数理解析研究所講究録
1531 巻 2007 年 129-133

Submetrizability and Interpolations

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All topological spaces considered here are Tychonoff. For a topological space X, $C(X)$ is the Banach space of all bounded real-valued continuous functions with the sup norm: $\| f \| = \sup\{|f(x)| : x \in X\}$ for $f \in C(X)$. The space $F(X \times \mathbf{R})$ is the hyperspace consisting of all finite subsets of the product space $X \times \mathbf{R}$. Of course, its topology is the Vietoris topology. Let $S(X)$ be the subspace of $F(X \times \mathbf{R})$ defined by

$$S(X) = \{\{(x_1, r_1), \cdots, (x_n, r_n)\} : x_i \neq x_j \quad \text{for} \quad i \neq j\}.$$

For each $n = 1, 2, \cdots$, define $F_n(X \times \mathbf{R})$ and $S_n(X)$ by:

$$F_n(X \times \mathbf{R}) = \{D \in F(X \times \mathbf{R}) : D \text{ has at most } n \text{ points}\},$$

$$S_n(X) = S(X) \cap F_n(X \times \mathbf{R}).$$

For a point $D = \{(x_1, r_1), (x_2, r_2), \cdots, (x_n, r_n)\} \in S(X)$, a function f_D in $C(X)$ is called an interpolation function for D if

$$f_D(x_1) = r_1, f_D(x_2) = r_2, \cdots, f_D(x_n) = r_n$$

are satisfied. A map $\Theta : S(X) \to C(X)$ is called interpolation (algorithm) if $\Theta(D) = f_D$ is an interpolation function for each $D \in S(X)$. Further, if Θ satisfies the condition that the restriction $\Theta|_{S_n(X) - S_{n-1}(X)}$ is continuous for each $n = 1, 2, \cdots$, we call Θ to be a weakly continuous interpolation. Let a topology τ_1 on a set X is stronger than a topology τ_2 on the same set. If Θ is a weakly continuous interpolation on τ_2, then the same Θ is a weakly continuous interpolation on τ_1. Every metrizable space has a weakly continuous interpolation, and hence every submetrizable space has a weakly continuous interpolation. In [1], a weakly continuous interpolation on a metric space (X, d) is constructed as follows: For any $D = \{(x_1, r_1), \cdots, (x_n, r_n)\} \in S(X)$, let

$$m = \min\{d(x_i, x_j) : i \neq j\}$$

and

$$f_D(x) = \begin{cases} 0 & \text{if } d(x,x_i) \geq m/4 \text{ for each } i = 1,\cdots,n \\ r_i - \frac{4r_i}{m} d(x,x_i) & \text{if } d(x,x_i) < m/4 \text{ for some } i = 1,\cdots,n. \end{cases}$$

Then the map $\Theta : S(X) \to C(X)$ defined by $\Theta(D) = f_D$ is weakly continuous. For this interpolation, we can show the following. Here, for any distinct $p, q \in X$ D_{pq} denotes the sample $\{(p,-1),(q,1)\}$ in $S_2(X) - S_1(X)$.

Proposition 1 *The weakly continuous interpolation $\Theta : S(X) \to C(X)$ on a metric space (X,d) constructed above satisfies the following condition: There exists a constant M such that*

$$\| f_{D_{wz}} - f_{D_{xy}} \| \leq M \max\{\| f_{D_{wy}} - f_{D_{xy}} \|, \| f_{D_{xz}} - f_{D_{xy}} \|\}$$

for any $x, y, z, w \in X$.

Proof. We can assume that metric function is bounded. Assume that$d(x,y) < 1$ for any $x, y \in X$. It suffices to show that $\| f_{D_{wy}} - f_{D_{xy}} \| < \epsilon, \| f_{D_{xz}} - f_{D_{xy}} \| < \epsilon$ imply $\| f_{D_{wz}} - f_{D_{xy}} \| < 6\epsilon$.

Claim 1. Let $0 < \alpha < \frac{1}{2}$. If $d(z,y) < \alpha d(w,y)$, then $\| f_{D_{wy}} - f_{D_{wz}} \| \leq 9\alpha$.

Since interpolation functions defined above are piecewise linear on the distance from data points, $\| f_{D_{wy}} - f_{D_{wz}} \|$ does not exceed the following 5 values:

(1) At z, $|f_{D_{wy}}(z) - f_{D_{wz}}(z)| = \frac{4}{d(w,y)} d(y,z) < \frac{4\alpha d(w,y)}{d(w,y)} = 4\alpha$.

(2) At y, $|f_{D_{wy}}(y) - f_{D_{wz}}(y)| = \frac{4}{d(w,z)} d(y,z) < \frac{4d(y,z)}{d(w,y)-d(y,z)} < \frac{4\alpha d(w,y)}{(1-\alpha)d(w,y)} = \frac{4\alpha}{1-\alpha} < 8\alpha$.

(3) When the distanse from y is $\frac{d(w,y)}{4}$, $\frac{4}{d(w,z)}(d(y,z) + \frac{d(w,z)}{4} - \frac{d(w,y)}{4}) = \frac{4d(y,z)}{d(w,z)} + 1 - \frac{d(w,y)}{d(w,z)} < 8\alpha + 1 - \frac{d(w,y)}{d(w,z)} \leq 8\alpha + 1 - \frac{d(w,y)}{d(w,y)+d(y,z)} < 8\alpha + 1 - \frac{1}{1+\alpha} < 9\alpha$.

(4) When the distance from z is $\frac{d(w,z)}{4}$, $\frac{4}{d(w,y)}(d(y,z) + \frac{d(w,y)}{4} - \frac{d(w,z)}{4}) = (\frac{4d(y,z)}{d(w,y)} + 1 - \frac{d(w,z)}{d(w,y)} < \alpha + 1 - \frac{d(w,y)-d(y,z)}{d(w,y)} = \alpha + 1 - 1 + \frac{d(y,z)}{d(w,y)} < 2\alpha$.

(5) When the distance from w is $\frac{d(w,z)}{4}$ in case $d(w,z) \leq d(w,y)$, $\frac{4}{d(w,y)}(\frac{d(w,y)}{4} - \frac{d(w,z)}{4}) = 1 - \frac{d(w,z)}{d(w,y)} \leq 1 - \frac{d(w,y)-d(y,z)}{d(w,y)} = \frac{d(y,z)}{d(w,y)} < \alpha$. When the distance from w is $\frac{d(w,y)}{4}$ in case $d(w,y) < d(w,z)$, $\frac{4}{d(w,z)}(\frac{d(w,z)}{4} - \frac{d(w,y)}{4}) = 1 - \frac{d(w,y)}{d(w,z)} \leq 1 - \frac{d(w,y)}{d(w,y)+d(y,z)} = 1 - \frac{1}{1+\alpha} = \frac{\alpha}{1+\alpha} < \alpha$.

Claim 2. Let $0 < \varepsilon < 1$. If $\| f_{D_{wy}} - f_{D_{xy}} \| < \varepsilon, \| f_{D_{xz}} - f_{D_{xy}} \| < \varepsilon$, then $d(y,z) < \frac{\varepsilon}{2} d(w,y)$.

Since $\| f_{D_{wy}} - f_{D_{xy}} \| < \varepsilon$, estimating the value at x, we obtain $d(x,w)\frac{4}{d(w,y)} < \varepsilon$. Hence $d(x,w) < \frac{\varepsilon}{4} d(w,y)$. Similarly, the value of $|f_{D_{xz}} - f_{D_{xy}}|$ at z is $d(y,z)\frac{4}{d(x,y)}$. Hence it follows that $d(y,z) < \frac{\varepsilon}{4} d(x,y)$. Then $d(y,z) < \frac{\varepsilon}{4}(d(x,w) + d(w,y)) < \frac{\varepsilon}{4}(\frac{\varepsilon}{4} d(w,y) + d(w,y)) = ((\frac{\varepsilon}{4})^2 + \frac{\varepsilon}{4}) d(w,y) < \frac{\varepsilon}{2} d(w,y)$.

By claim 2, $d(y,z) < \frac{\varepsilon}{2}d(w,y)$. Then $\| f_{D_{wy}} - f_{D_{wz}} \| < \frac{9}{2}\varepsilon < 5\varepsilon$ by claim 1. Hence $\| f_{D_{wz}} - f_{D_{xy}} \| \leq \| f_{D_{wz}} - f_{D_{wy}} \| + \| f_{D_{wy}} - f_{D_{xy}} \| < 5\varepsilon + \varepsilon = 6\varepsilon$.

Let us call a weakly continuous interpolation to be regular when the condition in the proposition is satisfied. A sequence $\{\mathcal{G}_n\}$ of open covers of X is called a G_δ-diagonal if for any $x \neq y \in X$, there exists n such that $y \notin st(x, \mathcal{G}_n)$. Further G_δ-diagonal sequence $\{\mathcal{G}_n\}$ is called regular if for $G, G' \in \mathcal{G}_{n+1}$, $G \cap G' \neq \emptyset$ implies that there exists $G \in \mathcal{G}_n$ such that $G \cup G' \subset G$. It is well known that a space X is submetrizable if and only if X has a regular G_δ-diagonal sequence(see[2]).

Theorem 1 *The following are equivalent.*
(1) X is submetrizable.
(2) $X \times (\omega + 1)$ has a weakly continuous regular interpolation.

Proof. If X is submetrizable, then $X \times (\omega + 1)$ is also submetrizable. Hence it follows that (1) implies (2) from the proposition above.

Assume that (2) is satisfied. We will show that X has a regular G_δ-diagonal sequence. Let $\Theta : S(X \times (\omega + 1)) \to C(X \times (\omega + 1))$ be a weakly continuous interpolation which satisfies the regular condition. Let us recall the notation that $D_{(p,m)(q,n)} = \{((p,m),-1),((q,n),1)\} \in S_2(X \times (\omega+1)) - S_1(X \times (\omega+1))$ for any $(p,m),(q,n) \in X \times (\omega+1)$ and $f_{D_{(p,m)(q,n)}} = \Theta(D_{(p,m)(q,n)})$. Then there exists a constant M such that

$$\| f_{D_{(w,l)(z,k)}} - f_{D_{(x,i)(y,j)}} \| \leq M \max\{\| f_{D_{(w,l)(y,j)}} - f_{D_{(x,i)(y,j)}} \|, \| f_{D_{(x,i)(z,k)}} - f_{D_{(x,i)(y,j)}} \|\}$$

for any $D_{(x,i)(y,j)}, D_{(w,l)(y,j)}, D_{(x,i)(z,k)}, D_{(w,l)(z,k)} \in S_2(X \times (\omega+1)) - S_1(X \times (\omega+1))$. It can be also assumed that $M > 1$.

Let $\mathcal{G}_n$ be the family of all open subsets U which satisfies

$$\| f_{D_{(x,i)(y,\omega)}} - f_{D_{(x',i)(y',\omega)}} \| < \frac{1}{(2M)^n}$$

for any $x, y, x', y' \in U$ and any $i = 0, 1, \cdots, n$.

Claim 1. $\mathcal{G}_n$ is an open cover of X.

For any $x \in X$, consider $D_{(x,i)(x,\omega)}$ for $i = 0, 1, \cdots, n$. Since Θ is weakly continuous, for each $i = 0, 1, \cdots, n$ there exists an open neighborhood U_i of x such that

$$\| f_{D_{(u,i)(v,\omega)}} - f_{D_{(x,i)(x,\omega)}} \| < \frac{1}{2(2M)^n}$$

for any $u, v \in U_i$. Then $U = \cap_{i=0}^{n} U_i \in \mathcal{G}_n$ is an open neighborhood of x in X.

Claim 2. $\{\mathcal{G}_n\}$ is a G_δ-diagonal sequence of X.

Assume that $\{\mathcal{G}_n\}$ is not a G_δ-diagonal sequence. Then there exist two distinct point x_0, y_0 such that for each n, $x_0, y_0 \in U_n$ for some $U_n \in \mathcal{G}_n$. Let us take $D_{(x_0,\omega)(y_0,\omega)} = \{((x_0,\omega),-1),((y_0,\omega),1)\}$. Then $f_{D_{(x_0,\omega)(y_0,\omega)}}((x_0,\omega)) = -1$. Let W be a neighborhood of $D_{(x_0,\omega)(y_0,\omega)}$ in $S_2(X\times(\omega+1)) - S_1(X\times(\omega+1))$ such that $\| f_D - f_{D_{(x_0,\omega)(y_0,\omega)}} \| < 1$ for any $D \in W$. Then there exist n such that $D_{(x_0,i)(y_0,\omega)} \in W$ for any $i \geq n$. Especially, for such $D_{(x_0,i)(y_0,\omega)}$, the value $|f_{D_{(x_0,i)(y_0,\omega)}}((x_0,\omega)) - f_{D_{(x_0,\omega)(y_0,\omega)}}((x_0,\omega))| < 1$ at (x_0,ω), and hence

$$f_{D_{(x_0,i)(y_0,\omega)}}((x_0,\omega)) < 0.$$

On the other hand, since $x_0, y_0 \in U_i$ for $i \geq n$, $\| f_{D_{(x_0,i)(y_0,\omega)}} - f_{D_{(x_0,i)(x_0,\omega)}} \| < \frac{1}{(2M)^i}$ and $f_{D_{(x_0,i)(x_0,\omega)}}((x_0,\omega)) = 1$, it must be

$$f_{D_{(x_0,i)(y_0,\omega)}}((x_0,\omega)) > 0.$$

This is a contradiction.

Claim 3. $\{\mathcal{G}_n\}$ is regular.

Assume that $U_1, U_2 \in \mathcal{G}_{n+1}$ satisfy $U_1 \cap U_2 \neq \emptyset$. Then there exist $p \in U_1 \cap U_2$. For any $x, y \in U_1 \cup U_2$, it is shown that $\| f_{D_{(x,i)(y,\omega)}} - f_{D_{(p,i)(p,\omega)}} \| < \frac{1}{2^{n+1}M^n}$ for any $i = 0, 1, \cdots n$. In fact, in case $x, y \in U_1$ or $x, y \in U_2$, it is obvious. In other case, since $\| f_{D_{(x,i)(p,\omega)}} - f_{D_{(p,i)(p,\omega)}} \| < \frac{1}{(2M)^{n+1}}$, $\| f_{D_{(p,i)(y,\omega)}} - f_{D_{(p,i)(p,\omega)}} \| < \frac{1}{(2M)^{n+1}}$, it follows that $\| f_{D_{(x,i)(y,\omega)}} - f_{D_{(p,i)(p,\omega)}} \| < \frac{1}{2^{n+1}M^n}$ by the regularity condition of Θ. Then for $i = 0, 1, \cdots, n$ and for any $x, y, x', y' \in U_1 \cup U_2$,

$$\| f_{D_{(x,i)(y,\omega)}} - f_{D_{(x',i)(y',\omega)}} \| = \| f_{D_{(x,i)(y,\omega)}} - f_{D_{(p,i)(p,\omega)}} \| + \| f_{D_{(p,i)(p,\omega)}} - f_{D_{(x',i)(y',\omega)}} \|$$

$$\leq \frac{1}{2^{n+1}M^n} + \frac{1}{2^{n+1}M^n} < \frac{1}{(2M)^n}.$$

This shows that $U_1 \cup U_2 \in \mathcal{G}_n$.

In the proof of the above theorem, we used the regularity condition on the interpolation only to show the regularity of the G_δ-diagonal sequence. Hence the following theorem is also obtained.

Theorem 2 *If $X \times (\omega+1)$ has a weakly continuous interpolation, then X has a G_δ-diagonal. In particular, for a paracompact space X, $X \times (\omega+1)$ has a weakly continuous interpolation if and only if X has a G_δ-diagonal.*

Further, we used interpolation functions essentially for only $D \in S_2(X \times (\omega+1)) - S_1(X \times (\omega+1))$ to show the submetrizability of X in Theorem 1. Let us call a map $\Theta_2 : S_2(X) - S_1(X) \to C(X)$ to be a continuous S_2-interpolation if it is continuous and $\Theta_2(D)$ is an interpolation function for every $D \in S_2(X) - S_1(X)$. Theorem 1 can be rewitten as the following.

Theorem 3 *The following are equivalent.*
(1) X is submetrizable.
(2) $X \times (\omega+1)$ has a weakly continuous regular interpolation.
(3) $X \times (\omega+1)$ has a continuous regular S_2-interpolation.

Remark. It may be generally shown that a space X has a weakly continuous interpolation if and only if X has a continous S_2-interpolation.

参考文献

[1] G. Gruenhage. *Generalized metric spaces.* Handbook of Set-theoretic Topology, 423-502, North-Holland, 1984.

[2] T. Terada. *Continuity of interpolations.* Tsukuba J. Math., vol. 30, (2006), 225-236.