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グラフ写像の強推移性について

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1. INTRODUCTION

The purpose of this résumé is to describe (strong) transitivity properties for graph self-maps in my recent works. W. Parry [7] pointed out a sufficient condition for the existence of a special measure on a symbolic dynamics, which has a close relation to a linearization of the dynamics on intervals. Then, as an application, he introduced the concept of strong transitivity that is one of conditions under which an interval map is conjugate to a uniformly piecewise linear map [7, §5, §6]. E. Coven and I. Mulvey [6, Theorem B and C] stated the relation between transitivity and strong transitivity properties for interval (or circle) self-maps.

We extend the above relation to graph self-maps (see §3). A motivation for studying graph maps is that higher-dimensional dynamics can often be reduced to one-dimensional dynamics.

Throughout this paper, by a *graph*, we mean a *connected* compact one-dimensional polyhedron, and a *tree* is a graph which contains no loops. We also assume that any graph G is endowed with a metric d; we define $\mathbb{B}(x;\varepsilon)$, $x\in G$, $\varepsilon>0$ to be the set of points of G whose distance from x is less than ε. $\mathrm{B}(G)$ and $\mathrm{E}(G)$ denote the sets of branch points and of endpoints of G, respectively. A *map* f is a continuous function from a space X to itself; f^0 is the identity map, and for every $n\geq 0$, $f^{n+1}=f^n\circ f$. We denote by $\mathrm{Fix}(f)$ and $\mathrm{Per}(f)$ the sets of fixed points and of periodic points of f, respectively. For a subset K of X, $\mathrm{Int}\,K$ and $\mathrm{Cl}\,K$ denote the interior and closure of K in X.

2. STRONG TRANSITIVITY

An onto map $f:X\to X$ is called (*topologically*) *transitive* if any of the following equivalent conditions holds.

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(1) There exists a point with dense orbit.
(2) Whenever U, V are non-empty open sets, there exists an $n \geq 1$ such that $f^{-n}(U) \cap V \neq \emptyset$.
(3) The only closed invariant set K with $\operatorname{Int} K \neq \emptyset$ is $K = X$.

Remark. We note that, in the case of a graph map $f : G \to G$, f is transitive if and only if for every pair of non-empty open sets U and V in G, there exists a $k \geq 1$ such that $U \cap \operatorname{Int} f^k(V) \neq \emptyset$.

In the study of transitive maps, the subclass of those maps having all iterates transitive plays a significant role. A map f is *totally transitive* if f^n is transitive for all $n \geq 1$ (see [1]); note that a transitive map is not always totally transitive.

A map $f : X \to X$ is called *strongly transitive* if for every non-empty open set J of X, there exists an n such that $\cup_{k=0}^{n} f^k(J) = X$.

We first call a useful proposition which shows a backward structure of a strongly transitive map for each point.

Proposition 2.1. *Let $f : X \to X$ be a map of X to itself. Then the following are equivalent.*

(1) *For each $x \in X$, $\operatorname{Cl} \bigcup_{n=0}^{\infty} f^{-n}(x) = X$.*
(2) *For every non-empty open set U of X, $\bigcup_{n=0}^{\infty} f^n(U) = X$.*

Furthermore, if f is open, then (1) *and* (2) *are equivalent to*

(3) *If $E \subseteq X$ is a closed set with $f^{-1}(E) \subseteq E$, then $E = \emptyset$ or X.*

The examples below clarify the difference between transitivity and strong transitivity properties.

Example 1. There exists a transitive map of the interval which is not strongly transitive. This example appears in [3, Example 3] to illustrate another property. For completeness, we give a construction of the map here.

Let $\{p_n \mid n \in \mathbb{Z}\}$ be a two-sided sequence of real numbers in $[0, 1]$ such that

$$\cdots < p_{-2} < p_{-1} < p_0 < p_1 < p_2 < \cdots,$$

and $p_n \to 1$ and $p_{-n} \to 0$ when $n \to \infty$. For $n \in \mathbb{Z}$ put $I_n = [p_n, p_{n+1}]$. Define the map $f_n : I_n \to I_{n-1} \cup I_n \cup I_{n+1}$ by $f_n(p_n) = p_n$, $f_n(p_{n+1}) = p_{n+1}$, $f_n(\frac{2p_n+p_{n+1}}{3}) = p_{n+2}$, $f_n(\frac{p_n+2p_{n+1}}{3}) = p_{n-1}$, and f_n is linear on the intervals complementary to these points. $f : [0,1] \to [0,1]$ is given by $f(0) = 0$, $f(1) = 1$, and $f(x) = f_n(x)$ if $x \in I_n$ (see Figure 2 in [3]).

By Example 1 taken mod 1, we also have

Example 2. There exists a transitive map of the circle which is not strongly transitive.

Let B_n be the bouquet with n-petals generated by n copies of the unit circle, where $n \geq 1$. Using Example 1 taken mod 1 and a rotation among petals with respect to the origin, we can easily have an example on B_n.

Example 3. There exists a transitive map of B_n which is not strongly transitive.

Example 4. Since the map f in Example 1 is actually totally transitive as stated in [3, Example 3], we have a totally transitive interval map which is not strongly transitive. On the other hand, the interval map g below is strongly transitive, but not totally transitive. $g(x) = 2x + 1/2, (0 \leq x \leq 1/4); -2x + 3/2, (1/4 \leq x \leq 3/4); 2x - 3/2, (3/4 \leq x \leq 1)$.

3. Main results

Here is our main result.

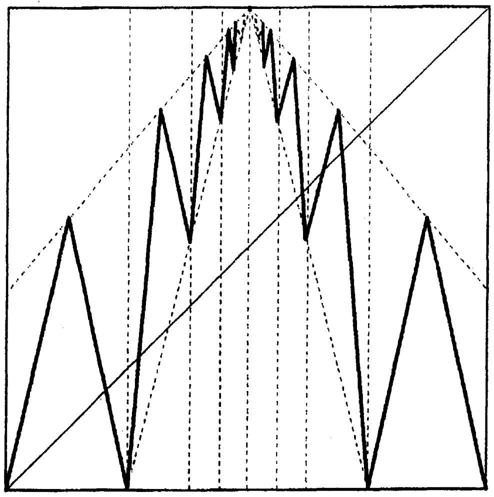
Theorem 3.1. *Let $f : G \to G$ be a graph map with $\#\mathrm{Fix}(f^k) < \infty$ for each $k \geq 1$. If f is transitive, then it is strongly transitive.*

A map f on a graph G is *piecewise monotone* if there is a finite set A in G such that f is monotone on each component of $G \setminus A$.

Corollary 3.2. *Let $f : G \to G$ be a piecewise monotone graph map. If f is transitive, then it is strongly transitive.*

Remark. The interval case of the corollary above was proved by Coven-Mulvey [6].

Example 5. Let $f : [0,1] \to [0,1]$ be the map whose graph appears below. Then f is transitive and the set of fixed points of f^k is finite for each $k \geq 1$. Therefore f is strongly transitive, in fact, for each non-degenerate subinterval J of $[0,1]$, there exists an n such that $f^n(J) = [0,1]$.



Proposition 3.3. *Let $f : T \to T$ be a totally transitive tree map. Then f is strongly transitive if and only if for every non-degenerate connected set J of T, there exists an M such that for any $m \geq M$, $f^m(J) = T$.*

The following generalizes the result for interval maps of Coven-Mulvey [6] to one for tree maps.

Theorem 3.4. *Let $f : T \to T$ be an onto tree map. Let $\mathrm{v}(T)$ be the maximum order of any branch point in T and $N_{\mathrm{v}(T)}$ the least common multiple of $\{2, \ldots, \mathrm{v}(T)\}$. Then the following are equivalent.*

(1) *f is transitive and has a point of period which is prime to $2, \ldots, \mathrm{v}(T)$.*
(2) *$f^{N_{\mathrm{v}(T)}}$ is transitive.*
(3) *f is totally transitive.*
(4) *f is topologically mixing.*

Furthermore, if $\# \mathrm{Fix}(f^k)$ is finite for each $k \geq 1$, then the following is equivalent to

(5) *for every non-degenerate connected set J of T, there exists an M such that for any $m \geq M$, $f^m(J) = T$.*

Remark. The equivalences (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) are well-known [8, Theorem 4.1], [1].

4. Remarks

(I): It is useful to investigate the relation between the dynamics of a graph map and the dynamics of the induced self-homeomorphism of the inverse limit space [2], [3].

Let $f : X \to X$ be an onto map. Associated with f is the inverse limit space $(X, f) = \{(x_0, x_1, \ldots) \mid x_i \in X, \text{ and } f(x_{i+1}) = x_i\}$, and the induced homeomorphism $\hat{f} : (X, f) \to (X, f)$ (which is called the shift homeomorphism), given by $\hat{f}((x_0, x_1, \dots)) = (f(x_0), x_0, x_1, \dots)$.

Proposition 4.1. *Let $f : X \to X$ be an onto map of a metrizable compact space X. If the shift homeomorphism $\hat{f} : (X, f) \to (X, f)$ is strongly transitive, then f is strongly transitive.*

Unfortunately, the shift homeomorphism of a strongly transitive graph map is not always strongly transitive. In fact, we have the following.

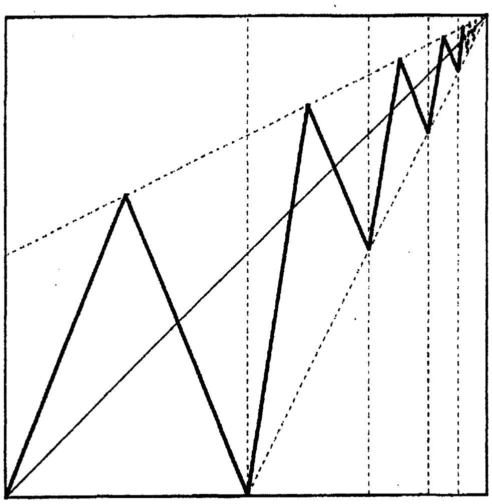
Proposition 4.2. *Let G be a non-degenerate graph and $f : G \to G$ be an onto map. Then the shift homeomorphism $\hat{f} : (G, f) \to (G, f)$ is strongly transitive if and only if G is the circle and f is conjugate to an irrational rotation.*

(II): We note that statement (2) in Proposition 2.1, which was introduced by Parry [7], implies strong transitivity for *tree* maps.

Proposition 4.3. *Let $f : T \to T$ be an onto tree map. Then f is strongly transitive if and only if for every non-empty open set U of T, $\bigcup_{n=0}^{\infty} f^n(U) = T$.*

However, it is not always true for a general graph map.

Example 6. Let $f : [0, 1] \to [0, 1]$ be the map whose graph appears below. Using this map f, we define the circle map $g : S^1 \to S^1$ by $g(e^{2\pi i \theta}) = e^{2\pi i f(\theta)}$, where $0 \le \theta \le 1$. Then the map g is transitive and satisfies statement (2) in Proposition 2.1, but is not strongly transitive.



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