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A Note on Certain Analytic Functions

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Abstract

The object of the present paper is to obtain some interesting properties of analytic functions.

1 Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n$$

which are analytic in the unit disc $\mathbb{E}=\{z||z|<1\}$. Sakaguchi [1] proved the following theorem.

Theorem A. *If $f(z)\in\mathcal{A}$ satisfies the condition*

$$\mathrm{Re}\frac{zf'(z)}{f(z)-f(-z)}>0 \text{ in } \mathbb{E}, \tag{1}$$

then $f(z)$ is univalent and starlike with respect to symmetrical points in $\mathbb{E}$.

We call $f(z)$ a Sakaguchi functions which satisfies the condition (1).

In this paper, we need the following lemma.

Lemma 1. *Let $f(z)\in\mathcal{A}$ and*

$$\mathrm{Re}\frac{zf'(z)}{f(z)}>K \text{ in } \mathbb{E}$$

where K is a real and bounded constant, then we have

$$f(z)\neq 0 \text{ in } 0<|z|<1.$$

2 Results

Theorem 1. *For arbitrary positive real number α, $0<\alpha\leqq\pi$, if $f(z)\in\mathcal{A}$ satisfies the following condition*

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$$\mathrm{Re}\frac{z\left(e^{i\alpha}f'(ze^{i\alpha})-f'(z)\right)}{f(ze^{i\alpha})-f(z)}>0 \ in\ \mathbb{E}, \tag{1}$$

then $f(z)$ is univalent in $\mathbb{E}$.

Proof. If there exists a r, $0<r<1$ for which $f(z)$ is univalent in $|z|<r$ but $f(z)$ is not univalent on $|z|=r$, then there exists two points z_1, $z_2=z_1e^{i\alpha}$, $0<\alpha\leqq\pi$,

$$f(z_1)=f(z_2) \tag{2}$$

and $f(z)$ is univalent on the arc $\mathbb{C}$ where

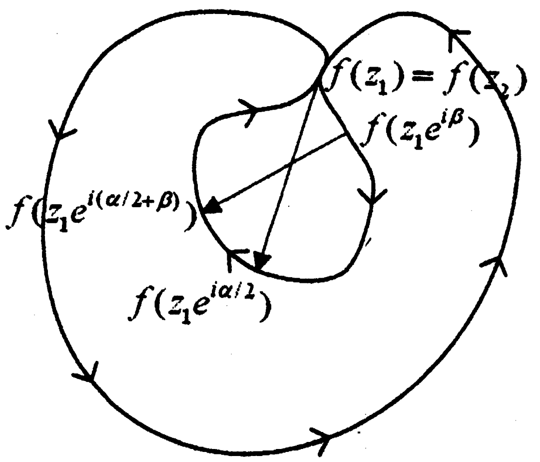
$$\mathbb{C}=\{z|z=z_1e^{i\theta},0\leqq\theta<\alpha\}. \tag{3}$$

From the assumption of Theorem 1, we have

$$\mathrm{Re}\frac{z\left(e^{i\alpha/2}f'(ze^{i\alpha/2})-f'(z)\right)}{f(ze^{i\alpha/2})-f(z)}>0 \ in\ \mathbb{E}. \tag{4}$$

This shows that $(f(ze^{i\alpha/2})-f(z))$ is starlike with respect to the origin.

From (2) and (3), we get the following image of $|z|=r$ under the mapping $w=f(z)$,



where β is sufficiently small positive real number.

Then vectors $(f(z_1e^{i\alpha/2})-f(z_1))$ and $(f(z_1e^{i(\alpha/2+\beta)})-f(z_1e^{i\beta}))$ move on the clockwise direction (the negative direction). This contradicts (4) and it completes the proof. □

Another proof of Theorem 1. If there exists a r, $0<r<1$ for which $f(z)$ is univalent in $|z|<r$ but $f(z)$ is not univalent on $|z|=r$, then there exists at least two points z_1, $z_2=z_1e^{i\alpha}$, $0<\alpha\leqq\pi$ and for which

$$f(z_1)=f(z_2).$$

Applying Lemma 1 and form the hypothesis (1), we have

$$f(ze^{i\alpha})-f(z)\neq 0.$$

This is a contradiction and therefore, it completes the proof.

□

Remark. If $f(z) \in \mathcal{A}$ satisfies the condition (1) only for the case $\alpha = \pi$, then $f(z)$ is a Sakaguchi function.

Theorem 2. *If $f(z) \in \mathcal{A}$ satisfy the following condition for sufficiently small and positive real number δ and arbitrary real number α, $0 < |\alpha| < \delta$ for which*

$$\mathrm{Re}\frac{z\left(e^{i\alpha}f'(ze^{i\alpha}) - f'(z)\right)}{f(ze^{i\alpha}) - f(z)} > 0 \text{ in } \mathbb{E}. \tag{5}$$

Then $f(z)$ is convex in $\mathbb{E}$ or

$$1 + \mathrm{Re}\frac{zf''(z)}{f'(z)} > 0 \text{ in } \mathbb{E}.$$

Proof. From the hypothesis (5), all the tangent vector of $\mathbb{C}$ which is the image of $|z| = r$, $0 < r < 1$ under the mapping $w = f(z)$ move in the counterclockwise direction. Geometrically, this shows that $f(z)$ is convex in $\mathbb{E}$ or

$$1 + \mathrm{Re}\frac{zf''(z)}{f'(z)} > 0 \text{ in } \mathbb{E}.$$

□

References

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