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Some properties of certain analytic functions

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Abstract

Defining the subclasses $\mathcal{MD}(\alpha,\beta)$ and $\mathcal{ND}(\alpha,\beta)$ of certain analytic functions $f(z)$ in the open unit disk $\mathbb{U}$, some properties for $f(z)$ belonging to the classes $\mathcal{MD}(\alpha,\beta)$ and $\mathcal{ND}(\alpha,\beta)$ are discussed. In this present paper, some coefficient estimates and some interesting applications of Jack's lemma for functions $f(z)$ in the classes $\mathcal{MD}(\alpha,\beta)$ and $\mathcal{ND}(\alpha,\beta)$ are given.

1 Introduction

Let $\mathcal{A}$ be the class of functions $f(z)$ of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n$$

which are analytic in the open unit disk $\mathbb{U}=\{z\in\mathbb{C}\,|\,|z|<1\}$. Shams, Kulkarni and Jahangiri [3] have considered the subclass $\mathcal{SD}(\alpha,\beta)$ of $\mathcal{A}$ consisting of $f(z)$ which satisfy

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right)>\alpha\left|\frac{zf'(z)}{f(z)}-1\right|+\beta \qquad (z\in\mathbb{U})$$

for some $\alpha(\alpha\geqq 0)$ and $\beta(0\leqq\beta<1)$. The class $\mathcal{KD}(\alpha,\beta)$ is defined by the subclass of $\mathcal{A}$ consisting of $f(z)$ such that $zf'(z)\in\mathcal{SD}(\alpha,\beta)$. In view of the classes $\mathcal{SD}(\alpha,\beta)$ and $\mathcal{KD}(\alpha,\beta)$, we introduce the subclass $\mathcal{MD}(\alpha,\beta)$ of $\mathcal{A}$ consisting of all functions $f(z)$ which satisfy

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right)<\alpha\left|\frac{zf'(z)}{f(z)}-1\right|+\beta \qquad (z\in\mathbb{U})$$

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for some $\alpha(\alpha \leqq 0)$ and $\beta(\beta > 1)$. The class $\mathcal{ND}(\alpha,\beta)$ is also defined by $f(z) \in \mathcal{ND}(\alpha,\beta)$ if and only if $zf'(z) \in \mathcal{MD}(\alpha,\beta)$. The classes $\mathcal{MD}(\alpha,\beta)$ and $\mathcal{ND}(\alpha,\beta)$ were introduced by Nishiwaki and Owa [2]. We discuss some properties of functions $f(z)$ belonging to the classes $\mathcal{MD}(\alpha,\beta)$ and $\mathcal{ND}(\alpha,\beta)$.

We note if $f(z) \in \mathcal{MD}(\alpha,\beta)$, then $\dfrac{zf'(z)}{f(z)} = u + iv$ maps $\mathbb{U}$ onto elliptic domain such that

$$\left(u - \frac{\alpha^2-\beta}{\alpha^2-1}\right)^2 + \frac{\alpha^2}{\alpha^2-1}v^2 < \frac{\alpha^2(\beta-1)^2}{(\alpha^2-1)^2}$$

for $\alpha < -1$, the parabolic domain such that

$$u < -\frac{1}{2(\beta-1)}v^2 + \frac{\beta+1}{2}$$

for $\alpha = -1$, and the hyperbolic domain such that

$$\left(u - \frac{\alpha^2-\beta}{\alpha^2-1}\right)^2 - \frac{\alpha^2}{1-\alpha^2}v^2 > \frac{\alpha^2(\beta-1)^2}{(\alpha^2-1)^2}$$

for $-1 < \alpha < 0$.

2 Coefficient estimates for the classes $\mathcal{MD}(\alpha,\beta)$ and $\mathcal{ND}(\alpha,\beta)$

By definitions of $\mathcal{MD}(\alpha,\beta)$ and $\mathcal{ND}(\alpha,\beta)$, we derive

Theorem 2.1. *If $f(z) \in \mathcal{MD}(\alpha,\beta)$, then*

$$f(z) \in \mathcal{MD}\left(0, \frac{\beta-\alpha}{1-\alpha}\right).$$

Proof. If $f(z) \in \mathcal{MD}(\alpha,\beta)$,

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) < \alpha\left|\frac{zf'(z)}{f(z)} - 1\right| + \beta \leqq \alpha \operatorname{Re}\left(\frac{zf'(z)}{f(z)} - 1\right) + \beta \qquad (z \in \mathbb{U})$$

implies that

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) < \frac{\beta-\alpha}{1-\alpha} \qquad (\alpha \leqq 0, \beta > 1).$$

Since $\dfrac{\beta-\alpha}{1-\alpha} > 1$, we prove the Theorem. □

Corollary 2.1. *If $f(z) \in \mathcal{ND}(\alpha,\beta)$, then*

$$f(z) \in \mathcal{ND}\left(0, \frac{\beta-\alpha}{1-\alpha}\right).$$

Our result for the coefficient estimates of $\mathcal{MD}(\alpha,\beta)$ and $\mathcal{ND}(\alpha,\beta)$ is contained in

Theorem 2.2. *If $f(z) \in \mathcal{MD}(\alpha,\beta)$, then*

$$|a_2| \leqq \frac{2(\beta-1)}{1-\alpha}$$

and

$$|a_n| \leqq \frac{2(\beta-1)}{(n-1)(1-\alpha)} \prod_{j=1}^{n-2}\left(1+\frac{2(\beta-1)}{j(1-\alpha)}\right) \qquad (n \geqq 3).$$

Proof. If $f(z) \in \mathcal{MD}(\alpha,\beta)$, then

$$\beta-\alpha+(\alpha-1)\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0$$

from Theorem2.1. And let us define the function $p(z)$ by

$$p(z) = \frac{\beta-\alpha+(\alpha-1)\dfrac{zf'(z)}{f(z)}}{\beta-1}. \tag{2.1}$$

Then $p(z)$ is analytic in $\mathbb{U}$, $p(0)=1$ and $\mathrm{Re}\,p(z) > 0$ $(z \in \mathbb{U})$. Therefore, if we write

$$p(z) = 1+\sum_{n=1}^{\infty} p_n z^n, \tag{2.2}$$

then $|p_n| \leqq 2$ $(n \geqq 1)$. From (2.1) and (2.2), we obtain that

$$(\alpha-1)\sum_{n=2}^{\infty}(n-1)a_n z^n = (\beta-1)\sum_{n=1}^{\infty}p_n z^n\left(z+\sum_{n=2}^{\infty}a_n z^n\right).$$

Therefore we have

$$a_n = \frac{\beta-1}{(n-1)(\alpha-1)}(p_{n-1}+p_{n-2}a_2+\cdots+p_2 a_{n-2}+p_1 a_{n-1})$$

for all $n \geqq 2$. When $n=2$,

$$|a_2| \leqq \frac{\beta-1}{1-\alpha}|p_1| \leqq \frac{2(\beta-1)}{1-\alpha}.$$

And when $n = 3$,

$$|a_3| \leqq \frac{\beta - 1}{2(1-\alpha)}(|p_2| + |p_1||a_2|)$$
$$\leqq \frac{2(\beta-1)}{2(1-\alpha)}\left(1 + \frac{2(\beta-1)}{1-\alpha}\right).$$

Let us suppose that

$$|a_k| \leqq \frac{2(\beta-1)}{(k-1)(1-\alpha)}(1 + |a_2| + \cdots + |a_{k-2}| + |a_{k-1}|) \tag{2.3}$$
$$\leqq \frac{2(\beta-1)}{(k-1)(1-\alpha)}\prod_{j=1}^{k-2}\left(1 + \frac{2(\beta-1)}{j(1-\alpha)}\right) \qquad (k \geqq 3).$$

Then we see

$$1 + |a_2| + \cdots + |a_{k-2}| + |a_{k-1}| \leqq \prod_{j=1}^{k-2}\left(1 + \frac{2(\beta-1)}{j(1-\alpha)}\right). \tag{2.4}$$

By using (2.3) and (2.4),

$$|a_{k+1}| \leqq \frac{2(\beta-1)}{k(1-\alpha)}(1 + |a_2| + \cdots + |a_{k-2}| + |a_{k-1}| + |a_k|)$$
$$\leqq \left(1 + \frac{2(\beta-1)}{(k-1)(1-\alpha)}\right)\frac{2(\beta-1)}{k(1-\alpha)}\prod_{j=1}^{k-2}\left(1 + \frac{2(\beta-1)}{j(1-\alpha)}\right)$$
$$\leqq \frac{2(\beta-1)}{k(1-\alpha)}\prod_{j=1}^{k-1}\left(1 + \frac{2(\beta-1)}{j(1-\alpha)}\right).$$

This completes the proof of the Theorem. □

Corollary 2.2. *If $f(z) \in \mathcal{ND}(\alpha,\beta)$, then*

$$|a_2| \leqq \frac{2(\beta-1)}{2(1-\alpha)}$$

and

$$|a_n| \leqq \frac{2(\beta-1)}{n(n-1)(1-\alpha)}\prod_{j=1}^{n-2}\left(1 + \frac{2(\beta-1)}{j(1-\alpha)}\right) \qquad (n \geqq 3).$$

Proof. From $f(z) \in \mathcal{ND}(\alpha,\beta)$ if and only if $zf'(z) \in \mathcal{MD}(\alpha,\beta)$, replacing a_n by na_n in Theorem2.2, we have the corollary. □

3 Applications of Jack's lemma for the classes $\mathcal{MD}(\alpha,\beta)$ and $\mathcal{ND}(\alpha,\beta)$

In this section, some applications of Jack's lemma for $f(z)$ belonging to the classes $\mathcal{MD}(\alpha,\beta)$ and $\mathcal{ND}(\alpha,\beta)$ are discussed. Next lemma was given by Jack [1].

Lemma 3.1. *Let the function $w(z)$ be analytic in $\mathbb{U}$ with $w(0)=0$. If*

$$\max_{|z|\leqq|z_0|}|w(z)|=|w(z_0)|,$$

then

$$z_0w'(z_0)=kw(z_0),$$

where k is a real number and $k\geqq 1$.

Theorem 3.1. *If $f(z)\in\mathcal{MD}(\alpha,\beta)$, then*

$$\left|\left(\frac{f(z)}{z}\right)^{\frac{(1+\delta)(1-\alpha)}{(2+\delta)(\beta-1)}}-1\right|<1+\delta \qquad (\delta\geqq 0)$$

for some $\alpha(\alpha\leqq 0)$ and $\beta(\beta>1)$, or

$$\left|\left(\frac{f(z)}{z}\right)^{\frac{(1+\delta)(1+\alpha)}{(2+\delta)(\beta-1)}}-1\right|<1+\delta \qquad (\delta\geqq 0)$$

for some $\alpha(\alpha\leqq -1)$ and $\beta(\beta>1)$.

Proof. Let us define

$$\gamma=\frac{(1+\delta)(1-\alpha)}{(2+\delta)(\beta-1)}>0,$$

for $\alpha\leqq 0$ and $\beta>1$, and

$$\gamma=\frac{(1+\delta)(1+\alpha)}{(2+\delta)(\beta-1)}<0$$

for $\alpha\leqq -1$ and $\beta>1$. Further, let the funcion $w(z)$ be defined by

$$w(z)=\frac{\left(\frac{f(z)}{z}\right)^{\gamma}-1}{1+\delta} \qquad (\delta\geqq 0)$$

which is equivalent to

$$\frac{zf'(z)}{f(z)}-1=\frac{(1+\delta)zw'(z)}{\gamma\{(1+\delta)w(z)+1\}}.$$

Then we see that $w(z)$ is analytic in $\mathbb{U}$, and $w(0) = 0$. On the other hand, if $f(z) \in \mathcal{MD}(\alpha,\beta)$ $(\alpha \leqq 0, \beta > 1)$, then

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) - \alpha\left|\frac{zf'(z)}{f(z)} - 1\right| = 1 + \frac{1}{\gamma}\mathrm{Re}\left(\frac{(1+\delta)zw'(z)}{(1+\delta)w(z)+1}\right) - \frac{\alpha}{|\gamma|}\left|\frac{(1+\delta)zw'(z)}{(1+\delta)w(z)+1}\right| < \beta.$$

Furthermore, if there is a point z_0 $(z_0 \in \mathbb{U})$, which satisfies

$$\max_{|z| \leqq |z_0|} |w(z)| = |w(z_0)| = 1,$$

then Lemma3.1 gives us that

$$\begin{aligned} &1 + \frac{1}{\gamma}\mathrm{Re}\left(\frac{(1+\delta)z_0w'(z_0)}{(1+\delta)w(z_0)+1}\right) - \frac{\alpha}{|\gamma|}\left|\frac{(1+\delta)z_0w'(z_0)}{(1+\delta)w(z_0)+1}\right| \\ &= 1 + \frac{k(1+\delta)}{\gamma}\mathrm{Re}\left(\frac{1}{(1+\delta)+e^{-i\theta}}\right) - \frac{\alpha k(1+\delta)}{|\gamma|}\left|\frac{1}{(1+\delta)+e^{-i\theta}}\right| \\ &= 1 + \frac{k(1+\delta)}{\gamma}\cdot\frac{1+\delta+cos\theta - \alpha\sqrt{(1+\delta)^2+2(1+\delta)cos\theta+1}}{(1+\delta)^2+2(1+\delta)cos\theta+1} = F(\theta). \end{aligned}$$

When $\gamma > 0$,

$$\begin{aligned} F(\theta) &\geqq 1 + \frac{k(1+\delta)(1-\alpha)}{\gamma(2+\delta)} \\ &\geqq 1 + \frac{(1+\delta)(1-\alpha)}{\gamma(2+\delta)} = \beta, \end{aligned}$$

because

$$\gamma = \frac{(1+\delta)(1-\alpha)}{(2+\delta)(\beta-1)}.$$

Further, when $\gamma < 0$,

$$\begin{aligned} F(\theta) &\geqq 1 + \frac{k(1+\delta)(1+\alpha)}{\gamma(2+\delta)} \\ &\geqq 1 + \frac{(1+\delta)(1+\alpha)}{\gamma(2+\delta)} = \beta. \end{aligned}$$

This contradicts our condition of the Theorem. Thus there is no $z_0 \in \mathbb{U}$ such that $|w(z_0)| = 1$. This completes the proof of the Theorem. □

Corollary 3.1. *If $f(z) \in \mathcal{ND}(\alpha,\beta)$, then*

$$\left|(f'(z))^{\frac{(1+\delta)(1-\alpha)}{(2+\delta)(\beta-1)}} - 1\right| < 1+\delta \qquad (\delta \geqq 0)$$

for some $\alpha(\alpha \leqq 0)$ and $\beta(\beta > 1)$, or

$$\left|(f'(z))^{\frac{(1+\delta)(1+\alpha)}{(2+\delta)(\beta-1)}} - 1\right| < 1+\delta \qquad (\delta \geqq 0)$$

for some $\alpha(\alpha \leqq -1)$ and $\beta(\beta > 1)$.

Proof. Replacing $f(z)$ by $zf'(z)$ in Theorem3.1, we have the corollary3.1. □

References

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