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Formulas with only one atomic formula in Grzegorczyk logic and provability logic *

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Abstract. Here we discuss the set $\mathbf{S}(p)$ of the formulas having only one atomic formula p in Grzegorczyk logic **Grz** and the set $\mathbf{S}(\perp)$ of the formulas having only one atomic formula $\perp$ in provability logic **GL**. We give an inductive construction of representatives in the quotient set $\mathbf{S}(p)/\equiv_{\mathbf{Grz}}$ modulo the provability of **Grz**. On the other hand, in Boolos [1], it was shown that any formula $A \in \mathbf{S}(\perp)$ is equivalent to some truth-functional combination of formulas of the form $\Box^k\perp$ in **GL**. We modify it and give representatives in the quotient set $\mathbf{S}(\perp)/\equiv_{\mathbf{GL}}$, which correspond to the representatives for **Grz**. By these representatives, we clarify structures $\langle \mathbf{S}(p)/\equiv_{\mathbf{Grz}}, \leq_{\mathbf{Grz}}\rangle$ and $\langle \mathbf{S}(\perp)/\equiv_{\mathbf{GL}}, \leq_{\mathbf{GL}}\rangle$, where $\leq_{\mathbf{L}}$ is the derivation in $\mathbf{L} \in \{\mathbf{Grz}, \mathbf{GL}\}$. Comparing these two structures, we also give a way to express the **GL**-provability of formulas in $\mathbf{S}(\perp)$ in **Grz**. In spite of the simplicity of $\mathbf{S}(p)$ and $\mathbf{S}(\perp)$, it is worth considering since the quotient sets are infinite. There is few result on such structures with infinite quotient sets. One result was given in Nishimura [7] in intuitionistic propositional logic, however, the target set of formulas are also simple, with only two atomic formulas p and $\perp$. Shehtman [11] considered more general structure for **Grz**. however, in our simple case, our results have more infomation.

1 Introduction

In this section, we introduce Grzegorczyk logic **Grz** and provability logic (or Gödel-Löb logic) **GL**. We use lower case Latin letters $p, q, \cdots$, possibly with suffixes, for propositional variables. Formulas are defined inductively, as usual, from the propositional variables and $\perp$ (contradiction) by using logical connectives $\wedge$ (conjunction), $\vee$ (disjunction), $\supset$ (implication) and $\Box$ (necessitation). We use upper case Latin letters $A, B, \cdots$, possibly with suffixes, for formulas. The expression $\Box^n A$ is defined inductively as $\Box^0 A = A$ and $\Box^{k+1}A = \Box(\Box^k A)$. For a finite $\mathbf{S}$ of formulas, we put $\Box\mathbf{S} = \{\Box A \mid A \in \mathbf{S}\}$. We fix the enumeration **ENU** of formulas. For a finite non-empty set $\mathbf{S}$ of formulas, the expressions $\bigwedge \mathbf{S}$ and $\bigvee \mathbf{S}$ denotes the formulas $A_1 \wedge A_2 \wedge \cdots \wedge A_n$ and $A_1 \vee A_2 \vee \cdots \vee A_n$, respectively,

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where $\mathbf{S} = \{A_1, \cdots, A_n\}$ and A_i occurs earlier than A_j in **ENU** if $i < j$. Also we put $\bigwedge \emptyset = \bot \supset \bot$ and $\bigvee \emptyset = \bot$. The depth $d(A)$ of a formula A is defined inductively as $d(D) = 0$ for any atomic formula D, $d(B \wedge C) = d(B \vee C) = d(B \supset C) = \max\{d(B), d(C)\}$ and $d(\Box B) = d(B) + 1$. Let D be an atomic formula in $\{p, \bot\}$. By $\mathbf{S}(D)$, we mean the set of formulas constructed from D by using $\wedge$, $\vee$, $\supset$ and $\Box$. We put $\mathbf{S}^n(D) = \{B \in \mathbf{S}(D) \mid d(B) \leq n\}$.

By **Grz**, we mean the smallest set of formulas containing all the tautologies, and the axioms $K : \Box(A \supset B) \supset (\Box A \supset \Box B)$, $T : \Box A \supset A$, $grz : \Box(\Box(A \supset \Box A) \supset A) \supset \Box A$ (Grzegorczyk axiom), and closed under modus ponens and necessitation. By **GL**, we mean the smallest set of formulas containing all the tautologies, and the axioms K and $L : \Box(\Box A \supset A) \supset \Box A$ (Löb's axiom), and closed under modus ponens and necessitation.

Let $\mathbf{L}$ be either **Grz** or **GL**. We write $A \equiv_{\mathbf{L}} B$ if $(A \supset B) \wedge (B \supset A) \in \mathbf{L}$. Also for any equivalent classes $[A]$ and $[B]$ modulo the provability of $\mathbf{L}$, we write $[A] \leq_{\mathbf{L}} [B]$ if $A \supset B \in \mathbf{L}$. We also use this kind of notations for other logics.

A Kripke model is a triple $M = \langle W, R, P \rangle$, where W is a non-empty set, R is a binary relation on W and P is a mapping from the set of propositional variables to 2^W. The truth valuation $(M, \alpha) \models A$, a formula A is true at $\alpha \in W$ in M, is defined by an induction on A in the usual way. The expression $M \models A$ denotes $(M, \alpha) \models A$ for every $\alpha \in W$. Since $P(p) = \{\alpha \mid (M, \alpha) \models p\}$, we can extend the mapping P to the set of formulas as $P(A) = \{\alpha \mid (M, \alpha) \models A\}$.

Lemma 1.1.[1]

(1) $A \in \mathbf{Grz}$ *iff* $M \models A$ *for any finite Kripke model* $M = \langle W, R, P \rangle$ *with partial orders (i. e.* W *is finite and* R *is a partial order).*

(2) $A \in \mathbf{GL}$ *iff* $M \models A$ *for any finite Kripke model* $M = \langle W, R, P \rangle$ *with strict partial orders (i. e.* W *is finite and* R *is a strict partial order).*

Lemma 1.2. *For* $\mathbf{L} \in \{\mathbf{Grz}, \mathbf{GL}\}$,

(1) $\Box(A \wedge B) \equiv_{\mathbf{L}} \Box A \wedge \Box B$,

(2) $\Box^k A \supset \Box^{k+i} A \in \mathbf{L}$ *for* $k > 0, i > 0$,

(3) $A \supset B \in \mathbf{L}$ *implies* $\Box A \supset \Box B \in \mathbf{L}$,

(4) $\Box A \supset B \in \mathbf{L}$ *implies* $\Box A \supset \Box B \in \mathbf{L}$.

2 The structure $\langle \mathbf{S}(p)/\equiv_{\mathbf{Grz}}, \leq_{\mathbf{Grz}} \rangle$

In this section, we construct representatives of equivalent classes in $\mathbf{S}^n(p)/\equiv_{\mathbf{Grz}}$ and clarify the structure $\langle \mathbf{S}(p)/\equiv_{\mathbf{Grz}}, \leq_{\mathbf{Grz}} \rangle$. It is known, however, structures $\langle \mathbf{S}^n(p)/\equiv_{\mathbf{Grz}}, \leq_{\mathbf{Grz}} \rangle$ are boolean(cf. [3]). Also the quotient set is finite. So, we have only to construct representatives of generators of the boolean.

[1] cf. K. Segerberg [10] and A. Chagrov and M. Zakharyaschev [3].

Definition 2.1. A list $F_0, F_1, \cdots$ of formulas are defined inductively as $F_0 = p$ and $F_{k+1} = F_k \supset \Box F_k$.

Lemma 2.2. *For* $k \geq 0$,
(1) $\Box F_k \supset \Box F_{k+i} \in \mathbf{Grz}$ $(i \geq 0)$,
(2) $F_k \wedge F_{k+1} \equiv_{\mathbf{Grz}} \Box F_k$,
(3) $\Box(\Box F_{k+1} \supset \Box F_k) \equiv_{\mathbf{Grz}} \Box F_k$,
(4) $F_{k+2} \wedge (\Box F_{k+1} \supset \Box F_k) \equiv_{\mathbf{Grz}} F_k$,
(5) $\Box F_k \equiv_{\mathbf{Grz}} \bigwedge(\{\Box F_{k+\ell+1} \supset \Box F_{k+\ell} \mid \ell = 0, 1, \cdots, i-1\} \cup \{F_{k+i}, F_{k+i+1}\})$ $(i \geq 0)$.
Proof. By Lemma 1.2(4), we have (1). By the axiom T, we have (2).
We show (3). By the axiom T, we have $(\Box F_{k+1} \supset \Box F_k) \supset (\Box F_{k+1} \supset F_k) \in \mathbf{Grz}$, and using Lemma 1.2(3), $\Box(\Box F_{k+1} \supset \Box F_k) \supset \Box(\Box F_{k+1} \supset F_k) \in \mathbf{Grz}$. By the axiom grz, we have $\Box(\Box F_{k+1} \supset F_k) \supset \Box F_k \in \mathbf{Grz}$. Hence $\Box(\Box F_{k+1} \supset \Box F_k) \supset \Box F_k \in \mathbf{Grz}$. The converse is from Lemma 1.2(4).
We show (4). Let be that $A = F_{k+2} \wedge (\Box F_{k+1} \supset \Box F_k)$. We note that $F_{k+1} \supset (A \supset \Box F_k)$ is a tautology. Using the axiom T, we have $F_{k+1} \supset (A \supset F_k) \in \mathbf{Grz}$. We also note that $F_k \supset (A \supset F_k)$ and $F_k \vee F_{k+1}$ are tautologies. Hence we have $A \supset F_k \in \mathbf{Grz}$. On the other hand, we note that $(F_k \wedge F_{k+1}) \supset \Box F_k$ is a tautology, and using (1) and the axiom T, we have $(F_k \wedge F_{k+1}) \supset \Box F_{k+1} \in \mathbf{Grz}$ and $(F_k \wedge \Box F_{k+1}) \supset \Box F_k \in \mathbf{Grz}$. So, $F_k \supset (F_{k+1} \supset \Box F_{k+1}) = F_k \supset F_{k+2} \in \mathbf{Grz}$ and $F_k \supset (\Box F_{k+1} \supset \Box F_k) \in \mathbf{Grz}$.
We show (5). By (1), we have $\Box F_k \equiv_{\mathbf{Grz}} \bigwedge(\{\Box F_{k+\ell+1} \supset \Box F_{k+\ell} \mid \ell = 0, 1, \cdots, i-1\} \cup \{\Box F_{k+i}\})$, and using (2), we obtain (5). ⊣

Definition 2.3. The sets $\mathbf{G}_n$ $(n = 0, 1, 2, \cdots)$ of formulas are defined as $\mathbf{G}_0 = \{F_0\}$ and $\mathbf{G}_{k+1} = \{F_k, F_{k+1}, \Box F_k \supset \Box F_{k-1}, \cdots, \Box F_1 \supset \Box F_0\}$.

Theorem 2.4.
(1) $\mathbf{S}^n(p)/\equiv_{\mathbf{Grz}} = \{[\bigwedge \mathbf{S}] \mid \mathbf{S} \subseteq \mathbf{G}_n\}$.
(2) *For any subsets* $\mathbf{S}_1$ *and* $\mathbf{S}_2$ *of* $\mathbf{G}_n$, $\mathbf{S}_1 \subseteq \mathbf{S}_2$ *iff* $[\bigwedge \mathbf{S}_2] \leq_{\mathbf{Grz}} [\bigwedge \mathbf{S}_1]$.

To prove the theorem above, we need some preparations.

By **N**, **odd** and **even**, we mean the set of integers $0, 1, 2, \cdots$, the set of odd numbers $1, 3, 5, \cdots$, and the set of even numbers $0, 2, 4, \cdots$, respectively. We define the Kripke model $M_{Grz} = \langle W_{Grz}, R_{Grz}, P_{Grz} \rangle$, where $W_{Grz} = \mathbf{N}$, $R_{Grz} = \{(k, \ell) \mid k \geq \ell\}$ and $P_{Grz}(p) = \mathbf{odd}$.

Lemma 2.5.
(1) $(M_{Grz}, k) \models F_i$ *iff either* $i > k$ *or* $i + k \in \mathbf{odd}$,
(2) $(M_{Grz}, k) \models \Box F_i$ *iff* $i > k$.
Proof. We use an induction on i.
Basis($i = 0$): We note that $0 > k$ does not hold. So, by the definition of P_{Grz}, we have (1). By $kR_{Grz}0$ and $(M_{Grz}, 0) \not\models F_0$, we have $(M_{Grz}, k) \not\models \Box F_0$, and so, we obtain (2).
Induction step($i > 0$): We show (1). By the definition of $\models$, we have

$$(M_{Grz}, k) \not\models F_i \text{ iff } (M_{Grz}, k) \models F_{i-1} \text{ and } (M_{Grz}, k) \not\models \Box F_{i-1}.$$

Using the induction hypothesis,

$$(M_{Grz}, k) \not\models F_i \text{ iff "} i-1 > k \text{ or } i-1+k \in \textbf{odd}\text{" and } i-1 \le k.$$

We note that $i-1 > k$ and $i-1 \le k$ do not hold simultaneously. So,

$$(M_{Grz}, k) \not\models F_i \text{ iff } i-1+k \in \textbf{odd} \text{ and } i-1 \le k.$$

If $i-1 = k$, then $i-1+k(= k+1-1+k = 2k) \notin \textbf{odd}$. So,

$$(M_{Grz}, k) \not\models F_i \text{ iff } i-1+k \in \textbf{odd} \text{ and } i-1 < k.$$

So,

$$(M_{Grz}, k) \not\models F_i \text{ iff } i+k \notin \textbf{odd} \text{ and } i \le k.$$

We show (2). Suppose that $i \not> k$. Then we have $k \ge i$, and so $kR_{Grz}i$. On the other hand, by (1), we have $(M_{Grz}, i) \not\models F_i$. Hence we obtain $(M_{Grz}, k) \not\models \Box F_i$. Suppose that $i > k$. If $kR_{Grz}k'$, then we have $i > k'$, and using (1), $k' \models F_i$. Hence we obtain $(M_{Grz}, k) \models \Box F_i$. ⊣

Lemma 2.6.
(1) *None of the formulas in* $\mathbf{G}_n$ *is provable in* **Grz**.
(2) $\mathbf{S} = \{F_{2n}\} \cup \{\Box F_{2k+1} \supset \Box F_{2k} \mid k = 0, 1, \cdots, n-1\}$ *is a subset of* $\mathbf{G}_{2n}$ *and a subset of* $\mathbf{G}_{2n-1}$*, and* $\bigwedge \mathbf{S} \equiv_{\mathbf{Grz}} p$*,*
(3) *For any* $A, B \in \mathbf{G}_n$*,* $A \ne B$ *implies* $A \vee B \in \mathbf{Grz}$.
(4) *For any* $A, B \in \mathbf{G}_n$*,* $A \ne B$ *implies* $B \equiv_{\mathbf{Grz}} A \supset B$.
(5) $n \ne 0$ *implies* $\bigwedge \mathbf{G}_n \equiv_{\mathbf{Grz}} \Box p$.
Proof. For (1). By Lemma 2.5(2), we have $(M_{Grz}, i) \models \Box F_{i+1}$ and $(M_{Grz}, i) \not\models \Box F_i$, and so, $(M_{Grz}, i) \not\models \Box F_{i+1} \supset \Box F_i$. Also by Lemma 2.5(1), we have $(M_{Grz}, i) \not\models F_i$. Using Lemma 1.1(1), we obtain (1).
For (2). By an induction on n and Lemma 2.2(4).
For (3). We use an induction on n. If $n = 0$, then (3) is clear. Suppose that $n > 0$. We note that $F_n \vee F_{n-1}$ and $(\Box F_{n-1} \supset \Box F_{n-2}) \vee F_{n-1}$ are tautologies. By Lemma 2.2(1), $F_n \vee (\Box F_{k+1} \supset \Box F_k)$ $(k = 0, 1, \cdots, n-2)$ and $(\Box F_{n-1} \supset \Box F_{n-2}) \vee (\Box F_{k+1} \supset \Box F_k)$ $(k = 0, 1, \cdots, n-3)$ are tautologies. The other cases can be shown by the induction hypothesis. Hence we obtain (3).
For (4). By (3).
For (5). By Lemma 2.2(5). ⊣

Lemma 2.7. *Let* $\mathbf{S}_1$ *and* $\mathbf{S}_2$ *be subsets of* $\mathbf{G}_n$. *Then*
(1) $(\bigwedge \mathbf{S}_1) \wedge (\bigwedge \mathbf{S}_2) \equiv_{\mathbf{Grz}} \bigwedge(\mathbf{S}_1 \cup \mathbf{S}_2)$,
(2) $(\bigwedge \mathbf{S}_1) \vee (\bigwedge \mathbf{S}_2) \equiv_{\mathbf{Grz}} \bigwedge(\mathbf{S}_1 \cap \mathbf{S}_2)$,
(3) $(\bigwedge \mathbf{S}_1) \supset (\bigwedge \mathbf{S}_2) \equiv_{\mathbf{Grz}} \bigwedge(\mathbf{S}_2 - \mathbf{S}_1)$,
(4) *if* $\mathbf{S}_1 \ne \emptyset$*, then* $\Box(\bigwedge \mathbf{S}_1) \equiv_{\mathbf{Grz}} \Box F_k$*, where* $k = \min(\{i \mid F_i \in \mathbf{S}_1\} \cup \{i \mid \Box F_{i+1} \supset \Box F_i \in \mathbf{S}_1\})$.

Proof. (1) is from associative law and commutative law of $\wedge$. For (2) and (3), we use Lemma 2.6(3) and Lemma 2.6(4), respectively. We show (4). By Lemma 1.2(1) and Lemma 2.2(3),

$$\Box(\bigwedge \mathbf{S}_1) \equiv_{\mathbf{Grz}} \bigwedge(\Box \mathbf{S}_1)$$

$$\equiv_{\mathbf{Grz}} \bigwedge(\{\Box F_i \mid F_i \in \mathbf{S}_1\} \cup \{\Box(\Box F_{i+1} \supset \Box F_i) \mid \Box F_{i+1} \supset \Box F_i \in \mathbf{S}_1\})$$

$$\equiv_{\mathbf{Grz}} \bigwedge(\{\Box F_i \mid F_i \in \mathbf{S}_1\} \cup \{\Box F_i \mid \Box F_{i+1} \supset \Box F_i \in \mathbf{S}_1\}).$$

Using $k = \min(\{i \mid F_i \in \mathbf{S}_1\} \cup \{i \mid \Box F_{i+1} \supset \Box F_i \in \mathbf{S}_1\})$, and Lemma 2.2(1), we obtain $\Box(\bigwedge \mathbf{S}_1) \equiv_{\mathbf{Grz}} \Box F_k$. ⊣

Lemma 2.8. *Let A be a formula in $\mathbf{S}^n(p)$. Then there exists a subset $\mathbf{S}$ of $\mathbf{G}_n$ such that $A \equiv_{\mathbf{Grz}} \bigwedge \mathbf{S}$.*

Proof. We use an induction on A. If $A = p$, then by Lemma 2.6(2) we obtain the lemma. If $A \neq p$, then by the induction hypothesis, Lemma 2.7 and Lemma 2.2(5), we obtain the lemma. ⊣

Proof of Theorem 2.4. (1) is from Lemma 2.8. The "only if" part of (2) is clear. We show the "if part" of (2). Suppose that $[\bigwedge \mathbf{S}_2] \leq_{\mathbf{Grz}} [\bigwedge \mathbf{S}_1]$ and $\mathbf{S}_1 \not\subseteq \mathbf{S}_2$. By $\mathbf{S}_1 \not\subseteq \mathbf{S}_2$, there exists a formula A in $\mathbf{S}_1 - \mathbf{S}_2$. Using $[\bigwedge \mathbf{S}_2] \leq_{\mathbf{Grz}} [\bigwedge \mathbf{S}_1]$, we have $\bigwedge \mathbf{S}_2 \supset A \in \mathbf{Grz}$. Since $A \notin \mathbf{S}_2$, using Lemma 2.6(4), we have $\bigwedge \mathbf{S}_2 \supset A \equiv_{\mathbf{Grz}} A$, and so, we have $A \in \mathbf{Grz}$. This is in contradiction with Lemma 2.6(1). ⊣

Theorem 2.4 provides representatives of $\mathbf{S}^n(p)/\equiv_{\mathbf{Grz}}$. Next, we clarify the structure $\langle \mathbf{S}(p)/\equiv_{\mathbf{Grz}}, \leq_{\mathbf{Grz}}\rangle$. We first introduce an exact model, which is useful to clarify this kind of structures if the quotient set is finite. Let $\mathbf{S}$ be a set of formulas closed under $\supset$. A Kripke model $\langle W, R, P\rangle$ is said to be an exact model for $\mathbf{S}$ in a logic $\mathbf{L}$ if the following two conditions hold:

(1) P maps $\mathbf{S}$ onto 2^W,

(2) for any $A \in \mathbf{S}$, $A \in \mathbf{L}$ iff $P(A) = W$.

The condition (2) above is equivalent to

(3) for any $A \in \mathbf{S}$, $A \supset B \in \mathbf{L}$ iff $P(A) \subseteq P(B)$.

So, P is a homomorphism from $\langle \mathbf{S}, R\rangle$ to $\langle 2^W, \subseteq\rangle$, where $R = \{(A, B) \mid A \supset B \in \mathbf{L}\}$. If an exact model for a set $\mathbf{S}$ in a logic $\mathbf{L}$ is given, then we can construct a structure isomorphic to $\langle \mathbf{S}/\equiv_{\mathbf{L}}, \leq_{\mathbf{L}}\rangle$ as follows. By (3), for any $B \in [A]_{\equiv_{\mathbf{L}}}$, we have $P(A) = P(B)$. Hence we can define a one-to-one mapping f from $\mathbf{S}/\equiv_{\mathbf{L}}$ to 2^W as $f([A]) = P(A)$. By (1), f is onto, and so, an isomorphism. Hence $\langle \mathbf{S}/\equiv_{\mathbf{L}}, \leq_{\mathbf{L}}\rangle$ is isomorphic to $\langle 2^W, \subseteq\rangle$.

So, giving a concrete exact model for $\mathbf{S}$ in $\mathbf{L}$ is an effective way to clarify a structure $\langle \mathbf{S}/\equiv_{\mathbf{L}}, \leq_{\mathbf{L}}\rangle$. Bruijn [2], Hendriks [6] used this model for the set of disjunction free formulas with finite number of atomic formulas in intuitionistic propositional logic, and gave precise description on the structure, while Diego [4], Urquhart [12] and [8] treated the same structure. [9] also used this model for the set of disjunction free formulas with finite number of atomic formulas in a normal modal logic, called propositional lax logic, and gave precise description on the structure. Also exact models are useful to clarify such kind of structures if quotient sets are finite.

However, if $\mathbf{S}/\equiv_{\mathbf{L}}$ is not finite, then there is no exact model for $\mathbf{S}$ in $\mathbf{L}$. Suppose that there is an exact model. Then $\langle \mathbf{S}/\equiv_{\mathbf{L}}, \leq_{\mathbf{L}}\rangle$ is isomorphic to $\langle 2^W, \subseteq\rangle$. So, 2^W is not finite, and neither is W. So, 2^W is not countable. On the other hand, since $\mathbf{S}$ is countable, so are the quotient set and 2^W. This is a contradiction. Although Nishimura [7] clarified the structure, with infinite quotient set, for the set of formulas with only two atomic formulas p and $\perp$ in intuitionist logic, results on such structures with infinite quotient sets are few.

The quotient set $\mathbf{S}(p)/\equiv_{\mathbf{Grz}}$ is not finite. This makes our problem difficult. There is no exact model for $\mathbf{S}(p)$ in $\mathbf{Grz}$. So, we modify exact model in order to treat such infinite case. The idea has already used in general frame described in [3]. The structure $\langle W, R, \mathcal{Q}\rangle$ in the definition below is called a general frame in [3].

Definition 2.9. A structure $\langle W, R, P, \mathcal{Q}\rangle$ is said to be a general exact model for a set $\mathbf{S}$ of formulas in a logic $\mathbf{L}$ if the following four conditions hold:

(1) $\langle W, R, P\rangle$ is a Kripke model,
(2) $\mathcal{Q}$ is a subset of 2^W
(3) $Im(P)(=\{P(A) \mid A \in \mathbf{S}\}) = \mathcal{Q}$,
(4) for any $A \in \mathbf{S}$, $A \in \mathbf{L}$ iff $P(A) = W$.

Similarly to the description of exact models, we have

Lemma 2.10. *Let $\langle W, R, P, \mathcal{Q}\rangle$ be a general exact model for a set $\mathbf{S}$ of formulas in a logic $\mathbf{L}$. Then*

(1) *we can define a one-to-one mapping f from $\mathbf{S}/\equiv_{\mathbf{L}}$ to $\mathcal{Q}$ as $f([A]) = P(A)$, and f is an isomorphism from $\langle \mathbf{S}/\equiv_{\mathbf{L}}, \leq_{\mathbf{L}}\rangle$ to $\langle \mathcal{Q}, \subseteq\rangle$,*
(2) *$\langle \mathbf{S}/\equiv_{\mathbf{L}}, \leq_{\mathbf{L}}\rangle$ is isomorphic to $\langle \mathcal{Q}, \subseteq\rangle$.*

[11] constructed the structure $\langle W, R, P\rangle$ satisfying the first two conditions in Definition 2.9 and

(5) $\{\{w\} \mid w \in W\} \subseteq Im(P)$,

for the set $\mathbf{S}$ of formulas having only n propositional variables $p_1, \cdots, p_n$. However, he did not clarify $Im(P)(=\mathcal{Q})$, in a sense, and so, we have not known the structure $\langle \mathbf{S}/\equiv_{\mathbf{L}}, \leq_{\mathbf{L}}\rangle$. After the proof of Theorem 2.13, we will give an example as this reason

We will clarify $\langle \mathbf{S}(p)/\equiv_{\mathbf{Grz}}, \leq_{\mathbf{Grz}}\rangle$ by giving a concrete general exact model for $\mathbf{S}(p)$ in $\mathbf{Grz}$. Theorem 2.4 and Lemma 2.5 are useful for it. We put

$\mathcal{Q}_{fin}(\mathbf{S}) = \{\mathbf{S}_1 \mid \mathbf{S}_1 \text{ is a finite subset of } \mathbf{S}\}$,
$\mathbf{odd}_{2k+1} = \{i \in \mathbf{odd} \mid i \geq 2k+1\}$,
$\mathbf{even}_{2k} = \{i \in \mathbf{odd} \mid i \geq 2k\}$,
$\mathbf{N}_k = \{i \mid i \geq k\}$,
$\mathcal{Q}_{inf} = \{\mathbf{odd}_{2k+1} \mid k \in \mathbf{N}\} \cup \{\mathbf{even}_{2k} \mid k \in \mathbf{N}\} \cup \{\mathbf{N}_k \mid k \in \mathbf{N}\}$,
$\mathcal{Q}_{Grz} = \{\mathbf{S}_1 \cup \mathbf{S}_2 \mid \mathbf{S}_1 \in \mathcal{Q}_{fin}(\mathbf{N}), \mathbf{S}_2 \in \mathcal{Q}_{inf} \cup \{\emptyset\}\}$.

Lemma 2.11. $\mathbf{S}, \mathbf{T} \in \mathcal{Q}_{inf}$ *implies* $\mathbf{S} \cap \mathbf{T} \in \mathcal{Q}_{Grz}$.

Proof. Below, we only show the case that $\mathbf{S} \in \{\mathbf{odd}_{2k+1} \mid k \in \mathbf{N}\}$. The other cases can be shown similarly.

$\mathbf{odd}_{2k+1} \cap \mathbf{odd}_{2\ell+1} = \mathbf{odd}_{\max\{2k+1,2\ell+1\}}$,
$\mathbf{odd}_{2k+1} \cap \mathbf{even}_{2\ell} = \emptyset$,
$\mathbf{odd}_{2k+1} \cap \mathbf{N}_{2\ell+1} = \mathbf{odd}_{\max\{2k+1,2\ell+1\}}$,
$\mathbf{odd}_{2k+1} \cap \mathbf{N}_{2\ell} = \begin{cases} \mathbf{odd}_{2k+1} & \text{if } 2k+1 > 2\ell \\ \{2\ell\} \cup \mathbf{odd}_{2\ell+1} & \text{if } 2k+1 < 2\ell. \end{cases}$ ⊣

Lemma 2.12. *$\mathcal{Q}_{Grz}$ is closed under $\cap$.*

Proof. Let $\mathbf{S}$ and $\mathbf{T}$ be any sets in $\mathcal{Q}_{Grz}$. Then there exist $\mathbf{S}_1, \mathbf{T}_1 \in \mathcal{Q}_{fin}(\mathbf{N})$ and $\mathbf{S}_2, \mathbf{T}_2 \in \mathcal{Q}_{inf} \cup \{\emptyset\}$ such that $\mathbf{S} = \mathbf{S}_1 \cup \mathbf{S}_2$ and $\mathbf{T} = \mathbf{T}_1 \cup \mathbf{T}_2$. So,

$$\mathbf{S} \cap \mathbf{T} = (\mathbf{S}_1 \cup \mathbf{S}_2) \cap (\mathbf{T}_1 \cup \mathbf{T}_2) = (\mathbf{S}_1 \cap (\mathbf{T}_1 \cup \mathbf{T}_2)) \cup (\mathbf{S}_2 \cap \mathbf{T}_1) \cup (\mathbf{S}_2 \cap \mathbf{T}_2).$$

We put $\mathbf{U} = (\mathbf{S}_1 \cap (\mathbf{T}_1 \cup \mathbf{T}_2)) \cup (\mathbf{S}_2 \cap \mathbf{T}_1)$, and note that $\mathbf{U} \in \mathcal{Q}_{fin}(\mathbf{N})$. If either $\mathbf{S}_2$ or $\mathbf{T}_2$ is $\emptyset$, then we have $\mathbf{S} \cap \mathbf{T} = \mathbf{U} \cup \emptyset \in \mathcal{Q}_{Grz}$. So, we assume that $\mathbf{S}_2, \mathbf{T}_2 \in \mathcal{Q}_{inf}$. By Lemma 2.10, there exist $\mathbf{U}_1 \in \mathcal{Q}_{fin}(\mathbf{N})$ and $\mathbf{U}_2 \in \mathcal{Q}_{inf}$ such that $\mathbf{S}_2 \cap \mathbf{T}_2 = \mathbf{U}_1 \cup \mathbf{U}_2$. Clearly, $(\mathbf{U} \cup \mathbf{U}_1) \in \mathcal{Q}_{fin}(\mathbf{N})$, and so, $\mathbf{S} \cap \mathbf{T} = \mathbf{U} \cup (\mathbf{S}_2 \cap \mathbf{T}_2) = (\mathbf{U} \cup \mathbf{U}_1) \cup \mathbf{U}_2 \in \mathcal{Q}_{Grz}$. ⊣

Theorem 2.13. *The structure $\langle W_{Grz}, R_{Grz}, P_{Grz}, \mathcal{Q}_{Grz}\rangle$ is a general exact model for $\mathbf{S}(p)$ in* **Grz**.

Proof. The conditions (1) and (2) in Definition 2.9 are clear. So, it is sufficient to show the following:

(3) $Im(P_{Grz}) = \mathcal{Q}_{Grz}$,

(4) $A \in \mathbf{Grz}$ iff $P_{Grz}(A) = W_{Grz}$.

We show (3). Suppose that $\mathbf{S} \in Im(P_{Grz})$. Then there exists a formula A such that $P_{Grz}(A) = \mathbf{S}$. Let be that $n = \max\{d(A), 1\}$. Then using Lemma 2.8, there exists a subset $\mathbf{S}_1$ of $\mathbf{G}_n$ such that $A \equiv_{\mathbf{Grz}} \bigwedge \mathbf{S}_1$. So,

$$P_{Grz}(A) = \{k \mid (M_{Grz}, k) \models \bigwedge \mathbf{S}_1\} = \{k \mid (M_{Grz}, k) \models B, \text{ for any } B \in \mathbf{S}_1\}$$

$$= \bigcap_{B \in \mathbf{S}_1} \{k \mid (M_{Grz}, k) \models B\} = \bigcap_{B \in \mathbf{S}_1} P_{Grz}(B).$$

By Lemma 2.5, we have

(5) $P_{Grz}(\Box F_{k+1} \supset \Box F_k) = \mathbf{N} - \{k\} = \{i \mid 0 \le i < k\} \cup \mathbf{N}_{k+1} \in \mathcal{Q}_{Grz}$,

(6) $P_{Grz}(F_{2k}) = \{i \mid i < 2k\} \cup \mathbf{odd}_1 \in \mathcal{Q}_{Grz}$,

(7) $P_{Grz}(F_{2k+1}) = \{i \mid i < 2k+1\} \cup \mathbf{even}_0 \in \mathcal{Q}_{Grz}$.

So, for any formula $B \in \mathbf{G}_n$, we have $P_{Grz}(B) \in \mathcal{Q}_{Grz}$, and using Lemma 2.11, we obtain $\mathbf{S} = P_{Grz}(A) \in \mathcal{Q}_{Grz}$.

Suppose that $\mathbf{S} \in \mathcal{Q}_{Grz}$. Then there exist $\mathbf{S}_1 \in \mathcal{Q}_{fin}(\mathbf{N})$ and $\mathbf{S}_2 \in \mathcal{Q}_{inf} \cup \{\emptyset\}$ such that $\mathbf{S} = \mathbf{S}_1 \cup \mathbf{S}_2$. On the other hand, by Lemma 2.5, (5), (6) and (7), we have

$P_{Grz}(\Box p) = \emptyset$,

$P_{Grz}((\Box F_{k+1} \supset \Box F_k) \supset \Box p) = \{k\}$,

$P_{Grz}(F_{2k} \supset \Box p) = \mathbf{N} - (\{i \mid i < 2k\} \cup \mathbf{odd}_1) = \mathbf{even}_{2k}$,

$P_{Grz}(F_{2k+1} \supset \Box p) = \mathbf{N} - (\{i \mid i < 2k+1\} \cup \mathbf{even}_0) = \mathbf{odd}_{2k+1}$.

Also,

$P_{Grz}((F_{2k} \supset \Box p) \vee (F_{2k+1} \supset \Box p)) = \mathbf{N}_{2k}$,

$P_{Grz}((F_{2k+1} \supset \Box p) \vee (F_{2k+2} \supset \Box p)) = \mathbf{N}_{2k+1}$.
Here we note that $\mathbf{S}_2 = P_{Grz}(B)$ for some $B \in \mathbf{S}(p)$. We also note for any formulas C and D,

$$P_{Grz}(C) \cup P_{Grz}(D) = \{i \mid (M_{Grz}, i) \models C\} \cup \{i \mid (M_{Grz}, i) \models D\}$$
$$= \{i \mid (M_{Grz}, i) \models C \vee D\} = P_{Grz}(C \vee D).$$

Hence

$$\mathbf{S} = \mathbf{S}_1 \cup \mathbf{S}_2 = (\bigcup_{k \in \mathbf{S}_1} \{k\}) \cup P_{Grz}(B) = (\bigcup_{k \in \mathbf{S}_1} P_{Grz}((\Box F_{k+1} \supset \Box F_k) \supset \Box p))) \cup P_{Grz}(B)$$

$$= P_{Grz}(\bigvee\{(\Box F_{k+1} \supset \Box F_k) \supset \Box p \mid k \in \mathbf{S}_1\} \vee B).$$

If $\mathbf{S}_1 = \emptyset$, then the set above is $P_{Grz}(B)$, and hence $\mathbf{S} \in Im(P_{Grz})$; if not, we have $\bigvee\{(\Box F_{k+1} \supset \Box F_k) \supset \Box p \mid k \in \mathbf{S}_1\} \in \mathbf{S}(p)$, and hence we obtain $\mathbf{S} \in Im(P_{Grz})$.

We show (4). Form Lemma 1.1, we have the "only if" part. Suppose that $A \in \mathbf{S}^n(p) - \mathbf{Grz}$. Then $A \not\equiv_{\mathbf{Grz}} \bigwedge \emptyset$. Using Lemma 2.8, there exists a non-empty subset $\mathbf{S}$ of $\mathbf{G}_n$ such that $A \equiv_{\mathbf{Grz}} \bigwedge \mathbf{S}$. Since $\mathbf{S} \neq \emptyset$, using Lemma 2.6(1), $(M_{Grz}, k) \not\models \bigwedge \mathbf{S}$ for some $k \in W_{Grz}$. By $A \supset \bigwedge \mathbf{S}$ and the "only if" part of (4). we have $P_{Grz}(A \supset \bigwedge \mathbf{S}) = W_{Grz}$, and so, $(M_{Grz}, k) \not\models A$. Hence $P_{Grz}(A) \neq W_{Grz}$. ⊣

From the theorem above, we can see that the set $\{3n \mid n \in \mathbf{N}\}$ does not belongs to $\mathcal{Q}(= Im(P))$, while from [11] we can't see it directly.

Corollary 2.14. *$\langle \mathbf{S}(p)/\equiv_{\mathbf{Grz}}, \leq_{\mathbf{Grz}}\rangle$ is isomorphic to $\langle \mathcal{Q}_{Grz}, \subseteq\rangle$.*

Also by sketching the proof of Theorem 2.13, we also have

Theorem 2.15. *Let be that $M^n_{Grz} = \langle W^n_{Grz}, R^n_{Grz}, P^n_{Grz}\rangle$, where $W^n_{Grz} = \{i \mid 0 \leq i \leq n\}$, $R^n_{Grz} = R_{Grz} \cap (W^n_{Grz} \times W^n_{Grz})$ and $P^n_{Grz}(p) = P_{Grz}(p) \cap W^n_{Grz}$. Then*
(1) M^n_{Grz} is an exact model for $\mathbf{S}^n(p)$ in **Grz**,
(2) $\langle \mathbf{S}^n(p)/\equiv_{\mathbf{Grz}}, \leq_{\mathbf{Grz}}\rangle$ is isomorphic to $\langle 2^{W^n_{Grz}}, \subseteq\rangle$,
(3) for any $A \in \mathbf{S}^n(p)$, $A \equiv_{\mathbf{Grz}} \bigwedge\{f(k) \mid (M^n_{Grz}, k) \not\models A\}$, where

$$f(k) = \begin{cases} \Box F_{k+1} \supset \Box F_k & \text{if } 0 \leq k \leq n-2 \\ F_k & \text{if } n-1 \leq k \leq n. \end{cases}$$

3 The structure $\langle \mathbf{S}(\bot)/\equiv_{\mathbf{GL}}, \leq_{\mathbf{GL}}\rangle$

In this section, we treat the structure $\langle \mathbf{S}(\bot)/\equiv_{\mathbf{GL}}, \leq_{\mathbf{GL}}\rangle$ as in the previous section. In [1], we can see many useful results for our study, and most of the result here can be given by considering carefully the correspondence between proofs in [1] and the structure. However, since [1] does not aim at the structure, there are notions that we do not need.

Also we would like to compare the structures here with the one in the previous section. So, we treat the structure in a similar way to the previous sections. Some lemmas below can be proved using results in [1] but we will give their proof directly in the case that we have to define new notions in order to use the result in [1] and the case that it seems to be better in order to compare the structure with the one in the previous section.

First, we construct representatives in the quotient set $\mathbf{S}^n(\bot)/\equiv_{\mathbf{GL}}$.

Definition 3.1. The sets $\mathbf{G}_n^*$ $(n = 0, 1, 2, \cdots)$ of formulas are defined as $\mathbf{G}_n^* = \{\Box^n\bot, \Box^n\bot \supset \Box^{n-1}\bot, \cdots, \Box\bot \supset \bot\}$.

Lemma 3.2. *For* $k \geq 0$, $\Box(\Box^{k+1}\bot \supset \Box^k\bot) \equiv_{\mathbf{Grz}} \Box^{k+1}\bot$.
Proof. By Lemma 1.2(3) and the axiom L. ⊣

Theorem 3.3.
(1) $\mathbf{S}^n(\bot)/\equiv_{\mathbf{GL}} = \{[\bigwedge \mathbf{S}] \mid \mathbf{S} \subseteq \mathbf{G}_n^*\}$.
(2) *For any subsets* $\mathbf{S}_1$ *and* $\mathbf{S}_2$ *of* $\mathbf{G}_n^*$, $\mathbf{S}_1 \subseteq \mathbf{S}_2$ *iff* $[\bigwedge \mathbf{S}_2] \leq_{\mathbf{GL}} [\bigwedge \mathbf{S}_1]$.

Theorem 3.3 can be proved using the lemma in [1] below, but we have to check depth of formulas and independence of the elements in $\mathbf{G}_n^*$.

Lemma 3.4([1]). *If A is a formula in* $\mathbf{S}(\bot)$, *then there exists a truth-functional combination B of formulas of the form* $\Box^k\bot$ *such that* $A \equiv_{\mathbf{GL}} B$.

Here we prove Theorem 3.3 in a similar way to section 2. Some lemmas we will show are useful to the investigation in section 4.

We define the Kripke model $M_{GL} = \langle W_{GL}, R_{GL}, P_{GL}\rangle$, where $W_{GL} = \mathbf{N}$, $R_{GL} = \{(k, \ell) \mid k > \ell\}$ and $P_{GL}(A) = \emptyset$ for any propositional variable A.

Lemma 3.5. $(M_{Grz}, k) \models \Box^i\bot$ *iff* $i > k$.
Proof. By an induction i. ⊣

[1] introduced two notions *rank* and *trace*, and using the result in [1], we can show that the rank of $k \in W_{GL}$ is k and that the trace of a formula A is $P_{GL}(A)$. As the result, the lemma above is just the lemma in [1](Lemma 5 in Chapter 7).

Lemma 3.6.
(1) *None of the formulas in* $\mathbf{G}_n^*$ *is provable in* $\mathbf{GL}$.
(2) *For* $k \leq n$, $\bigwedge\{\Box^n\bot, \Box^n\bot \supset \Box^{n-1}\bot, \cdots, \Box^{k+1}\bot \supset \Box^k\bot\} \equiv_{\mathbf{GL}} \Box^k\bot$.
(3) *For any* $A, B \in \mathbf{G}_n^*$, $A \neq B$ *implies* $A \vee B \in \mathbf{GL}$,
(4) *For any* $A, B \in \mathbf{G}_n^*$, $A \neq B$ *implies* $B \equiv_{\mathbf{GL}} A \supset B$.
Proof. By Lemma 3.5 and Lemma 1.1(2), we have (1). By Lemma 1.2(2), we have (2) and (3). By (3), we have (4). ⊣

Lemma 3.7. *Let* $\mathbf{S}_1$ *and* $\mathbf{S}_2$ *be subsets of* $\mathbf{G}_n^*$. *Then*

(1) $(\bigwedge \mathbf{S}_1) \wedge (\bigwedge \mathbf{S}_2) \equiv_{\mathbf{GL}} \bigwedge(\mathbf{S}_1 \cup \mathbf{S}_2)$,

(2) $(\bigwedge \mathbf{S}_1) \vee (\bigwedge \mathbf{S}_2) \equiv_{\mathbf{GL}} \bigwedge(\mathbf{S}_1 \cap \mathbf{S}_2)$,

(3) $(\bigwedge \mathbf{S}_1) \supset (\bigwedge \mathbf{S}_2) \equiv_{\mathbf{GL}} \bigwedge(\mathbf{S}_2 - \mathbf{S}_1)$,

(4) *if* $\mathbf{S}_1 \neq \emptyset$, *then* $\Box(\bigwedge \mathbf{S}_1) \equiv_{\mathbf{GL}} \Box^k\bot$, *where* $k = \min(\{n+1 \mid \Box^n\bot \in \mathbf{S}_1\} \cup \{i+1 \mid \Box^{i+1}\bot \supset \Box^i\bot \in \mathbf{S}_1\})$.

Proof. (1),(2) and (3) can be shown similarly to Lemma 2.7. (4) can also be shown, but we use Lemma 3.2 and Lemma 1.2(2) instead of Lemma 2.2(3) and Lemma 2.2(1). ⊣

Lemma 3.8. *Let* A *be a formula in* $\mathbf{S}^n(\bot)$. *Then there exists a subset* $\mathbf{S}$ *of* $\mathbf{G}_n^*$ *such that* $A \equiv_{\mathbf{GL}} \bigwedge \mathbf{S}$.

Proof. We use an induction on A. If $A = \bot$, then by Lemma 3.6(2),

$$\bigwedge \mathbf{G}_n^* = \bigwedge\{\Box^n\bot, \Box^n\bot \supset \Box^{n-1}\bot, \cdots, \Box\bot \supset \bot\} \equiv_{\mathbf{GL}} \bot = A.$$

If $A \neq \bot$, then by the induction hypothesis, Lemma 3.7 and Lemma 3.6(2), we obtain the lemma. ⊣

Similarly to Theorem 2.4, Theorem 3.3 is proved by Lemma 3.8, Lemma 3.6(1) and Lemma 3.6(4).

Next, we clarify the structure $\langle \mathbf{S}(\bot)/\equiv_{\mathbf{GL}}, \leq_{\mathbf{GL}}\rangle$ by giving a concrete general exact model for $\mathbf{S}(\bot)$ in $\mathbf{GL}$. We put

$\mathcal{Q}_{GL} = \{\mathbf{S}_1 \cup \mathbf{S}_2 \mid \mathbf{S}_1 \in \mathcal{Q}_{fin}(\mathbf{N}), \mathbf{S}_2 \in \{\mathbf{N}_k \mid k \in \mathbf{N}\} \cup \{\emptyset\}\}$.

Lemma 3.9. $\mathcal{Q}_{GL}$ *is closed under* $\cap$.

Proof. Let $\mathbf{S}$ and $\mathbf{T}$ be any sets in $\mathcal{Q}_{GL}$. Then there exist $\mathbf{S}_1, \mathbf{T}_1 \in \mathcal{Q}_{fin}(\mathbf{N})$ and $\mathbf{S}_2, \mathbf{T}_2 \in \{\mathbf{N}_k \mid k \in \mathbf{N}\} \cup \{\emptyset\}$ such that $\mathbf{S} = \mathbf{S}_1 \cup \mathbf{S}_2$ and $\mathbf{T} = \mathbf{T}_1 \cup \mathbf{T}_2$. Similarly to the proof of Lemma 2.11, $\mathbf{S} \cap \mathbf{T} = \mathbf{U} \cup (\mathbf{S}_2 \cap \mathbf{T}_2)$, where $\mathbf{U} = (\mathbf{S}_1 \cap (\mathbf{T}_1 \cup \mathbf{T}_2)) \cup (\mathbf{S}_2 \cap \mathbf{T}_1) \in \mathcal{Q}_{fin}(\mathbf{N})$. On the other hand, we have $\mathbf{S}_2 \cap \mathbf{T}_2 \in \{\mathbf{N}_k \mid k \in \mathbf{N}\} \cup \{\emptyset\}$ since $\mathbf{N}_i \cap \mathbf{N}_j = \mathbf{N}_{\max\{i,j\}}$ and $\mathbf{N}_i \cap \emptyset = \emptyset$. Hence $\mathbf{S} \cap \mathbf{T} \in \mathcal{Q}_{GL}$. ⊣

Theorem 3.10. *The structure* $\langle W_{GL}, R_{GL}, P_{GL}, \mathcal{Q}_{GL}\rangle$ *is a general exact model for* $\mathbf{S}(\bot)$ *in* $\mathbf{GL}$.

Proof. The conditions (1) and (2) in Definition 2.9 are clear. Also (4) in Definition 2.9 can be shown similarly to the proof of Theorem 2.13 using Lemma 3.6(1) and Lemma 3.8. So, it is sufficient to show

(3) $Im(P_{GL}) = \mathcal{Q}_{GL}$.

Suppose that $\mathbf{S} \in Im(P_{Grz})$. Then there exists a formula A such that $P_{GL}(A) = \mathbf{S}$. Let be that $n = d(A)$. Then using Lemma 3.8, there exists a subset $\mathbf{S}_1$ of $\mathbf{G}_n^*$ such that $A \equiv_{\mathbf{GL}} \bigwedge \mathbf{S}_1$. So, similarly to Theorem 2.13, we have $P_{GL}(A) = \bigcap_{B \in \mathbf{S}_1} P_{GL}(B)$. By Lemma 3.5, we have

(5) $P_{GL}(\Box^{k+1}\bot \supset \Box^k\bot) = \mathbf{N} - \{k\} = \{i \mid 0 \leq i < k\} \cup \mathbf{N}_{k+1} \in \mathcal{Q}_{GL}$,

(6) $P_{GL}(\Box^k\bot) = \{i \mid i < k\} \in \mathcal{Q}_{GL}$.
So, for any formula $B \in \mathbf{G}^*_n$, we have $P_{GL}(B) \in \mathcal{Q}_{GL}$, and using Lemma 3.9, we obtain $\mathbf{S} = P_{Grz}(A) \in \mathcal{Q}_{Grz}$.

Suppose that $\mathbf{S} \in \mathcal{Q}_{GL}$. Then there exist $\mathbf{S}_1 \in \mathcal{Q}_{fin}(\mathbf{N})$ and $\mathbf{S}_2 \in \{\mathbf{N}_k \mid k \in \mathbf{N}\} \cup \{\emptyset\}$ such that $\mathbf{S} = \mathbf{S}_1 \cup \mathbf{S}_2$. On the other hand, by Lemma 3.5, (5) and (6), we have
(7) $P_{GL}(\bot) = \emptyset$,
(8) $P_{GL}((\Box^{k+1}\bot \supset \Box^k\bot) \supset \bot) = \{k\} \in \mathcal{Q}_{GL}$,
(9) $P_{GL}(\Box^k\bot \supset \bot) = \mathbf{N}_k \in \mathcal{Q}_{GL}$.
Here we note that $\mathbf{S}_2 = P_{GL}(B)$ for some $B \in \mathbf{S}(\bot)$. Also similarly to Theorem 2.14, for any formulas C and D, $P_{GL}(C) \cup P_{GL}(D) = P_{GL}(C \vee D)$. Hence

$$\mathbf{S} = \mathbf{S}_1 \cup \mathbf{S}_2 = (\bigcup_{k \in \mathbf{S}_1} \{k\}) \cup P_{GL}(B) = (\bigcup_{k \in \mathbf{S}_1} P_{GL}((\Box^{k+1}\bot \supset \Box^k\bot) \supset \bot) \cup P_{Grz}(B)$$

$$= P_{Grz}(\bigvee\{(\Box^{k+1}\bot \supset \Box^k\bot) \supset \bot \mid k \in \mathbf{S}_1\} \vee B).$$

Hence we obtain $\mathbf{S} \in Im(P_{GL})$. ⊣

Considering the relations that the *rank* of $k \in W_{GL}$ is k and that the *trace* of a formula A is $P_{GL}(A)$, the conditions (4), (8) and $Im(P_{GL}) \subseteq \mathcal{Q}_{GL}$, in the proof above, have been shown in [1].

Corollary 3.11. *$\langle \mathbf{S}(\bot)/\equiv_{\mathbf{GL}}, \leq_{\mathbf{GL}}\rangle$ is isomorphic to $\langle \mathcal{Q}_{GL}, \subseteq\rangle$.*

Also by sketching the proof of Theorem 3.10, we also have

Theorem 3.12. *Let be that $M^n_{GL} = \langle W^n_{GL}, R^n_{GL}, P^n_{GL}\rangle$, where $W^n_{GL} = \{i \mid 0 \leq i \leq n\}$, $R^n_{GL} = R_{GL} \cap (W^n_{GL} \times W^n_{GL})$ and $P^n_{GL}(q) = P_{GL}(q)$ for any q. Then*
(1) M^n_{GL} is an exact model for $\mathbf{S}^n(\bot)$ in $\mathbf{GL}$,
(2) $\langle \mathbf{S}^n(\bot)/\equiv_{\mathbf{GL}}, \leq_{\mathbf{GL}}\rangle$ is isomorphic to $\langle 2^{W^n_{GL}}, \subseteq\rangle$,
(3) for any $A \in \mathbf{S}^n(\bot)$, $A \equiv_{\mathbf{GL}} \bigwedge\{f(k) \mid (M^n_{GL}, k) \not\models A\}$, where

$$f(k) = \begin{cases} \Box^{k+1}\bot \supset \Box^k\bot & \text{if } 0 \leq k \leq n-1 \\ \Box^k\bot & \text{if } k = n. \end{cases}$$

4 GL-provability of formulas in S(⊥) in Grz

Comparing $\langle \mathbf{S}(p)/\equiv_{\mathbf{Grz}}, \leq_{\mathbf{Grz}}\rangle$ with $\langle \mathbf{S}(\bot)/\equiv_{\mathbf{GL}}, \leq_{\mathbf{GL}}\rangle$, we can see some kinds of correspondence between them. We show one kind of correspondence by giving a way to express the **GL**-provability of formulas in $\mathbf{S}(\bot)$ in **Grz**. More precisely, we give a mapping g from $\mathbf{S}(\bot)$ to $\mathbf{S}(p)$ satisfying for any formula $A \in \mathbf{S}(\bot)$,

$$A \in \mathbf{GL} \text{ iff } g(A) \in \mathbf{Grz}.$$

On the other hand, Goldblatt [5] gave a mapping $*$, satisfying for any formula A,

$$A^* \in \mathbf{GL} \text{ iff } A \in \mathbf{Grz}.$$

From this, we can know the **Grz**-provability of any formulas in **GL**. Our result is a kind of the converse of his result with restriction.

Definition 4.1. A list $g_0, g_1, \cdots$ of mappings from $\mathbf{S}(\bot)$ to $\mathbf{S}(p)$ are defined inductively as follows:
(1) $g_i(\bot) = \Box F_i$,
(2) $g_i(B \otimes C) = g_i(B) \otimes g_i(C)$, for $\otimes \in \{\wedge, \vee, \supset\}$,
(3) $g_i(\Box B) = \Box g_{i+1}(B)$.

Also we put $g_i(\mathbf{S}) = \{g_i(A) \mid A \in \mathbf{S}\}$. The mapping g_0 transforms the formula $\Box(\Box\bot \supset \bot) \supset \Box\bot$, an instance of the axiom L, into a formula

$$g_0(\Box(\Box\bot \supset \bot) \supset \Box\bot) = \Box(\Box\Box(F_1 \supset \Box F_1) \supset \Box F_1) \supset \Box\Box F_1).$$

Here we note that the image is similar to $\Box(\Box(F_1 \supset \Box F_1) \supset F_1) \supset \Box F_1$, an instance of the axiom grz, and that the image and the instance are equivalent in **Grz**. The mappings g_i also have the same property since

$$g_i(\Box(\Box\bot \supset \bot) \supset \Box\bot) = \Box(\Box\Box(F_{i+1} \supset \Box F_{i+1}) \supset \Box F_{i+1}) \supset \Box\Box F_{i+1}).$$

Lemma 4.2.
(1) $g_i(\Box^k\bot) = \Box^{k+1}F_{k+i} \equiv_{\mathbf{Grz}} \Box F_{k+i}$.
(2) $g_i(\Box^{k+1}\bot \supset \Box^k\bot) \equiv_{\mathbf{Grz}} \Box F_{k+i+1} \supset \Box F_{k+i}$.
Proof. We have (1) by an induction on k. By (1), we have (2). $\dashv$

Theorem 4.3. *For any formula $A \in \mathbf{S}(\bot)$, and for any i,*

$$A \in \mathbf{GL} \textit{ iff } g_i(A) \in \mathbf{Grz}.$$

To prove the theorem above, we define a mapping h and show three lemmas.

Definition 4.4. For a subset $\mathbf{S}$ of $\mathbf{G}_n^*$, we put

$$h_i(\mathbf{S}) = \{F_{n+i} \mid \Box^n\bot \in \mathbf{S}\} \cup \{F_{n+i+1} \mid \Box^n\bot \in \mathbf{S}\}$$
$$\cup \bigcup_{k=1}^{n} \{g_i(\Box^k\bot \supset \Box^{k-1}\bot) \mid \Box^k\bot \supset \Box^{k-1}\bot \in \mathbf{S}\}.$$

Lemma 4.5. *Let $\mathbf{S}$ and $\mathbf{S}_1$ be subsets of $\mathbf{G}_n^*$. Then for any i,*
(1) $h_i(\mathbf{S}) \subseteq \mathbf{G}_{n+i+1}$,
(2) $\bigwedge g_i(\mathbf{S}) \equiv_{\mathbf{Grz}} \bigwedge h_i(\mathbf{S})$,
(3) $\mathbf{S} \neq \mathbf{S}_1$ *implies* $\bigwedge h_i(\mathbf{S}) \not\equiv_{\mathbf{Grz}} \bigwedge h_i(\mathbf{S}_1)$.
Proof. (1) is clear from the Lemma 4.2. (2) is from Lemma 2.2(2). We show (3). Suppose that $\mathbf{S} \neq \mathbf{S}_1$. Then either $\mathbf{S} \not\subseteq \mathbf{S}_1$ or $\mathbf{S}_1 \not\subseteq \mathbf{S}$. Using (1) and Theorem 2.4(2), we

have either $[\bigwedge h_i(\mathbf{S})] \not\leq_{\mathbf{Grz}} [\bigwedge h_i(\mathbf{S}_1)]$ or $[\bigwedge h_i(\mathbf{S}_1)] \not\leq_{\mathbf{Grz}} [\bigwedge h_i(\mathbf{S})]$. So, either $(\bigwedge h_i(\mathbf{S})) \supset (\bigwedge h_i(\mathbf{S}_1)]) \notin \mathbf{Grz}$ or $(\bigwedge h_i(\mathbf{S}_1)) \supset (\bigwedge h_i(\mathbf{S})]) \notin \mathbf{Grz}$. Hence $\bigwedge h_i(\mathbf{S}) \not\equiv_{\mathbf{Grz}} \bigwedge h_i(\mathbf{S}_1)$. ⊣

Lemma 4.6. *Let* $\mathbf{S}_1$ *and* $\mathbf{S}_2$ *be subsets of* $\mathbf{G}_n^*$. *Then for any* i,

(1) $(\bigwedge h_i(\mathbf{S}_1)) \wedge (\bigwedge h_i(\mathbf{S}_2)) \equiv_{\mathbf{Grz}} \bigwedge h_i(\mathbf{S}_1 \cup \mathbf{S}_2)$,

(2) $(\bigwedge h_i(\mathbf{S}_1)) \vee (\bigwedge h_i(\mathbf{S}_2)) \equiv_{\mathbf{Grz}} \bigwedge h_i(\mathbf{S}_1 \cap \mathbf{S}_2)$,

(3) $(\bigwedge h_i(\mathbf{S}_1)) \supset (\bigwedge h_i(\mathbf{S}_2)) \equiv_{\mathbf{Grz}} \bigwedge h_i(\mathbf{S}_2 - \mathbf{S}_1)$,

(4) *if* $\mathbf{S}_1 \neq \emptyset$, *then* $\Box(\bigwedge h_i(\mathbf{S}_1)) \equiv_{\mathbf{Grz}} \Box F_k$, *where* $k = \min(\{n+i \mid \Box^n\bot \in \mathbf{S}_1\} \cup \{j+i \mid \Box^{j+1}\bot \supset \Box^j\bot \in \mathbf{S}_1\})$.

Proof. We note that $h_i(\mathbf{S}_1) \cup h_i(\mathbf{S}_2) = h_i(\mathbf{S}_1 \cup \mathbf{S}_2)$, $h_i(\mathbf{S}_1) \cap h_i(\mathbf{S}_2) = h_i(\mathbf{S}_1 \cap \mathbf{S}_2)$ and $h_i(\mathbf{S}_2) - h_i(\mathbf{S}_1) = h_i(\mathbf{S}_2 - \mathbf{S}_1)$. So, using Lemma 2.7 and Lemma 4.5(1) we obtain (1), (2) and (3).

We show (4). By Lemma 2.7(4) and Lemma 4.5(1), $\Box(\bigwedge h_i(\mathbf{S}_1)) \equiv_{\mathbf{Grz}} \Box F_k$, where $k = \min(\{j \mid F_j \in h_i(\mathbf{S}_1)\} \cup \{j \mid \Box F_{j+1} \supset \Box F_j \in h_i(\mathbf{S}_1)\})$. So, $k = \min(\{n+i \mid \Box^n\bot \in \mathbf{S}_1\} \cup \{j+i \mid \Box^{j+1}\bot \supset \Box^j\bot \in \mathbf{S}_1\})$. ⊣

Lemma 4.7. *Let* A *be a formula in* $\mathbf{S}^n(\bot)$ *and let* $\mathbf{S}$ *be a subset of* $\mathbf{G}_n^*$. *Then for any* i,

$$A \equiv_{\mathbf{GL}} \bigwedge \mathbf{S} \text{ iff } g_i(A) \equiv_{\mathbf{Grz}} \bigwedge g_i(\mathbf{S}).$$

Proof. We use an induction on A.

Basis($A = \bot$): By Lemma 3.6(2), we have $A = \bot \equiv_{\mathbf{GL}} \bigwedge \mathbf{G}_n^*$. By Lemma 4.5(2), Lemma 4.6(4), we have

$$\bigwedge g_i(\mathbf{G}_n^*) \equiv_{\mathbf{Grz}} \bigwedge h_i(\mathbf{G}_n^*) \equiv_{\mathbf{Grz}} \Box F_i = g_i(\bot) = g_i(A).$$

So, if $\mathbf{S} = \mathbf{G}_n^*$, then we have both of $A \equiv_{\mathbf{GL}} \bigwedge \mathbf{S}$ and $g_i(A) \equiv_{\mathbf{Grz}} \bigwedge g_i(\mathbf{S})$. If not, then by Theorem 3.3(2), $A \not\equiv_{\mathbf{GL}} \bigwedge \mathbf{S}$, and by Lemma 4.5(2) and Lemma 4.5(3),

$$\bigwedge g_i(\mathbf{S}) \equiv_{\mathbf{Grz}} \bigwedge h_i(\mathbf{S}) \not\equiv_{\mathbf{Grz}} \bigwedge h_i(\mathbf{G}_n^*) \equiv_{\mathbf{Grz}} \bigwedge g_i(\mathbf{G}_n^*) \equiv_{\mathbf{Grz}} g_i(A),$$

and so, $g_i(A) \not\equiv_{\mathbf{Grz}} \bigwedge g_i(\mathbf{S})$.

Induction step($A \neq \bot$): We divide the cases.

The case that $A = A_1 \wedge A_2$: We note $A_1, A_2 \in \mathbf{S}^n(\bot)$. So, by Lemma 3.8, there exist subsets $\mathbf{S}_1, \mathbf{S}_2$ of $\mathbf{G}_n^*$ such that $A_1 \equiv_{\mathbf{GL}} \bigwedge \mathbf{S}_1$ and $A_2 \equiv_{\mathbf{GL}} \bigwedge \mathbf{S}_2$. Using the induction hypothesis, $g_i(A_1) \equiv_{\mathbf{Grz}} \bigwedge g_i(\mathbf{S}_1)$ and $g_i(A_2) \equiv_{\mathbf{Grz}} \bigwedge g_i(\mathbf{S}_2)$. By Lemma 3.7(1),

$$A = A_1 \wedge A_2 \equiv_{\mathbf{GL}} (\bigwedge \mathbf{S}_1) \wedge (\bigwedge \mathbf{S}_2) \equiv_{\mathbf{GL}} \bigwedge(\mathbf{S}_1 \cup \mathbf{S}_2).$$

Also by Lemma 4.5(2) and Lemma 4.6(1),

$$g_i(A) = g_i(A_1) \wedge g_i(A_2) \equiv_{\mathbf{Grz}} (\bigwedge g_i(\mathbf{S}_1)) \wedge (\bigwedge g_i(\mathbf{S}_2)) \equiv_{\mathbf{Grz}} (\bigwedge h_i(\mathbf{S}_1)) \wedge (\bigwedge h_i(\mathbf{S}_2))$$

$$\equiv_{\mathbf{Grz}} \bigwedge h_i(\mathbf{S}_1 \cup \mathbf{S}_2) \equiv_{\mathbf{Grz}} \bigwedge g_i(\mathbf{S}_1 \cup \mathbf{S}_2).$$

So, considering the cases that $\mathbf{S} = \mathbf{S}_1 \cup \mathbf{S}_2$ and that $\mathbf{S} \neq \mathbf{S}_1 \cup \mathbf{S}_2$, we obtain the lemma similarly to Basis.

The case that $A = A_1 \vee A_2$ can be shown similarly, but we use Lemma 3.7(2) and Lemma 4.6(2) instead of Lemma 3.7(1) and Lemma 4.6(1). Also the case that $A = A_1 \supset A_2$ we use Lemma 3.7(3) and Lemma 4.6(3) instead of them.

The case that $A = \Box A_1$: Similarly to the above cases, there exists a subset $\mathbf{S}_1$ of $\mathbf{G}^*_{n-1}$ such that for any k, $A_1 \equiv_{\mathbf{GL}} \bigwedge \mathbf{S}_1$ and $g_k(A_1) \equiv_{\mathbf{Grz}} \bigwedge g_k(\mathbf{S}_1)$.

If $\mathbf{S}_1 = \emptyset$, then we have

$$A = \Box A_1 \equiv_{\mathbf{GL}} \Box \bigwedge \emptyset \equiv_{\mathbf{GL}} \Box(\bot \supset \bot) \equiv_{\mathbf{GL}} \bot \supset \bot \equiv_{\mathbf{GL}} \bigwedge \emptyset$$

and

$$g_i(A) = \Box g_{i+1}(A_1) \equiv_{\mathbf{Grz}} \Box \bigwedge g_{i+1}(\emptyset) = \Box \bigwedge \emptyset \equiv_{\mathbf{Grz}} \bigwedge h_i(\emptyset) \equiv_{\mathbf{Grz}} \bigwedge g_i(\emptyset).$$

So, considering the cases that $\mathbf{S} = \emptyset$ and that $\mathbf{S} \neq \emptyset$, we obtain the lemma similarly to Basis.

If $\mathbf{S}_1 = \{\Box^{n-1}\bot\}$, then we have

$$A = \Box A_1 \equiv_{\mathbf{GL}} \Box \bigwedge \{\Box^{n-1}\bot\} \equiv_{\mathbf{GL}} \bigwedge \{\Box^n \bot\}$$

and

$$g_i(A) = g_i(\Box A_1) = \Box g_{i+1}(A_1) \equiv_{\mathbf{Grz}} \Box \bigwedge g_{i+1}(\{\Box^{n-1}\bot\})$$

$$\equiv_{\mathbf{Grz}} \Box g_{i+1}(\Box^{n-1}\bot) \equiv_{\mathbf{Grz}} g_i(\Box^n\bot) \equiv_{\mathbf{Grz}} \bigwedge g_i(\{\Box^n\bot\}) \equiv_{\mathbf{Grz}} \bigwedge h_i(\{\Box^n\bot\}).$$

So, considering the cases that $\mathbf{S} = \{\Box^n\bot\}$ and that $\mathbf{S} \neq \{\Box^n\bot\}$, we obtain the lemma similarly to Basis.

Suppose that $\mathbf{S}_1 \notin \{\emptyset, \{\Box^{n-1}\bot\}\} = \mathcal{P}(\{\Box^{n-1}\bot\})$. Then we have $\mathbf{S}_1 \not\subseteq \{\Box^{n-1}\bot\}$. Since $\mathbf{S}_1 \subseteq \mathbf{G}^*_{n-1}$, we have $\emptyset \neq \mathbf{S}_1 - \{\Box^{n-1}\bot\} \subseteq \{\Box^{n-1}\bot \supset \Box^{n-1}\bot, \cdots, \Box\bot \supset \bot\}$. So, there exists the minimum k of $\{\ell \mid \Box^\ell\bot \supset \Box^{\ell-1} \in \mathbf{S}_1\}$. By Lemma 3.7(4) and Lemma 3.6(2),

$$A = \Box A_1 \equiv_{\mathbf{GL}} \Box \bigwedge \mathbf{S}_1 \equiv_{\mathbf{GL}} \Box^k \bot$$

$$\equiv_{\mathbf{GL}} \bigwedge \{\Box^n\bot, \Box^n\bot \supset \Box^{n-1}\bot, \cdots, \Box^{k+1}\bot \supset \Box^k\bot\}.$$

Also by Lemma 4.5(2) and Lemma 4.6(4),

$$g_i(A) = g_i(\Box A_1) = \Box g_{i+1}(A_1) \equiv_{\mathbf{Grz}} \Box \bigwedge g_{i+1}(\mathbf{S}_1) \equiv_{\mathbf{Grz}} \Box \bigwedge h_{i+1}(\mathbf{S}_1) \equiv_{\mathbf{Grz}} \Box F_{k+i}$$

$$\equiv_{\mathbf{Grz}} \bigwedge \{F_{n+i+1}, F_{n+i}, \Box F_{n+i} \supset \Box F_{n+i-1}, \cdots, \Box F_{k+i+1} \supset \Box F_{k+i}\}$$

$$\equiv_{\mathbf{Grz}} \bigwedge h_i(\{\Box^n\bot, \Box^n\bot \supset \Box^{n-1}\bot, \cdots, \Box^{k+1}\bot \supset \Box^k\bot\})$$

$$\equiv_{\mathbf{Grz}} \bigwedge g_i(\{\Box^n\bot, \Box^n\bot \supset \Box^{n-1}\bot, \cdots, \Box^{k+1}\bot \supset \Box^k\bot\}).$$

So, considering the case that $S = \{\Box^n\bot, \Box^n\bot \supset \Box^{n-1}\bot, \cdots, \Box^{k+1}\bot \supset \Box^k\bot\}$ and the other case, we obtain the lemma similarly to Basis. $\dashv$

Considering the case that $S = \emptyset$ in Lemma 4.7, we obtain Theorem 4.3.

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