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Numerical analysis of normal sequences of finite open covers and Pontrjagin-Schnirelmann's theorem

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1 Introduction

Recently, there has been an increase in the importance of fractal sets in the sciences, and fractal dimension theory has been studied by many scientists and mathematicians (e.g. see [1], [5], [10] and [15]). Fractal dimensions depend on the metrics on spaces and hence the analysis of metrics of the spaces are very important. In this note, we study some properties of topological dimension, metrics and box-counting dimensions of separable metric spaces from a point of view of general topology. In general topology, the notion of normal sequence of open covers is one of the most useful tools for the study (e.g. see [11, 12, 13]). For example, the notion is the essence of metrizability of spaces (see Theorem 2.1). The key word is "normal sequence" of finite open covers. In this note, we study directly the numerical properties of normal sequences of "finite" open covers on a given separable metric space X and we will give another proof of Pontrjagin and Schnirelmann's theorem. Furtheremore, by use of normal sequences we construct metrics ρ which can control the values of $\frac{\log N(\epsilon,\rho)}{-\log\epsilon}$. In particular, we can construct *chaotic metrics* with respect to the determination of the box-counting dimensions. The methods used in this note are based on dimensional theoretical techniques in an abstract topological space.

In fractal dimension theory, Pontrjagin and Schnirelmann [16] proved the following fundamental result involving topological dimension $\dim X$ and (lower) box-counting dimension $\dim_B(X,\rho)$ for a compact metric space (X,ρ): For a metric ρ on X and $\epsilon>0$, let

$$N(\epsilon,\rho)=\min\{|\mathcal{U}| \mid \mathcal{U} \text{ is a finite open cover of } X \text{ with } \mathrm{mesh}_\rho(\mathcal{U})\leq\epsilon\}$$

and

$$\dim_B(X,\rho)=\sup\{\inf\{\frac{\log N(\epsilon,\rho)}{-\log\epsilon}\mid 0<\epsilon<\epsilon_0\}\mid 0<\epsilon_0\}(=\liminf_{\epsilon\to 0}\frac{\log N(\epsilon,\rho)}{-\log\epsilon}),$$

where $|A|$ denotes the cardinality of a set A. Then

$$\dim X=\min\{\dim_B(X,\rho)|\ \rho \text{ is a metric for } X\}.$$

More generally, Bruijning ([2] or [12, p.81, Corollary]) showed that if X is a separable metric space, then

$$\dim X=\min\{\dim_B(X,\rho)|\ \rho \text{ is a totally bounded metric for } X\}.$$

Pontrjagin and Schnirelmann proved their theorem by use of geometric arguments in an Euclidean space. In fact, such a metric ρ on X with $\dim X=\dim_B(X,\rho)$ was obtained by use of geometric arguments (embedding arguments) on polyhedra approximations of n-dimensional sets in the $(2n+1)$-dimensional Euclidean space R^{2n+1} (see [12] and [16]).

2 Normal sequences of open covers

In this note, we need the following terminology and concepts. Let $\mathcal{U}$ and $\mathcal{V}$ be open covers of a space X. We assume that each element of any open cover of a space is not an empty set. If $\mathcal{V}$ refines $\mathcal{U}$, then we denote $\mathcal{V} \leq \mathcal{U}$ (e.g. see [11] and [13]). Suppose that A is a subset of a space X and $\mathcal{U}$ is an open cover of X. Then we denote

$$St(A,\mathcal{U}) = \bigcup\{U \in \mathcal{U} \mid U \cap A \neq \phi\}.$$

Inductively, we define $St^0(A,\mathcal{U}) = A$, $St^1(A,\mathcal{U}) = St(A,\mathcal{U})$ and

$$St^{p+1}(A,\mathcal{U}) = St(St^p(A,\mathcal{U}),\mathcal{U}) = \bigcup\{U \in \mathcal{U} \mid U \cap St^p(A,\mathcal{U}) \neq \phi\} \ (p \geq 1).$$

We put

$$\mathcal{U}^\star = \{St(U,\mathcal{U}) \mid U \in \mathcal{U}\} \text{ and } \mathcal{U}^\Delta = \{St(x,\mathcal{U}) \mid x \in X\}.$$

Note that if $|\mathcal{U}|$ is finite, then $|\mathcal{U}^\star|$ and $|\mathcal{U}^\Delta|$ are finite. Also, we put $\mathcal{U}^{\star^0} = \mathcal{U}$, $\mathcal{U}^{\Delta^0} = \mathcal{U}$, $\mathcal{U}^{\star^1} = \mathcal{U}^\star$, and $\mathcal{U}^{\Delta^1} = \mathcal{U}^\Delta$. Inductively, we define

$$\mathcal{U}^{\star^{p+1}} = (\mathcal{U}^{\star^p})^\star = \{St(W,\mathcal{U}^{\star^p}) \mid W \in \mathcal{U}^{\star^p}\}$$

and

$$\mathcal{U}^{\Delta^{p+1}} = (\mathcal{U}^{\Delta^p})^\Delta = \{St(x,\mathcal{U}^{\Delta^p}) \mid x \in X\}.$$

An open cover $\mathcal{V}$ of X is a *star p-refinement* of an open cover $\mathcal{U}$ of X if $\mathcal{V}^{\star^p} \leq \mathcal{U}$. An open cover $\mathcal{V}$ of X is a *delta p-refinement* of an open cover $\mathcal{U}$ of X if $\mathcal{V}^{\Delta^p} \leq \mathcal{U}$. An open cover $\mathcal{V}$ of X is a *star-refinement* of an open cover $\mathcal{U}$ of X if $\mathcal{V}$ is a *star* 1*-refinement* of $\mathcal{U}$. An open cover $\mathcal{V}$ of X is a *delta-refinement* of an open cover $\mathcal{U}$ of X if $\mathcal{V}$ is a *delta* 1*-refinement* of $\mathcal{U}$. Note that $\mathcal{V} \leq \mathcal{V}^\Delta \leq \mathcal{V}^\star \leq \mathcal{V}^{\Delta^2}$.

Let $\mathcal{U}_i$ $(i = 1, 2, ..,)$ be open covers of X. Then the sequence $\{\mathcal{U}_i\}_{i=1}^\infty$ is called a *normal star-sequence* (e.g. see [11], [12] and [13]) if $\mathcal{U}_{i+1}$ is a star-refinement of $\mathcal{U}_i$ $(i = 1, 2, ...,)$. Also, the sequence $\{\mathcal{U}_i\}_{i=1}^\infty$ is called a *normal delta-sequence* if $\mathcal{U}_{i+1}$ is a delta-refinement of $\mathcal{U}_i$ $(i = 1, 2, ...,)$. The sequence $\{\mathcal{U}_i\}_{i=1}^\infty$ is called a *normal sequence* (e.g. see [11], [12] and [13]) if either $(\star)$ $\{\mathcal{U}_i\}_{i=1}^\infty$ is a normal star-sequence or (Δ) $\{\mathcal{U}_i\}_{i=1}^\infty$ is a normal delta-sequence. The sequence $\{\mathcal{U}_i\}_{i=1}^\infty$ is called a *development* of X if $\{St(x,\mathcal{U}_i) \mid i = 1, 2, ...,\}$ is a neighborhood base for each point x of X.

The following theorem is well known as Alexandroff-Urysohn's metrization theorem (e.g. see [11, 12, 13]). We need some constructions of metrics in the proof of the theorem.

Theorem 2.1 (Alexandroff-Urysohn's metrization theorem) *A T_1-space X is metrizable if and only if there exists a sequence $\{\mathcal{U}_i\}_{i=1}^\infty$ of open covers of X such that $\{\mathcal{U}_i\}_{i=1}^\infty$ is a normal sequence and a development of X.*

For any normal space X and natural numbers k and p, we define the following indices:

(1) The function $\star_k^p(X)$ is defined as the least natural number m such that for every open cover $\mathcal{U}$ of X with $|\mathcal{U}| = k$, there is an open cover $\mathcal{V}$ of X such that $|\mathcal{V}| \leq m$ and $\mathcal{V}^{\star^p} \leq \mathcal{U}$ (see [14]).

(2) The function $\Delta_k^p(X)$ is defined as the least natural number m such that for every open cover $\mathcal{U}$ of X with $|\mathcal{U}| = k$, there is an open cover $\mathcal{V}$ of X such that $|\mathcal{V}| \leq m$ and $\mathcal{V}^{\Delta^p} \leq \mathcal{U}$ (see [14]).

By C_m^k, we shall denote the set of all m-element subsets of the set $\{1, 2, .., k\}$ and by $\begin{pmatrix} k \\ m \end{pmatrix}$ its cardinality, i.e.,

$$\begin{pmatrix} k \\ m \end{pmatrix} = \frac{k!}{m!(k-m)!}.$$

For natural numbers $k, m, p \geq 1$ with $k \geq m$, we define the natural numbers

$$\tilde{\Delta}(k; m; p) = \Sigma_{m \geq j_1 \geq j_2 \geq \ldots \geq j_p \geq 1} \begin{pmatrix} k \\ j_1 \end{pmatrix} \begin{pmatrix} j_1 \\ j_2 \end{pmatrix} \cdots \begin{pmatrix} j_{p-1} \\ j_p \end{pmatrix}$$

and

$$\tilde{\star}(k; m; p) = \Sigma_{m \geq j_1 \geq j_2 \geq \ldots \geq j_p \geq 1} \begin{pmatrix} k \\ j_1 \end{pmatrix} \begin{pmatrix} j_1 \\ j_2 \end{pmatrix} \cdots \begin{pmatrix} j_{p-1} \\ j_p \end{pmatrix} j_p.$$

In [3], Bruijning and Nagata determined the index $\Delta_k^1(X)$, and in [6], Hashimoto and Hattori determined the index $\star_k^1(X)$. Finally, in [9, Corollary 3.11] we determined the indices $\Delta_k^p(X)$ and $\star_k^p(X)$ for all $p \geq 1$ as follows, which is the key lemma of this note.

Lemma 2.2 *Let X be an infinite normal space with* $\dim X = m$ *and let k and p be natural numbers. Then*

$$\star_k^p(X) = \begin{cases} \tilde{\star}(k; k; (1/2)(3^p - 1)) = k[(1/2)(3^p - 1) + 1)]^{k-1}, & \text{if } k \leq m+1 \\ \tilde{\star}(k; m+1; (1/2)(3^p - 1)), & \text{if } k \geq m+1. \end{cases}$$

and

$$\Delta_k^p(X) = \begin{cases} \tilde{\Delta}(k; k; 2^{p-1}) = (2^{p-1} + 1)^k - (2^{p-1})^k, & \text{if } k \leq m+1 \\ \tilde{\Delta}(k; m+1; 2^{p-1}), & \text{if } k \geq m+1. \end{cases}$$

3 Topological dimension and normal sequences of finite open covers

By use of Lemma 2.2, we obtain the next theorem which means that topological dimension is characterized in terms of the growth of the global cardinality $|\mathcal{U}_i|$ of members $\mathcal{U}_i$ of normal sequences.

Theorem 3.1 *Let X be a separable metric space. Then*

$$(1)\quad \dim X = \min\{\liminf_{i\to\infty} \frac{\log_3 |\mathcal{U}_i|}{i} \mid \{\mathcal{U}_i\}_{i=1}^{\infty} \textit{ is a normal star-sequence of finite open covers of } X \textit{ and a development of } X\}$$

and

$$(2)\quad \dim X = \min\{\liminf_{i\to\infty} \frac{\log_2 |\mathcal{U}_i|}{i} \mid \{\mathcal{U}_i\}_{i=1}^{\infty} \textit{ is a normal delta-sequence of finite open covers of } X \textit{ and a development of } X\}.$$

For the another proof of Pontrjagin and Schnirelmann's theorem, we need the followings.

Proposition 3.2 *Let X be a separable metric space. Then*

(1) $\dim X = \min\{\dim_B(X,\rho_\star) \mid \rho_\star$ *is an Alexandroff-Urysohn's metric for X induced by a sequence $\{\mathcal{U}_i\}_{i=1}^{\infty}$ which is a normal star-sequence of finite open covers of X and a development of X*$\}$,

(2) $\dim X = \min\{\dim_B(X,d_\Delta) \mid d_\Delta$ *is an Alexandroff-Urysohn's metric for X induced by a sequence $\{\mathcal{U}_i\}_{i=1}^{\infty}$ which is a normal delta-sequence of finite open covers of X and a development of X*$\}$.

Let X be a metrizable space and let ρ_1 and ρ_2 be two metrics on X. Then ρ_1 is *Lipschitz equivalent* to ρ_2 if the identity maps $Id_1 : (X,\rho_1) \to (X,\rho_2)$ and $Id_2 : (X,\rho_2) \to (X,\rho_1)$ are Lipschitz homeomorphisms.

Proposition 3.3 *Let (X,ρ_1) be a metric space such that ρ_1 is bounded, i.e., $\mathrm{diam}_{\rho_1} X < \infty$. Suppose that $\{\mathcal{U}_i\}_{i=1}^{\infty}$ is a normal star (resp. delta)-sequence of open covers of X and a development of X. Then the followings are equivalent.*

(1) The Alexandroff-Urysohn's metric ρ_2 induced by $\{\mathcal{U}_i\}_{i=1}^{\infty}$ is Lipschitz equivalent to ρ_1.

(2) There are positive numbers $c_2 \geq c_1 > 0$ such that for each i,

$$\{U_{\rho_1}(x, c_1/3^i) \mid x \in X\} \leq \mathcal{U}_i \leq \{U_{\rho_1}(x, c_2/3^i) \mid x \in X\}$$

$$(\textit{resp. } \{U_{\rho_1}(x, c_1/2^i) \mid x \in X\} \leq \mathcal{U}_i \leq \{U_{\rho_1}(x, c_2/2^i) \mid x \in X\}).$$

The next proposition implies that for any separable metric space X there is a natural bijection from the set of all totally bounded metrics on X to the set of Alexandroff-Urysohn's metrics on X induced by normal sequences of finite open covers which are developments of X, up to Lipschitz equivalence.

Proposition 3.4 *Let X be a separable metric space and let ρ_1 be a totally bounded metric on X. Then there is a normal star (resp. delta)-sequence $\{\mathcal{U}_i\}_{i=1}^{\infty}$ of finite open covers of X such that $\{\mathcal{U}_i\}_{i=1}^{\infty}$ is a development of X and ρ_1 is Lipschitz equivalent to ρ_2, where ρ_2 is the Alexandroff-Urysohn's metric induced by $\{\mathcal{U}_i\}_{i=1}^{\infty}$. In particular,* $\dim_B(X,\rho_1)=\dim_B(X,\rho_2)$.

Theorem 3.5 (Pontrjagin-Schnirelmann and Bruijning's theorem) *Let X be a separable metric space. Then*

$$\dim X=\min\{\dim_B(X,\rho) \mid \rho \text{ is a totally bounded metric for } X\}.$$

Proof. Put $\dim X=m$. By Proposition 3.4, we see that if ρ_1 is any totally bounded metric on X, then there is a normal star-sequence $\{\mathcal{U}_i\}_{i=1}^{\infty}$ of finite open covers of X such that $\{\mathcal{U}_i\}_{i=1}^{\infty}$ is a development of X and $\dim_B(X,\rho_1)=\dim_B(X,\rho_2)$, where ρ_2 is the Alexandroff-Urysohn's metric induced by $\{\mathcal{U}_i\}_{i=1}^{\infty}$. By use of this fact, we can prove Pontrjagin-Schnirelmann and Bruijning's theorem.

4 Chaotic metrics with respect to the determination of the box-counting dimensions

In this section, we construct *chaotic metrics* with respect to the determination of the box-counting dimensions. By Theorem 3.1, we know that for any separable metric space X, there is a normal star (resp.delta)-sequence $\{\mathcal{U}_i\}_{i=1}^{\infty}$ of finite open covers of X and a development of X such that $\liminf_{i\to\infty}\frac{\log_3|\mathcal{U}_i|}{i}=\dim X$ (resp. $\liminf_{i\to\infty}\frac{\log_2|\mathcal{U}_i|}{i}=\dim X$). We call such a normal sequence $\{\mathcal{U}_i\}_{i=1}^{\infty}$ *a fundamental normal sequence* of X.

Theorem 4.1 *Let X be a separable metric space with $\dim X=m\geq 1$. Suppose that $\{\mathcal{U}_i\}_{i=1}^{\infty}$ is a fundamental normal star-sequence of X (i.e., $\liminf_{i\to\infty}\frac{\log_3|\mathcal{U}_i|}{i}=\dim X$.) Let α be any real number with $\alpha\geq m$ $(=\dim X)$ or $\alpha=\infty$. Then there is a subsequence $\{\mathcal{U}_{i_j}\}_{j=1}^{\infty}$ of $\{\mathcal{U}_i\}_{i=1}^{\infty}$ such that*

$$[\alpha,\infty]=\{\liminf_{k\to\infty}\frac{\log_3|\mathcal{U}_{i_{j_k}}|}{j_k} \mid \{j_k\}_{k=1}^{\infty} \text{ is an increasing subsequence of natural numbers}\}.$$

Also, there is a totally bounded metric ρ_α on X such that

$$[\alpha,\infty]=\{\liminf_{k\to\infty}\frac{\log N(\epsilon_k,\rho_\alpha)}{-\log\epsilon_k} \mid \{\epsilon_k\}_{k=1}^{\infty} \text{ is a decreasing sequence of positive numbers with } \lim_{k\to\infty}\epsilon_k=0\},$$

where $N(\epsilon,\rho_\alpha)=\min\{|\mathcal{U}| \mid \mathcal{U}$ is a finite open cover of X with $mesh_{\rho_\alpha}(\mathcal{U})\leq\epsilon\}$.

Remark. Let X be a separable metric space with $\dim X = m \geq 1$. Suppose that $\{\mathcal{U}_i\}_{i=1}^{\infty}$ is a fundamental normal star-sequence of X. Then

$$[\dim X, \infty] = \{\liminf_{j\to\infty} \frac{\log_3 |\mathcal{U}_{i_j}|}{j} \mid \{\mathcal{U}_{i_j}\}_{j=1}^{\infty} \text{ is a subsequence of } \{\mathcal{U}_i\}_{i=1}^{\infty}\}$$

$$= \{\dim_B(X, \rho_*) \mid \rho_* \text{ is the Alexandroff-Urysohn's metric for } X \text{ induced by a subsequence } \{\mathcal{U}_{i_j}\}_{j=1}^{\infty} \text{ of } \{\mathcal{U}_i\}_{i=1}^{\infty}\}.$$

In other words, all box-counting dimensions of X are generated by each fundamental normal star-sequence of X. In case of normal delta-sequence of finite open covers of X, we have the following theorem.

Theorem 4.2 *Let X be a separable metric space with $\dim X = m \geq 1$. Suppose that $\{\mathcal{U}_i\}_{i=1}^{\infty}$ is a fundamental normal delta-sequence of X (i.e., $\liminf_{i\to\infty} \frac{\log_2 |\mathcal{U}_i|}{i} = \dim X$.) Let α be any real number with $\alpha \geq m$ $(= \dim X)$ or $\alpha = \infty$. Then there is a subsequence $\{\mathcal{U}_{i_j}\}_{j=1}^{\infty}$ of $\{\mathcal{U}_i\}_{i=1}^{\infty}$ such that*

$$[\alpha, \infty] = \{\liminf_{k\to\infty} \frac{\log_2 |\mathcal{U}_{i_{j_k}}|}{j_k} \mid \{j_k\}_{k=1}^{\infty} \textit{ is an increasing subsequence of natural numbers}\}.$$

Also, there is a totally bounded metric d_α on X such that

$$[\alpha, \infty] = \{\liminf_{k\to\infty} \frac{\log N(\epsilon_k, d_\alpha)}{-\log \epsilon_k} \mid \{\epsilon_k\}_{k=1}^{\infty} \textit{ is a decreasing sequence of positive numbers with } \lim_{k\to\infty} \epsilon_k = 0\}.$$

Corollary 4.3 *Let X be a separable metric space with $\dim X \geq 1$. If α is a real number with $\alpha \geq \dim X$ or $\alpha = \infty$, then there is a totally bounded metric ρ_α on X such that for any subset A of X with $\dim A = \dim X$,*

$$[\alpha, \infty] = \{\liminf_{k\to\infty} \frac{\log N(\epsilon_k, \rho_\alpha; A)}{-\log \epsilon_k} \mid \{\epsilon_k\}_{k=1}^{\infty} \textit{ is a decreasing sequence of positive numbers with } \lim_{k\to\infty} \epsilon_k = 0\},$$

where $N(\epsilon, \rho_\alpha; A) = \min\{|\mathcal{U}| \mid \mathcal{U}$ is a finite open cover of A with $mesh_{\rho_\alpha}(\mathcal{U}) \leq \epsilon\}$. In particular, $\dim_B(A, \rho_\alpha|A) = \alpha$.

Corollary 4.4 *Let X be a separable metric space with $\dim X \geq 1$. Suppose that $\{A_n\}_{n=1}^{\infty}$ is a family of mutually disjoint closed subsets A_n of X with $\dim A_n = \dim X$ for each n. If α_n is a real number with $\alpha_n \geq \dim X$ or $\alpha_n = \infty$ for each n, then there is a totally bounded metric ρ on X such that $\dim_B(A_n, \rho|A_n) = \alpha_n$ for each n.*

5 Upper box-counting dimension $\overline{\dim}_B(X,\rho)$ and normal sequences of finite open covers

In this sectin, we study relations between upper box-counting dimension and normal sequence of finite open covers. For a separable metric space (X,ρ), we consider the upper box-counting dimension of (X,ρ) (e.g. see [5] and [15]):

$$\overline{\dim}_B(X,\rho) = \limsup_{\epsilon\to 0} \frac{\log N(\epsilon,\rho)}{-\log \epsilon}.$$

Theorem 5.1 *Let X be a separable metric space with $\dim X = m \geq 1$. Suppose that there is a sequence $\{\mathcal{U}_i\}_{i=1}^{\infty}$ which is a normal star (resp.delta)-sequence of finite open covers of X and a development of X such that $\lim_{i\to\infty} \frac{\log_3 |\mathcal{U}_i|}{i} = m$ (resp. $\lim_{i\to\infty} \frac{\log_2 |\mathcal{U}_i|}{i} = m$). For any α,β with $m \leq \alpha \leq \beta \leq \infty$, there is a totally bounded metric $\rho_{\alpha,\beta}$ on X such that*

$$[\alpha,\beta] = \{\liminf_{k\to\infty} \frac{\log N(\epsilon_k,\rho_{\alpha,\beta})}{-\log \epsilon_k} \mid \{\epsilon_k\} \textit{ is a decreasing sequence of positive numbers with } \lim_{k\to\infty} \epsilon_k = 0\}.$$

In particular, $\dim_B(X,\rho_{\alpha,\beta}) = \alpha \leq \beta = \overline{\dim}_B(X,\rho_{\alpha,\beta})$.

Corollary 5.2 *Let $I = [0,1]$ be the unit interval and let $X = I^m$ be the m-cube ($m \geq 1$). Then there is a sequence $\{\mathcal{U}_i\}_{i=1}^{\infty}$ which is a normal star (resp.delta)-sequence of finite open covers and a development of X such that $\lim_{i\to\infty} \frac{\log_3 |\mathcal{U}_i|}{i} = m$ (resp. $\lim_{i\to\infty} \frac{\log_2 |\mathcal{U}_i|}{i} = m$). Moreover, for any α,β with $m \leq \alpha \leq \beta \leq \infty$, there is a metric $\rho_{\alpha,\beta}$ on X such that*

$$[\alpha,\beta] = \{\liminf_{k\to\infty} \frac{\log N(\epsilon_k,\rho_{\alpha,\beta})}{-\log \epsilon_k} \mid \{\epsilon_k\} \textit{ is a decreasing sequence of positive numbers with } \lim_{k\to\infty} \epsilon_k = 0\}.$$

In particular, $\dim_B(X,\rho_{\alpha,\beta}) = \alpha \leq \beta = \overline{\dim}_B(X,\rho_{\alpha,\beta})$.

For the case of $\dim X = 0$, we have the following.

Theorem 5.3 *Let X be an infinite 0-dimensional separable metric space. Then there is a sequence $\{\mathcal{U}_i\}_{i=1}^{\infty}$ which is a sequence of mutually disjoint clopen covers and a development of X such that $|\mathcal{U}_i| = i$ for each $i = 1,2,\cdots$. For any α,β with $0 \leq \alpha \leq \beta \leq \infty$, there is a totally bounded metric $\rho_{\alpha,\beta}$ on X such that*

$$[\alpha,\beta] = \{\liminf_{k\to\infty} \frac{\log N(\epsilon_k,\rho_{\alpha,\beta})}{-\log \epsilon_k} \mid \{\epsilon_k\} \textit{ is a decreasing sequence of positive numbers with } \lim_{k\to\infty} \epsilon_k = 0\}.$$

In particular, $\dim_B(X,\rho_{\alpha,\beta}) = \alpha \leq \beta = \overline{\dim}_B(X,\rho_{\alpha,\beta})$.

Compared with our results of this note and the Pontrjagin-Schnirelmann' theorem, finally we have the following problem.

Problem 5.4 *Let X be a separable metric space with* $\dim X = m \geq 1$. *Does there exist a sequence* $\{\mathcal{U}_i\}_{i=1}^{\infty}$ *which is a normal star (resp.delta)-sequence of finite open covers of* X *and a development of* X *such that* $\lim_{i\to\infty} \frac{\log_3 |\mathcal{U}_i|}{i} = m$ *(resp.* $\lim_{i\to\infty} \frac{\log_2 |\mathcal{U}_i|}{i} = m$*) ? Does there exist a totally bounded metric* ρ *for* X *such that* $\overline{\dim}_B(X, \rho) = \dim X$ *? In particular, if* X *is the Menger m-dimensional compactum* ($m \geq 1$) *(e.g. see* [4] *for the Menger m-dimensional compactum), is it true ?*

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