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# Noncommutative hull-kernels for topological dynamical systems and their applications

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## 1 Introduction

This is an exposittory note about the titled subject. As is well known, for a topological space $X$ (not necessarily compact) and an appropriate algebras of continuous functions such as $C(X), C_0(X)$ or $C_b(X)$ notions of hulls and kernels play an important role in functional analysis. Having this situation in mind weregard a topological dynamical system $\Sigma = (X, \sigma)$ where $X$ is an arbitrary compact space with a homeomorphism $\sigma$ as the subject $X$ with an action of the integer group $Z$ on $X$ by $\sigma$. We then consider, as a noncommutative counterpart of usual hull-kernels, the pair $\{\Sigma, A(\Sigma)\}$ where $A(\Sigma)$ is a homeomorpism C*-algebra, namely the C*-crossed product of $C(X)$ by the automorphism $\alpha$ on it induced by $\sigma$. We write this as $A(\Sigma) = C^*(C(X), \delta)$ where $\delta$ is a generating unitary such that $\delta f \delta^* = \alpha(f)$ for all $f \in C(X)$.

Thus, by using generalized Fourier coefficients $\{a(n)\}$ of an element $a$ of $A(\Sigma)$ we define Hulls and Kernels (making difference from usual hull-kernel) in the following way. Let $S$ be a subset of $X$ and $I$ a closed ideal of of $A(\Sigma)$. Then

$$Ker(S) = \{a \in A(\Sigma) \quad | \quad a(n)(x) = 0 \quad \forall x \in S, n \in Z\}$$

and

$$\begin{aligned} Hull(I) &= \{x \in X \quad | \quad a(n)(x) = 0 \quad a \in I, n \in Z\} \\ &= \{x \in X \quad | \quad E(a)(x) = 0 \quad a \in I\} \end{aligned}$$

, where $E$ is the canonical expectation from $A(\Sigma)$ to $C(X)$. In this article we shall mainly discuss the following problems.

Problem A. What are the C*-algebraic meanings of the Kernels of those elementary sets as well as their gaps for the dynamical system $\Sigma$?

Problem B.What are the dynamical meanings of the Hulls of those structural ideals of the C*-algebra $A(\Sigma)$?

# 2 Notations and preliminary results

We write the elementary sets of $\Sigma$ as follows;

$Per(\sigma)$: the set of periodic points.

$Aper(\sigma)$: the set of aperiodic points.

$c(\sigma)$: the set of recurrent points.

A point $x$ is called a recurrent point if there exists a subnet $\{\sigma^{n_\alpha}(x)\}$ converging to $x$.

$\Omega(\sigma)$: the set of nonwandering points.

A point $x$ is said to be nonwandering if for any neighbprhood $U$ of $x$ there exists an $n$ such that $\sigma^n(U) \cap U \neq \phi$.

$R(\sigma)$: the set of chain recurrent points.

A point $x$ is here said to be chain recurrent if for any positive $\varepsilon$ there exists a cyclic $\varepsilon$-shadowing orbit for $x$.

We say that $\Sigma$ is topologically free if the set $Aper(\sigma)$ is dense in $X$. This terminology is not found in usual literature of topological dynamics, perhaps because most topological dynamical systems in manifolds become topologically free (as the set $Per(\sigma)$ is usually at most countable). We emphasize however this wide class is well suited to C*-theory as we see notably from the result [5, theorem 5.4]. Now recall that $\Sigma$ is topologically transitive if for any pair of open sets $\{U, V\}$ there exists an $n$ such that$\sigma^n(U) \cap V \neq \phi$. When $X$ is metrizable the property is known to be equivalent to have a point $x$ with dense orbit. This equivalency is however not valid when $X$ is not metrizable. In fact, every topological dynamical system arised from an ergodic transformation gives a counter example for this fact. We note that a topologically transitive dynamical system for an infinite set is known to become necessarily topologically free.

We denote as usual a representation of $A(\Sigma)$, $\tilde{\pi}$ on a Hilbert space $H$ as $\tilde{\pi} = \pi \times u$, where $\pi$ is the restriction of $\tilde{\pi}$ to $C(X)$ and $u$ is the unitary on $H$ as the image of $\tilde{\pi}(\delta)$. We can then define the dynamical system $\Sigma_\pi = (X_\pi, \sigma_\pi)$ induced by $\tilde{\pi}$ as follows. Put $X_\pi = k(\pi^{-1}(0))$, which turns out to be an invariant closed subset of $X$ and write $\sigma_\pi = \sigma | X_\pi$. Note that the quotient algebra of $C(X)$ by the kernel of $\pi$ is identified as $C(X_\pi)$. On the other hand, consider the compact space $X'_\pi$ defined by the Gelfand representation of $\pi(C(X))$ with the homeomorphism $\sigma'_\pi$ of $X'_\pi$ induced by the automorphism $Adu$ on $C(X'_\pi)$. It follows that the latter dynamical system is topologically conjugate to the system $\Sigma_\pi = \{X_\pi, \sigma_\pi\}$. Hence we naturally identify these two dynamical systems as the induced dynamical systrem by the representation $\tilde{\pi}$. A notable fact about this dynamical system is the following

**Proposition 2.1** *If $\tilde{\pi}$ is a factor repesentation (in particular, irreducible representation), the system $\Sigma_\pi$ becomes topologically transitive. Hence if $\tilde{\pi}$ is infinite dimensional the system becomes topologically free.*

Therefore by Theorem 5.1 of [5] the image $\tilde{\pi}(A(\Sigma))$ has the crossed product structure as $A(\Sigma_\pi)$. Another preparation we need here is the irreducible representation of $A(\Sigma)$ induced by a point $x$ of $X$. Namely the point evaluation $\mu_x$ at $x$ gives a pure state on $C(X)$, and it extends to a pure state $\varphi$ on $A(\Sigma)$. Here when $x$ is aperiodic the extension is unique and the unitary equivalence of its GNS-representation is determined by the orbit $O(x)$. On the other hand if $x$ is periodic the family of pure state extensions is parametrized by the torus $T$.Moreover their unitary equivalences are determined by the orbit and those parameters. Thus we denote their kernels by $P(\bar{x})$ if $x$ is aperiodic and by $P(\bar{x}, \lambda)$ if $x$ is periodic. Put the intersection of $P(\bar{x}, \lambda)$ through all parameters by $Q(\bar{x})$ [7]. We have then

**Proposition 2.2** *(cf.Proposition 2 in [7]) Every closed ideal of $A(\Sigma)$ is expressed as the intersection of those families $\{P(\bar{x}_\alpha)\}$ for aperiodic points and $\{P(\bar{y}_\beta, \lambda_\gamma)\}$ for periodic points.*

Notice that we impose no countability condition for $X$.

# 3 Results and discussions

Henceforth we mean an ideal a closed ideal of $A(\Sigma)$. We must mention first basic differences between our noncommutative Hull-Kernels from usual hull-kernels. In the present situation, for a given subset $S$ of $X$, we only see at first that $Ker(S)$ is just a closed linear subspace of $A(\Sigma)$. By using Cèsaro general polynomials $\sigma_n(a)$ with respect to an element $a$ of $A(\Sigma)$ which converges to $a$ in norm we however obtain

**Proposition 3.1** *(1) If $S$ is invariant, $Ker(S)$ becomes an ideal of $A(\Sigma)$,*
*(2) For an ideal $I$ of $A(\Sigma)$, $Hull(I)$ is always an invariant closed subset of $X$.*

Thus we first meet the usual situation;

$$Hull(Ker(S)) = S \quad \text{for a closed invariant set } S.$$

The other relation is however not valid, namely starting from an ideal $I$ we have in general that $Ker(Hull(I))$ contains $I$ strictly. Moreover, sometimes we meet the worst fact such as

$$Ker(Hull(I)) = A(\Sigma).$$

In fact, when $I = P(\bar{y}, \lambda)$ for a periodic point $y$ we see that

$$E(P(\bar{y},\lambda)) = C(X) \quad \text{and} \quad Ker(Hull(P(\bar{y},\lambda))) = A(\Sigma).$$

Therefore, the first important thing is to characterize an ideal $I$ such that $Ker(Hull(I)) = I$, for which we get the following result.

**Theorem 3.2** *The following assertions are equivalent;*
*(1)* $I = Ker(S)$ *for an invariant subset* $S$,
*(2)* $I = Ker(Hull(I))$, *(3)* $E(I) \subset I$,
*(4)* $I$ *is invariant by the dual action* $\{\hat{\alpha}_t| \quad t \in T\}$,
*(5) There exist families* $\{x_\alpha\}$ *in* $Aper(\sigma)$ *and* $\{y_\beta\}$ *in* $Per(\sigma)$ *such that* $I = \bigcap_\alpha P(\bar{x}_\alpha) \bigcap_\beta Q(\bar{y}_\beta)$.

An immediate consequence of this theorem is that when $\Sigma$ is free, i.e. no periodic points, every ideal of $A(\Sigma)$ has this good property. This kind of situation would be quite favarite for operator algebraists. We however remind that appearence of periodic points is the most common assumption for those people working on dynamical systems at present.

In the above case, the quotient algebra $A(\Sigma)/I$ is shown to have the crossed product structure and the map $E_I$ defined as $E_I([a]) = [E(a)]$ turns out to be the canonical expectation from $A(\Sigma)/I$ to $C(X)/I$. In fact, putting

$$X_I = h(E(I)) = h(C(X) \cap I) \quad \text{and} \quad \sigma_I = \sigma|X_I,$$

one may realize that the quotient algebra is the homeomorphism C*-algebra with respect to this dynamical system $\sigma_I$. This is nothing but the dynamical system $\Sigma_\pi$ induced by the representation $\tilde{\pi} = \pi \times u$ such that $\tilde{\pi}^{-1}(0) = I$.

In the theorem the assertion (5) clarifies the particularity of this kind of ideal among other ideals of $A(\Sigma)$, and the assertion (4) provides a good criterion to distinguish this ideal from others in C*-theory.

Let $I_F$ be the intersection of all kernels of finite dimensional irreducible representations and let $I_\infty$ be that of all infinite dimensional irreducible representations. We have then the following results as the first step of our discussions about Hull-Kernels for those elementary sets attached to the dynamical system.

**Theorem 3.3** *(1)* $I_F = Ker(Per(\sigma))$,
*(2)* $I_\infty = Ker(Aper(\sigma))$.

The assertion (1) looks rather reasonable because every finite dimensional irreducible representation arises from the one induced by a periodic point y so that the kernel has the form of $P(\bar{y}, \lambda)$. We emphasize however that since we impose no countability condition for the space $X$ the second assertion (that depends on Proposition 2.2) is not so trivial.

An advantage of this type of formulation together with forthcoming results of Hull-Kernels is at the point that we can see by this theorem the (approximate) sizes of $Per(\sigma)$ and $Aper(\sigma)$ as algebraic invariants (for instance density of them). As of now, we are far from final conclusion of general isomorphism theorem. In fact, except for the torus $T$ , we know only a little thing about the relation of two homeomorphisms $\sigma$ and $\tau$ even in the case $T^2$ when their corresponding homeomorphism C*-algebras are isomorphic each other. Thus it is important to know what items of dynamical systems are algebraic invariants.

Now recall here that the case $I_F = 0$ is known to be as $A(\Sigma)$ being a residually finite dimensional C*-algebra ( a nice class within quasidiagonal C*-algebras), whereas topological freeness of the system $\Sigma$ reflects in an algebraic way as the case $I_\infty = 0$. Hence we see that topological freeness is an algebraic invariance. Note that the Bernoulli shifts provide both examples that $I_F = 0$ and $I_\infty = 0$ as topologically transitive dynamical systems.

A C*-algebra A is called a CCR or liminal algebra if the image of every irreducible representation consists of compact operators. A piling C*-algebra A of CCR algebras is called a GCR or postliminal algebra, which is characterized as a C*-algebra of type 1. Equivalently, $A$ is of type 1 if the image of every irreducible representation contains the algebra of compact operators. We recall here that any C*-algebra $A$ contains the largest ideal $K$ of type 1 such that the quotient algebra by $K$ does not have no nonzero ideal of type 1. It is also known that $A$ contains the largest CCR ideal L. Denote the largest ideal of type 1 and the largest CCR ideal of $A(\Sigma)$ by $K(\sigma)$ and $L(\sigma)$ respectively. In the following we shall determine the structure of these ideals in terms of dynamical systems. We emphasize here that since the work by Effros and Hahn there are many liteletures to discuss when those transformation group C*-algebras become GCR or CCR algebras in the broad contexts, and most people have assumed now tthat this is an already solved old problem, but there are no work except for the author's joint work [3] and the result here for the estimation of the size of $K(\sigma)$ and $L(\sigma)$ to describe the topological backgrounds.

The following is a refined version of the result in [3]. We regret to have to state the result with the separability assumption as in the comming characterization of the ideal $L(\sigma)$. Note first both ideals $K(\sigma)$ and $L(\sigma)$ satisfy the condition (4) of Theorem 3.2, so that Kernels of their Hulls come back to the original ideals.

**Theorem 3.4** *$Hull(K(\sigma))$ contains the difference $c(\sigma) \setminus Per(\sigma)$.*
*When $X$ is metrizable we have*

$$Hull(K(\sigma)) = \overline{c(\sigma) \setminus Per(\sigma)}.$$

*Hence,*

$$K(\sigma) = Ker(\overline{c(\sigma) \setminus Per(\sigma)})$$

Thus, when $X$ is metrizable we can immediately tell how is the structure of $A(\Sigma)$, that is, how it is near to the algebra of type 1 or with no type 1 portion (antiliminal) once a dynamical system is given. In fact ,it is of type 1 if and only if there are no proper recurrent points and so on.

A key fact for this result is the following

**Proposition 3.5** *Let $\tilde{\pi} = \pi \times u$ be an irreducible representation of $A(\Sigma)$ on $H$, then $\tilde{\pi}(A(\Sigma))$ contains the algebra of compact operators if and only if $X_\pi = \overline{O(x_0)}$ for an isolated point $x_0$ not belonging to the set $c(\sigma) \setminus Per(\sigma)$.*

Here when $\tilde{\pi}$ is irreducible the induced dynamical system $\Sigma_\pi$ becomes topologically transitive, hence if the space is metrizable there exists a point in $X_\pi$ with dense orbit. Thus we can find a candidate point in the above proposition, but as we have mentioned before we can not make use of this advantage for nonseparable topologically transitive dynamical systems In contrast with this situation, we notice that in the above proposition the representing space $H$ becomes always separable.

Next, we consider the following property (*) of orbits with respect to a closed invariant set $S$ in $X$.

$(*)$ For every point $x$ in $X \setminus S$ the boundary set $\partial O(x) = \overline{O(x)} \setminus O(x)$ is contained in $S$.

Note that the above definition allows some periodic points outside of $S$ with the property (*). We can characterize this kind of a subset $S$. Namely,

**Proposition 3.6** *If $Ker(S)$ is a CCR ideal, then $S$ satisfies (*). The converse holds if $X$ is metrizable.*

A typical closed invariant set with this property is the nonwandering set $\Omega(\sigma)$. Hence $Ker(\Omega(\sigma))$ is a CCR ideal if $X$ is metrizable. This fact as well as the fact for $K(\sigma)$ shows that in spite of the present stage of the theory of C*-algebras centering around (purely) infinite C*-algebras so far the interplay between topological dynamics and C*-theory is concerned Kernel ideals corresponding to important elementary sets of dynamical systems belong necessarily to those old classes of type 1 and CCR algebras.Moreover, we further recall the remark after Theorem 3.2 that besides the above situation we often meet the cases where their quotient algebras have again the structure of homeomorphism algebras.

Now with these things in mind we shall determine the structure of the ideal $L(\sigma)$. Put

$$S_0 = \overline{\bigcup_{x \notin c(\sigma)} \partial O(x)} \cup \overline{c(\sigma) \setminus Per(\sigma)}.$$

We have then

**Theorem 3.7** *$Hull(L(\sigma))$ contains the set $S_0$.*

*When $X$ is metrizable, the equality holds, that is,*

$$Hull(L(\sigma)) = \overline{\bigcup_{x \notin c(\sigma)} \partial O(x)} \cup \overline{c(\sigma) \setminus Per(\sigma)}.$$

*Hence,*

$$L(\sigma) = \overline{\bigcup_{x \notin c(\sigma)} \partial O(x)} \cap K(\sigma).$$

As in the case of the ideal $K(\sigma)$, we meet here the same difficulty of countability assumption, which is concerned with the equivalency between topological transitivity and the dense orbit property in metrizable case.

As we have noticed above, $Hull(L(\sigma))$ may contain some periodic points besides the set of proper recurrent points. On the other hand.the difference between $\Omega(\sigma)$ and $Hull(L(\sigma))$ becomes more clear if we consider the extreme case where $X$ only consists of periodic points (such as the case of rational rotations). In fact, in this case $\Omega(\sigma) = X$ whereas $Hull(L(\sigma))$ becomes empty. It should be further noticed here that in spite of the countability restriction in the above theorem the topological condition when $A(\Sigma)$ becomes CCR algebra holds without such restriction.

**Theorem 3.8** *The algebra $A(\Sigma)$ becomes CCR if and only if $X$ consists of only periodic points.*

Now we come again to the nonwandering set $\Omega(\sigma)$. We note first that contrary to other elementary sets if we consider the nonwandering set for the retricted dynamical system to $\Omega(\sigma)$ it usually shrinks. Moreover, this steps will continue and when $X$ is metrizable it is known that these shrinking steps end at the Birkhoff center $\overline{c(\sigma)}$. Right now we do not know whether this is also true in general.

Thus, through the following discussions we assume on that $X$ is metrizable. To be precise then, write $\Omega_0 = X$ and $\Omega_1 = \Omega(\sigma)$. In this way we obtain a decreasing series of closed invariant sets $\{\Omega_\alpha\}$ indexed by ordinal numbers $\alpha (0 \leq \alpha \leq \gamma)$ for a countable ordinal number $\gamma$ having the properties that

$$\Omega_{\alpha+1} = \Omega(\sigma|\Omega_\alpha)$$

and if $\alpha$ is a limit ordinal number

$$\Omega_\alpha = \bigcap_{\lambda<\alpha} \Omega_\lambda.$$

The steps end at $\gamma$ as $\Omega_{\gamma+1} = \Omega_\gamma = \overline{c(\sigma)}$, and such $\gamma$ is called the depth of the center written as $d(\sigma) = \gamma$.

Now consider the ideal

$$J(\sigma) = Ker(\overline{c(\sigma)}) = K(\sigma) \cap I_F.$$

It is the largest ideal of type 1 with no finite dimensional irreducible representations. Write $Ker(\Omega_\alpha)$ as $Ker_\alpha(\sigma)$. We see then the net $\{Ker_\alpha(\sigma)| \quad 0 \leq \alpha \leq \gamma\}$ is just a composition series of the type 1 ideal $J(\sigma)$. Namely, they are increasing net of the ideals of $J(\sigma)$ such that

$$Ker_\alpha(\sigma) = \overline{\bigcup_{\lambda<\alpha} Ker_\lambda(\sigma)}$$

if $\alpha$ is a limit ordinal. These are in fact refined versions of the author's previous results in [8], and we have a characterization of this composition series (cf. [8, Theorem 1]). A standard composition series $\{I_\alpha\}$ for a C*-algebra $A$ of type 1 is that the quotient algebra $I_{\alpha+1}/I_\alpha$ is the largest CCR ideal of $A/I_\alpha$. Therefore, in this sense it is interesting to know whether $Ker(\Omega(\sigma))$ coincides with the ideal $L(\sigma)$ (in general $S_0 \subset \Omega(\sigma)$ and $Ker(\Omega(\sigma)) \subset L(\sigma)$ as a CCR ideal). We can see the case that $\Omega(\sigma)$ coincides with $S_0$ for the so-called horse-shoe diffeomorphisms on $S^2$. However, if we consider their perturbations we meet also the case where $\Omega(\sigma)$ exactly contains $S_0$, so that the shrinking steps do not generally fit to the standard composition series for $J(\sigma)$ (cf. Chap.6 of [1], particularly section 6 ibid). The author owes for these observations to Dr.N.Sumi.

A composition series $\{I_\alpha\}$ may be sharpened in general further that $I_{\alpha+1}/I_\alpha$ becomes a C*-algebra with continuous trace, and in our case we can also give a characterization of such a composition series $\{Ker_\alpha(\sigma)\}$ of $J(\sigma)$ in [8].

As of now we do not know the C*-algebraic meaning of the gap from $\Omega(\sigma)$ up to $R(\sigma)$. For the chain recurrent set $R(\sigma)$ and its gap from the space $X$ we recall first Pimsner's result [4]. We should notice here the highly nontrivial fact that the chain recurrent set $R(\sigma|R(\sigma))$ with respect to the restricted dynamical system coincides with $R(\sigma)$.Namely $R(\sigma)$ does not shrink as in the case of $\Omega(\sigma)$.

Theorem (Pimsner) The following assertions are equivalent.
(a) $A(\Sigma)$ can be imbedded into an AF-algebra,
(b) $A(\Sigma)$ is quasidiagonal,
(c) $R(\sigma) = X$.

Sharpenning this result as well as considering the gap from $R(\sigma)$ to $X$ we finally obtain the following result.

**Theorem 3.9** *The ideal $Ker(R(\sigma))$ is the smallest ideal among those ideals for which their quotient algebras become quasidiagonal algebras.*

In general for a C*-algebra $A$ and its ideal $I$ the obstruction when the quotient algebra $A/I$ becomes quasidiagonal has been remaining mysterious. In [10] we have clarified, to some extent, this situation at least for the homeomorphism C*-algebra $A(\Sigma)$.

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