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HAUSDORFF HYPERSPACES OF EUCLIDEAN SPACES AND THEIR DENSE SUBSPACES

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Here, we introduce the results obtained in the paper [11] and related problems. We consider metric spaces and their hyperspaces endowed with the Hausdorff metric. Specifically, given a metric space $X = \langle X, d\rangle$, we shall denote by $\mathrm{Cld}(X)$ and $\mathrm{Bd}(X)$ the hyperspaces consisting of all nonempty closed sets and of all nonempty bounded closed sets in X respectively and we denote by d_H the Hausdorff metric, which is infinite-valued on $\mathrm{Cld}(X)$ if X is unbounded. When X is compact, the space $\mathrm{Cld}(X)$ $(= \mathrm{Bd}(X))$ is equal to the hyperspace $\exp(X)$ of all nonempty compact sets with the Vietoris topology. Even if X is noncompact, on the space $\exp(X)$, the Hausdorff metric topology coincides with the Vietoris topology. However, in case X is noncompact, these topologies are very different on the spaces $\mathrm{Cld}(X)$ and $\mathrm{Bd}(X)$.

Vietoris hyperspaces $\exp(X)$ have been studied extensively for many years. Among the known results, let us mention the theorem of Curtis and Schori [8] (cf. [13, Chapter 8]), saying that $\exp(X)$ is homeomorphic to $(\cong)$ the Hilbert cube $\mathrm{Q} = [-1,1]^\omega$ if and only if X is a Peano continuum, that is, it is compact, connected and locally connected. Later, Curtis [7] characterized non-compact metric spaces X for which $\exp(X)$ is homeomorphic to the Hilbert cube minus a point $\mathrm{Q}\setminus 0$ $(= \mathrm{Q}\setminus\{0\})$ or the pseudo-interior $\mathrm{s} = (-1,1)^\omega$ of Q.[1] In particular, $\mathrm{Bd}(\mathbb{R}^m) = \exp(\mathbb{R}^m)$ is homeomorphic to $\mathrm{Q}\setminus 0$. For more information concerning Vietoris hyperspaces, we refer to the book of Ilanes and Nadler [10].

It is well known that the hyperspace $\exp(X)$ is an ANR (AR) if and only if X is locally connected (and connected). On the other hand, it is proved in [6] that the space $\mathrm{Bd}(X)$ is an ANR (AR) whenever the metric on X is *almost convex*, that is,

[1]It is well known that s is homeomorphic to the separable Hilbert space ℓ_2.

for every $\alpha > 0$, $\beta > 0$ and for every $x, y \in X$ such that $d(x,y) < \alpha+\beta$, there exists $z \in X$ with $d(x,z) < \alpha$ and $d(z,y) < \beta$. This condition was further weakened in [12], which has turned out to be actually a necessary and sufficient one by Banakh and Voytsitskyy [3]. In the last paper, several equivalent conditions are given, which are too technical to mention them here. We refer to [3] for the details. On the other hand, $\mathrm{Cld}(X)$ is not connected whenever X is a metric space which is not totally bounded. For example, $\mathrm{Cld}(\mathbb{R})$ has $2^{\aleph_0}$ many components.

The completion of a metric space $X = \langle X, d\rangle$ is denoted by $\tilde{X} = \langle \tilde{X}, d\rangle$. Then $\mathrm{Bd}(X)$ can be identified with the subspace of $\mathrm{Bd}(\tilde{X})$, via the isometric embedding $A \mapsto \mathrm{cl}_{\tilde{X}} A$. Thus we shall often write $\mathrm{Bd}(X) \subseteq \mathrm{Bd}(\tilde{X})$, having in mind this identification. In this case, $\mathrm{Bd}(\tilde{X})$ is the completion of $\mathrm{Bd}(X)$. By such a reason, we also consider a dense subspace D of a metric space $X = \langle X, d\rangle$. For each $0 \leqslant k < m$, let

$$\nu_k^m = \{x = (x_i)_{i=1}^m \in \mathbb{R}^m : x_i \in \mathbb{R}\setminus\mathbb{Q} \text{ except for at most } k \text{ many } i\},$$

which is the universal space for completely metrizable subspaces in $\mathbb{R}^m$ of $\dim \leqslant k$. In case $2k+1 < m$, ν_k^m is homeomorphic to the k-dimensional Nöbeling space ν_k^{2k+1}, which is the universal space for all separable completely metrizable spaces. Note that $\nu_0^m = (\mathbb{R}\setminus\mathbb{Q})^m \cong \mathbb{R}\setminus\mathbb{Q}$.

Theorem 1. *Suppose* $\langle m, k\rangle = \langle 1, 0\rangle$ *or* $0 \leqslant k < m-1$. *Then,*

$$\langle \mathrm{Bd}(\mathbb{R}^m), \mathrm{Bd}(\nu_k^m)\rangle \cong \langle \mathrm{Q}\setminus 0, \mathrm{s}\setminus 0\rangle.$$

Consequently, $\mathrm{Bd}(\nu_k^m) \cong \ell_2$.

This can be derived from the following:

Theorem 2. *Let D be a dense G_δ set in $\mathbb{R}^m$ such that $\mathbb{R}^m \setminus D$ is also dense in $\mathbb{R}^m$ and in case $m > 1$ it is assumed that $D = p[D] \times \mathbb{R}$, where $p : \mathbb{R}^m \to \mathbb{R}^{m-1}$ is the projection onto the first $m-1$ coordinates. Then,* $\langle \mathrm{Bd}(\mathbb{R}^m), \mathrm{Bd}(D)\rangle \cong \langle \mathrm{Q}\setminus 0, \mathrm{s}\setminus 0\rangle$.

Question 1. In case $m > 1$, under the only assumption that $D \subseteq \mathbb{R}^m$ is a dense G_δ set and $\mathbb{R}^m \setminus D$ is also dense in $\mathbb{R}^m$, is the pair $\langle \mathrm{Bd}(\mathbb{R}^m), \mathrm{Bd}(D)\rangle$ homeomorphic to $\langle \mathrm{Q}\setminus 0, \mathrm{s}\setminus 0\rangle$? In particular, is the pair $\langle \mathrm{Bd}(\mathbb{R}^m), \mathrm{Bd}(\nu_{m-1}^m)\rangle$ homeomorphic to $\langle \mathrm{Q}\setminus 0, \mathrm{s}\setminus 0\rangle$?

We also consider the following dense subspaces of $\mathrm{Bd}(X)$:

- $\mathrm{Nwd}(X)$ — all nowhere dense closed sets;
- $\mathrm{Perf}(X)$ — all perfect sets;[2]

[2] I.e., completely metrizable closed sets which are dense in itself.

- Cantor(X) — all compact sets homeomorphic to the Cantor set.

In case $X = \mathbb{R}^m$, we can also consider the following subspace:

- $\mathfrak{N}(\mathbb{R}^m)$ — all closed sets of the Lebesgue measure zero.

For these spaces, we have the following:

Theorem 3. *Let $\mathcal{F}$ be one of the following subspaces of* $\mathrm{Bd}(\mathbb{R}^m)$*:*

$$\mathrm{Nwd}(\mathbb{R}^m),\ \mathrm{Perf}(\mathbb{R}^m),\ \mathrm{Cantor}(\mathbb{R}^m)\ \textit{and}\ \mathfrak{N}(\mathbb{R}^m).$$

Then, $\langle \mathrm{Bd}(\mathbb{R}^m), \mathcal{F}\rangle \cong \langle \mathrm{Q}\setminus 0, \mathrm{s}\setminus 0\rangle$, *hence* $\mathcal{F} \cong \ell_2$.

To prove Theorems 2 and 3 above, we adopt the characterization of the pseudo-boundary $\mathrm{Q}\setminus \mathrm{s}$ of the Hilbert cube Q, see [5].

We also study the space $\mathrm{Cld}(\mathbb{R})$. It is very different from the hyperspace $\exp(\mathbb{R})$. It is not hard to see that $\mathrm{Cld}(\mathbb{R})$ has $2^{\aleph_0}$ many components, $\mathrm{Bd}(\mathbb{R})$ is the only separable one and any other component has weight $2^{\aleph_0}$. Applying Toruńczyk's Characterization of Hilbert space [14] (cf. [15]), we can prove

Theorem 4. *Let $\mathcal{H}$ be a nonseparable component of* $\mathrm{Cld}(\mathbb{R})$ *which does not contain* $\mathbb{R}$, $[0,+\infty)$, $(-\infty,0]$. *Then* $\mathcal{H} \cong \ell_2(2^{\aleph_0})$.

Question 2. Does Theorem 4 hold even if $\mathcal{H}$ contains $\mathbb{R}$, $[0,\infty)$ or $(-\infty,0]$?

Question 3. For $m > 1$, is $\mathrm{Cld}(\mathbb{R}^m) \setminus \mathrm{Bd}(\mathbb{R}^m)$ an $\ell_2(2^{\aleph_0})$-manifold?

Now, we consider the subspaces $\mathfrak{N}(\mathbb{R})$, $\mathrm{Nwd}(\mathbb{R})$, $\mathrm{Perf}(\mathbb{R})$ and $\mathrm{Cld}(\mathbb{R}\setminus\mathbb{Q})$ of $\mathrm{Cld}(\mathbb{R})$. Similarly to $\mathrm{Bd}(\mathbb{R})$, it can be shown that those complements are Z_σ-sets in $\mathrm{Cld}(\mathbb{R})$. Due to Negligibility Theorem ([1], [9]), if M is an $\ell_2(2^{\aleph_0})$-manifold and A is a Z_σ-set in M then $M \setminus A \cong M$. Thus, the following follows from Theorem 4:

Corollary 5. *Let $\mathcal{H}$ be a nonseparable component of* $\mathrm{Cld}(\mathbb{R})$ *which does not contain* $\mathbb{R}$, $[0,+\infty)$, $(-\infty,0]$. *Then, the following spaces are homeomorphic to* $\ell_2(2^{\aleph_0})$*:*

$$\mathcal{H}\cap\mathfrak{N}(\mathbb{R}),\ \mathcal{H}\cap\mathrm{Nwd}(\mathbb{R}),\ \mathcal{H}\cap\mathrm{Perf}(\mathbb{R})\ \textit{and}\ \mathcal{H}\cap\mathrm{Cld}(\mathbb{R}\setminus\mathbb{Q}).$$

Borel classes. Given a metric space $\langle X, d\rangle$, let $\langle \tilde{X}, d\rangle$ be its completion. Then, the hyperspace $\mathrm{Bd}(\tilde{X})$ is the completion of the hyperspace $\mathrm{Bd}(X)$. Concerning Borel classes of hyperspaces, the following are also shown in the paper [11]:

(1) $\mathrm{Bd}(X)$ is $F_{\sigma\delta}$ in $\mathrm{Bd}(\tilde{X})$ if X is σ-compact.
(2) $\mathrm{Bd}(X)$ is G_δ in $\mathrm{Bd}(\tilde{X})$ if X is Polish.[3]

[3]I.e., separable and completely metrizable

(3) $\mathrm{Bd}(X)$ is Polish for every Polish space X in which bounded sets are totally bounded.
(4) $\mathrm{Nwd}(X)$ is G_δ in $\mathrm{Bd}(X)$ for every separable metric space X.
(5) $\mathrm{Perf}(X)$ is G_δ in $\mathrm{Bd}(X)$ if X is separable and locally compact.
(6) $\mathrm{Perf}(X)$ is $F_{\sigma\delta}$ in $\mathrm{Bd}(X)$ for every Polish space X.
(7) $\mathrm{Bd}(X)$ is analytic for every analytic metric space X in which bounded sets are totally bounded.

Fix a dense set X in a separable Banach space E which admits the metric d induced from the norm of E. Then $\langle X, d\rangle$ is an almost convex metric space and therefore by a result of [6] the space $\mathrm{Bd}(X)$ is an AR. In case X is G_δ, the space $\mathrm{Bd}(X)$ is completely metrizable by (2). If additionally E is finite-dimensional then $\mathrm{Bd}(X)$ is Polish by (3). In case X is σ-compact, by (1), $\mathrm{Bd}(X)$ is absolutely $F_{\sigma\delta}$.

Remarks. Recently, Banakh and Voytsitskyy [4] proved that the space $\mathrm{Cld}(X)$ (resp. $\mathrm{Bd}(X)$) is homeomorphic to ℓ_2 if and only if X is a completely metrizable nowhere locally compact metric space such that each (resp. bounded) subset of X is totally bounded and the completion $\tilde{X}$ of X is connected and locally connected.

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