminus I_{n+1}^H$ and $\beta \in \omega_1$ such that $\beta_\alpha = \beta$ holds for every $\alpha \in S$. Since I_n^H is uncountable, there exists $\gamma \in I_n^H \setminus (\beta + 1)$ and then we take $\alpha \in S \setminus (\gamma + 1)$. We note that

$$\langle \gamma, n \rangle \in ((S_n \cap (\beta_\alpha, \alpha)) \times \{n\}) \cap H,$$

which is a contradiction. ⊣

From this claim and the argument in the proof of the previous theorem, we notice that $\mathfrak{X}$ satisfies **(D2)**. Moreover we note that if H and K are uncountable closed subsets of $\mathfrak{X}$, then H have to meet K. □

We have to note that the above $\mathfrak{X}$ is *not* regular, hence not normal. In our situation, we can find an $\alpha \in S_0$ and $\langle \beta_n; n \in \omega \setminus \{0\} \rangle$ such that

- $\beta_n \in S_n \cap \alpha$ for every $n \in \omega \setminus \{0\}$,
- $\beta_n < \beta_{n+1}$ for every $n \in \omega \setminus \{0\}$.

Then let $H := \{\langle \alpha, 0 \rangle\}$ and $K := \overline{\{\langle \beta_n, n \rangle ; n \in \omega \setminus \{0\}\}}$. We notice that H and K are disjoint closed subsets and cannot be separated by disjoint open subsets.

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