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On existence of continuous selection for finite sets

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Abstract

We prove a theorem which allows, under suitable conditions, to extend a Vietoris continuous selection for two-point subsets of a Hausdorff space X to a Vietoris continuous selection on all finite subsets of X. In particular, such an extension is always possible if X has at most one non-isolated point. We then apply this extension theorem to obtain the following result: If X is a scattered hereditarily paracompact Hausdorff space which has a Vietoris continuous selection for two-point subsets of X, then X also has a Vietoris continuous selection for all finite subsets of X. This gives a partial answer to a question of Gutev and Nogura [5].

1 Preliminaries

For a topological space X we use $\mathcal{F}(X)$ to denote the set of all non-empty closed subsets of X endowed with the Vietoris topology τ_V. Recall that τ_V is generated by all collections of the form

$$\langle \mathcal{V} \rangle = \{S \in \mathcal{F}(X) : S \subseteq \bigcup \mathcal{V} \text{ and } S \cap V \neq \emptyset \text{ whenever } V \in \mathcal{V}\},$$

where $\mathcal{V}$ runs over the finite families of open subsets of X. Define $\mathcal{K}(X) = \{S \in \mathcal{F}(X) : |S| < \omega\}$ and $\mathcal{F}_n(X) = \{S \in \mathcal{F}(X) : |S| \leq n\}$ for $n \in \omega$.

If $\mathcal{D} \subseteq \mathcal{F}(X)$, then a map $f : \mathcal{D} \to X$ is a *selection* on $\mathcal{D}$ provided that $f(S) \in S$ for every $S \in \mathcal{D}$. A selection $f : \mathcal{D} \to X$ is *continuous* if it is continuous with respect to the relative Vietoris topology on $\mathcal{D}$. A selection $f : \mathcal{F}_2(X) \to X$ on $\mathcal{F}_2(X)$ is called a *weak selection*.

Suppose that f is a weak selection. Then f defines a natural order-like relation "$\preceq$ "on X, by letting $x \preceq y$ if and only if $f(\{x, y\}) = x$. For convenience, we will write that $x \prec y$ if $x \preceq y$ and $x \neq y$. If B and C are (not necessarily non-empty) subsets of X , we write $B \prec C$ provided that $y \prec z$ for every $y \in B$ and $z \in C$.

Proposition 1.1. [4, Theorem 3.1] *Let X be a Hausdorff space, $f : \mathcal{F}_2(X) \to X$ be a selection, and let $\preceq$ be the order-like relation generated by f. Furthermore assume that $x, y \in X$ satisfy $x \prec y$. Then f is continuous at $\{x, y\}$ if and only if there exist disjoint open sets U and V such that $x \in U$, $y \in V$, and $U \preceq V$.*

Given a selection f on $\mathcal{F}_2(X)$, one can define a selection f^* on $\mathcal{F}_2(X)$ as follows: $f^*(\{x\}) = x$ for $x \in X$ and $f^*(\{x, y\}) = y$ if and only if $f(\{x, y\}) = x$ whenever $x, y \in X$, $x \neq y$ [4]. Since X is a Hausdorff space, Proposition 1.1 yields that f is a continuous weak selection if and only if f^* is a continuous weak selection. We will denote by $\preceq^*$ the order-like relation generated by f^*. Obviously, $x \preceq y$ if and only if $y \preceq^* x$.

Definition 1.2. [3] Given a weak selection $f : \mathcal{F}_2(X) \to X$ and a set $S \in \mathcal{F}(X)$, we will call a subset $B \subseteq S$ an *f-maximum* (*f-minimum*) of S if $B \in \mathcal{F}(X)$ and the following conditions hold:

(1) $S \setminus B \prec B$ $(B \prec S \setminus B)$,

(2) if $C \subseteq S$, $C \in \mathcal{F}(X)$, and $S \setminus C \prec C$ $(C \prec S \setminus C)$, then $B \subseteq C$.

Clearly, a set B is an f-maximum of S if and only if it is an f^*-minimum of S, so the following proposition is essentially due to Gutev and Nogura [3].

Proposition 1.3. *Let X be a Hausdorff space, $f : \mathcal{F}_2(X) \to X$ a weak selection and $\preceq$ the order-like relation generated by f. Then every non-empty compact subset $S \in \mathcal{F}(X)$ has an unique f-minimum and f-maximum.*

Definition 1.4. [3] Given a weak selection $f : \mathcal{F}_2(X) \to X$ and a compact (in particular, finite) set $S \in \mathcal{F}(X)$, we will use $\min_f S$ and $\max_f S$ to denote the f-minimum and f-maximum of S, respectively.

Proposition 1.5. *Suppose that $f : \mathcal{F}_2(X) \to X$ is a weak selection, $\preceq$ is an order-like relation generated by f, $S, D, E \subseteq X$, $S \subseteq D \cup E$, $D \prec E$, $S \cap D \in \mathcal{F}(X)$ and S is compact. Then $\min_f S \subseteq D$.*

2 Extension of weak selection to finite sets

The special case of item (i) of our next lemma (when $U = X$) has been proved in [3].

Lemma 2.1. *Let X be a Hausdorff space, U its open subset, $Bd\ U$ the boundary of U, $f : \mathcal{F}_2(X) \to X$ a continuous weak selection, and $\preceq$ the order-like relation generated by f. Let $\mathcal{E}_U = \{S \in \mathcal{K}(X) : S \cap U \neq \emptyset\}$.*

(i) If $U \prec Bd\, U$, then the map $\varphi : \mathcal{E}_U \to \mathcal{E}_U$ defined by $\varphi(S) = \min_f(S \cap U)$ is continuous.

(ii) If $Bd\, U \prec U$, then the map $\psi : \mathcal{E}_U \to \mathcal{E}_U$ defined by $\psi(S) = \max_f(S \cap U)$ is continuous.

By using Lemma 2.1, we get the following main theorem.

Theorem 2.2. *Let X be a Hausdorff space, $f : \mathcal{F}_2(X) \to X$ be a continuous weak selection, and let $\preceq$ be the order-like relation generated by f. Furthermore, assume that $p \in X$ and g is a continuous selection on $\mathcal{K}(X \setminus \{p\})$. Then there exists a continuous selection h on $\mathcal{K}(X)$ extending f.*

The following is an application of Theorem 2.2.

Corollary 2.3. *Let X be a Hausdorff space with a single non-isolated point $p \in X$, and let $f : \mathcal{F}_2(X) \to X$ be a weak selection. Then f can be extended to a continuous selection on $\mathcal{K}(X)$.*

3 Selections for finite subsets of paracompact scattered spaces

Let us recall the definition of a scattered space. For every ordinal number α, we define by transfinite induction the α-derivative of a space X: $X^{(0)} = X$; $X^{(\alpha+1)} = (X^{(\alpha)})' = \{x \in X : x \text{ is not an isolated point of } X^{(\alpha)}\}$; $X^{(\alpha)} = \bigcap_{\beta<\alpha} X^{(\beta)}$ if α is limit. A space X is called scattered if $X^{(\alpha)} = \emptyset$ for some ordinal α. For a scattered space X, the *height* $h(X)$ of X is the smallest ordinal α such that $X^{(\alpha)} = \emptyset$. For every $\alpha < h(X)$, each $x \in X^{(\alpha)} \setminus X^{(\alpha+1)}$ is an isolated point of $X^{(\alpha)}$, thus there exists a neighborhood V_x of x such that $V_x \cap X^{(\alpha)} = \{x\}$.

The following lemma is part of folklore, for example [2, Lemma 2.1].

Lemma 3.1. *Suppose that $\mathcal{U}$ is a clopen partition of a space X such that for every $U \in \mathcal{U}$, there exists a continuous selection $f_U : \mathcal{K}(U) \to U$ on $\mathcal{K}(U)$. Then $\mathcal{K}(X)$ has a continuous selection.*

The existence of continuous selections on scattered spaces has been studied in [1, 2]. A countable regular space has a continuous selection if and only if it is scattered [2, Theorem 2.4]. A paracompact scattered space admits a continuous selection provided that every point has a countable pseudo-base [2, Theorem 2.3]. By using transfinite induction with respect to the height $h(X)$ of X and applying Theorem 2.2, we can get our last theorem which contributes to this topic and provides a positive partial answer to Problem 5 from [5].

Theorem 3.2. *If a scattered hereditarily paracompact Hausdorff space X has a weak selection, then it also has a continuous selection on $\mathcal{K}(X)$.*

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