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Geometric Properties of Solutions of Ordinary Differential Equations

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ABSTRACT

The main object of this paper is to investigate several geometric properties, i.e. starlike, strongly starlike and convex, of the solutions of ordinary differential equation. Relevant connections of the results presented in this paper with those given earlier by, for example, Robertson, Miller and Saitoh are also considered.

Keywords—Starlike functions, Strongly starlike functions, Convex functions.

1 Introduction

Let A denote the class of functions f normalized by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$,

which are analytic in the open unit disk

$$U = \{ z : z \in \mathbb{C} \text{ and } |z| < 1 \}.$$

Also let S and S^* denote the subclasses of A consisting of functions which are, respectively, univalent and starlike with respect to the origin in U. Thus, by definition, we have (see, for detail, [1][6]).

(1.2) $$S^* = \left\{ f : f \in A \text{ and } \mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0 \quad (z \in U) \right\}.$$

Furthermore, $SS^*(\alpha)$ denote the subclasses of A consisting of functions which are strongly starlike of order α in U $(0 < \alpha \leq 1)$. By definition, we have

(1.3) $$SS^*(\alpha) := \left\{ f : f \in A \text{ and } \left|\arg\left\{\frac{zf'(z)}{f(z)}\right\}\right| \leq \frac{\pi}{2}\alpha \quad (z \in U;\ 0 < \alpha \leq 1) \right\}.$$

We begin by recalling the following result of Miller[3].

<u>Theorem A</u> (Miller[3]) Let $zp(z)$ be analytic in U with $|zp(z)| < 1$. Let $v(z)$, $z \in U$, be the unique solution of

(1.4) $$v''(z) + p(z)v(z) = 0$$

with $v(0) = 0$ and $v'(0) = 1$. Then

(1.5) $$\left| \frac{zv'(z)}{v(z)} - 1 \right| < 1,$$

and $v(z)$ is starlike conformal map of the unit disk.

2 A class of bounded functions and earlier results

Let B_J denote the class of bounded functions

(2.1) $$w(z) = \sum_{n=1}^{\infty} c_n z^n ,$$

analytic in U, for which

(2.2) $$|w(z)| < J \quad (z \in U;\ J > 0).$$

If $g(z) \in B_J$, then we can show that the function $w(z)$ defined by

(2.3) $$w(z) := z^{-\frac{1}{2}} \int_0^z g(t) t^{-\frac{1}{2}} dt$$

is also in the class B_J. Thus, in term of derivative, we have

(2.4) $$\left| \frac{1}{2} w(z) + z w'(z) \right| < J \quad (z \in U) \Rightarrow |w(z)| < J \ (z \in U).$$

Furthermore, by setting

(2.5) $$h(u, v) := \frac{1}{2} u + v ,$$

we can rewrite (2.4) in the form :

(2.6) $$| h(w(z), z w'(z)) | < J \quad (z \in U) \Rightarrow |w(z)| < J \ (z \in U).$$

In this section, we show that implication (2.6) holds true for functions $h(u,v)$ in the class H_J given by Definition 1 below (see also [3]).

<u>Definition 1.</u> Let H_J be the class of complex functions $h(u,v)$ satisfying each of the following conditions:

(i) $h(u,v)$ is continuous in a domain $\mathbb{D} \subset \mathbb{C} \times \mathbb{C}$;

(ii) $(0,0) \in \mathbb{D}$ and $|h(0,0)| < J$ $(J>0)$;

(iii) $|h(Je^{i\theta}, Ke^{i\theta})| \leq J$ whenever $(Je^{i\theta}, Ke^{i\theta}) \in \mathbb{D}$ $(\theta \in \mathbb{R};\ K \geq J > 0)$.

<u>Example 1.</u> It is easily seen that the function

$$h(u,v) = \gamma u + v \quad (\mathrm{Re}(\gamma) > 0;\ \mathbb{D} = \mathbb{C} \times \mathbb{C}) \tag{2.7}$$

is in the class H_J.

<u>Definition 2.</u> Let $h \in H_J$ with corresponding domain $\mathbb{D}$. We denote by $B_J(h)$ the class of functions $w(z)$ given by (2.1), which are analytic in U and satisfy each of the following conditions:

(i) $(w(z), zw'(z)) \in \mathbb{D}$;

(ii) $|h(w(z), zw'(z))| < J$ $(z \in U;\ J > 0)$.

Theorem B (Saitoh [7]) For any $h \in H_J$,

$$B_J(h) \subset B_J \quad (J>0).$$

Theorem C (Saitoh [7]) Let $h \in H_J$ and let the function $b(z)$ be analytic in U with

$$|b(z)| < J \quad (z \in U;\ J>0).$$

If the following initial-value problem :

$$(2.8) \quad h(w(z), zw'(z)) = b(z) \quad (w(0)=0)$$

has a solution $w(z)$ analytic in U, then

$$(2.9) \quad |w(z)| < J \quad (z \in U;\ J>0).$$

Recently, using Theorem C, we prove the following Theorem D.

Theorem D (Owa et. al [5]) Let $a(z)$ and $b(z)$ be analytic in U with

$$(2.10) \quad \left| b(z) - \frac{1}{2}a'(z) - \frac{1}{4}[a(z)]^2 \right| < J \quad (z \in U;\ J>0)$$

and

$$(2.11) \quad \mathrm{Re}\{z a(z)\} > -2J \quad (z \in U;\ J>0).$$

Also, let $w(z)$ denote the solution of initial-value problem :

$$(2.12) \quad w''(z) + a(z)w'(z) + b(z)w(z) = 0 ,$$

$w(0)=0$ and $w'(0)=1$ in U. Then we have

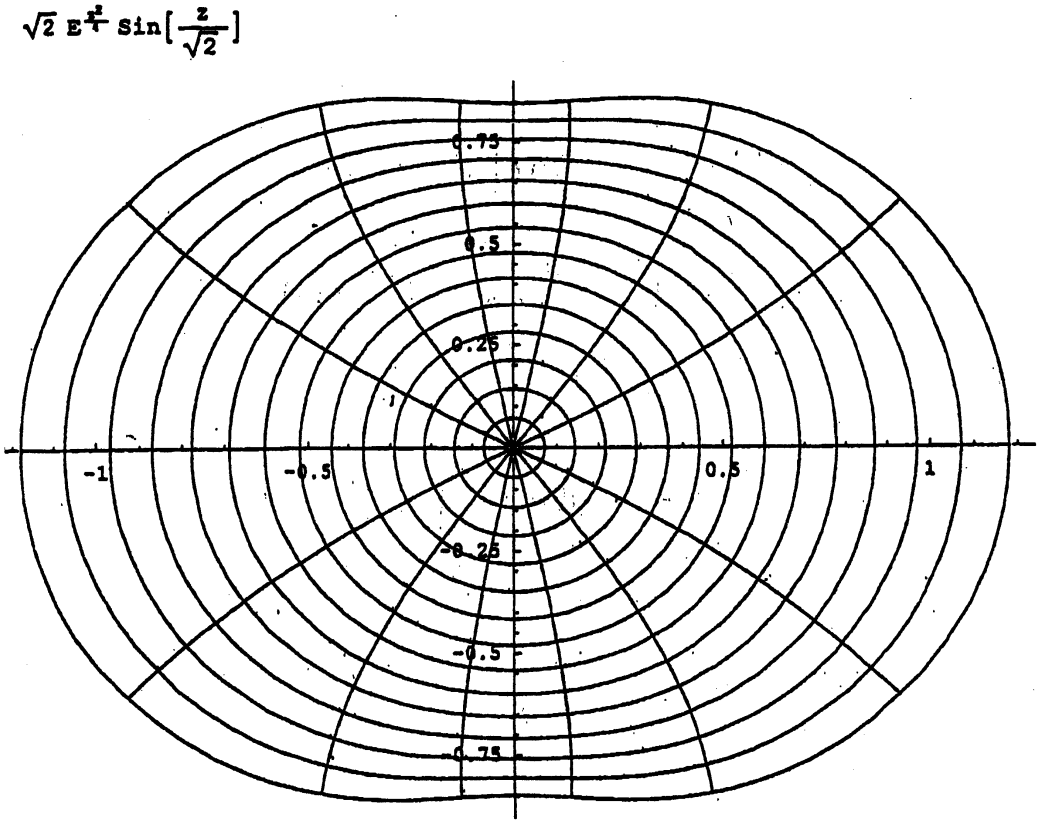
$$(2.13)\quad 1-J-\frac{1}{2}\mathrm{Re}\{za(z)\}<\mathrm{Re}\left\{\frac{zw'(z)}{w(z)}\right\}<1+J-\frac{1}{2}\mathrm{Re}\{za(z)\}$$

$$(z\in U;\ J>0).$$

Example 2. Let $a(z)=-z$, $b(z)=\frac{z^2}{4}$ in Theorem D, then the solution of

$$(2.14)\quad w''(z)-zw'(z)+\frac{z^2}{4}w(z)=0$$

is $w(z)=\sqrt{2}\exp\left(\frac{z^2}{4}\right)\sin\frac{z}{\sqrt{2}}\in S^*$.



3 Main results and their consequences

<u>Theorem 1.</u> Let $zp(z)$ be analytic in U with $|zp(z)| < J$ $(z \in U;\ 0 < J \leqq 1)$. Let $w(z)$, $z \in U$, be the solution of initial-value problem:

(3.1) $$w''(z) + p(z)\,w(z) = 0$$

with $w(0) = 0$ and $w'(0) \neq 0$. Then $w(z)$ is strongly starlike of order α, that is,

(3.2) $$\left|\arg\left\{\frac{zw'(z)}{w(z)}\right\}\right| < \frac{\pi}{2}\alpha$$

for some α $(0 < \alpha \leqq 1)$ and $\alpha = \frac{2}{\pi}\sin^{-1} J$ $(0 < J \leqq 1)$.

<Proof> If we put

(3.3) $$u(z) = \frac{zw'(z)}{w(z)} - 1 \qquad (z \in U),$$

then $u(z)$ is analytic in U, $u(0) = 0$ and (3.1) becomes

(3.4) $$[u(z)]^2 + u(z) + z\,u'(z) = z^2 p(z),$$

or equivalently

(3.5) $$h(u(z), zu'(z)) = z^2 p(z),$$

where $h(r, s) = r^2 + r + s$. It is easy to check $h(r, s) \in H_J$, i.e.,

(i) $h(r, s)$ is continuous in $\mathbb{D} = \mathbb{C} \times \mathbb{C}$;

(ii) $(0, 0) \in \mathbb{D}$, $|h(0,0)| = 0 < J$;

(iii) $|h(Je^{i\theta}, Ke^{i\theta})| \geqq J \quad (K \geqq J)$.

From assumption, we have

$$|z^2 p(z)| < J \quad (z \in U;\ 0 < J \leqq 1).$$

By using Theorem C, we have

$$|u(z)| < J \quad (z \in U;\ 0 < J \leqq 1),$$

which, in view of the relationship (3.3), yields

$$(3.6) \qquad \left| \frac{zw'(z)}{w(z)} - 1 \right| < J \quad (z \in U;\ 0 < J \leqq 1),$$

that is,

$$\left| \arg\left\{ \frac{zw'(z)}{w(z)} \right\} \right| < \frac{\pi}{2}\alpha$$

for some α $(0 < \alpha \leqq 1)$ and $\alpha = \frac{2}{\pi}\sin^{-1} J$ $(0 < J \leqq 1)$.

Q.E.D.

Furthermore, we have

<u>Theorem 2</u> Under same assumptions in Theorem 1, we have

$$(3.7) \qquad 1 - J < \mathrm{Re}\left\{ \frac{zw'(z)}{w(z)} \right\} < 1 + J \quad (z \in U;\ J > 0).$$

Proof is almost same in Theorem 1. Putting $J = 1$ in Theorem 1 and letting $J = 0$ in Theorem 2, we obtain Theorem A.

Example 3. Let $p(z)=\frac{3}{4}-\frac{z^2}{16}$ in Theorem 1 $(J=1)$, the solution of the following initial-value problem :

$$(3.8) \qquad w''(z)+\left(\frac{3}{4}-\frac{z^2}{16}\right)w(z)=0 \quad (w(0)=0,\ w'(0)=1)$$

is given by $w(z)=z\exp\left(-\frac{z^2}{8}\right)$. $w(z)$ is starlike function, that is, $w(z)\in S^*$.

In the case of $q(z)=\frac{6}{7}-\frac{4}{49}z^2$, the solution of

$$(3.9) \qquad w''(z)+\left(\frac{6}{7}-\frac{4}{49}z^2\right)w(z)=0$$

is given by $w(z)=z\cdot\exp\left(-\frac{z^2}{7}\right)\in S^*$.

For $a(z)=-z$ and $b(z)=\lambda$ $(\lambda\in\mathbb{C})$ in Theorem D, the initial-value problem (2.12) becomes

$$(3.10) \quad w''(z)-zw'(z)+\lambda w(z)=0 \quad (w(0)=0,\ w'(0)=1),$$

which, under the following transformation :

$$(3.11) \qquad w(z)=\exp\left(\frac{z^2}{4}\right)\cdot v(z),$$

assumes the normal form as given below

$$(3.12) \quad v''(z)+\left(\lambda+\frac{1}{2}-\frac{z^2}{4}\right)v(z)=0 \quad (v(0)=0,\ v'(0)=1).$$

These differential equations (3.10) and (3.12) are well-known, so called respectively Hermite's differential equation and Weber's differential equation.

Putting $p(z)=\lambda+\frac{1}{2}-\frac{z^2}{4}$ in Theorem 1, we have

Corollary 1 We consider Weber's differential equation (3.12). If

(3.13) $\quad |\lambda+\frac{1}{2}-\frac{z^2}{4}| < J \quad (z\in U;\ 0<J\leqq 1)$,

then $v(z)$ is strongly starlike of order α $(0<\alpha\leqq 1)$, that is, $v(z)\in SS^*(\alpha)$.

A function f, analytic in U, is said to be convex if it is univalent and $f(U)$ is convex. It is well-known that f is convex if and only if $f'(0)\neq 0$ and $\mathrm{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}>0$ in U. If in addition, $\mathrm{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}>\alpha$, $0\leqq\alpha<1$, then f is called convex of order α.

Next, we prove

Theorem 3 Let $u(z)$ be analytic in U and be the solution of Sonine's differential equation

(3.14) $\quad zu''(z)+(\mu+1-z)u'(z)+nu(z)=0$

with $u'(0) \neq 0$, $\mu \in \mathbb{R}$ and $n \in \mathbb{N}$.

If the following conditions are satisfied, i.e.,

(i) $\mu \geqq \sqrt{1+(n-1)^2} - 1$

(ii) $\mu \geqq \dfrac{2n+\alpha-3}{2(1-\alpha)}$,

then $u(z)$ is convex of order α $(0 \leqq \alpha < 1)$.

To prove this theorem, we prepare two lemmas due to Miller and Mocanu [4].

Lemma 1 (Miller and Mocanu [4]) Let E be a set in $\mathbb{C}$ and let a function $H : \mathbb{C}^2 \times U \to \mathbb{C}$ satisfy the condition:

(3.15) $\quad H(is, t; z) \notin E$, for $z \in U$ and real s, t with $t \leqq -\frac{1}{2}(1+s^2)$.

If $p(z)$ is analytic in U with $p(0)=1$ and $H(p(z), zp'(z); z) \in E$, $z \in U$, then $\mathrm{Re}\{p(z)\} > 0$ in U.

Lemma 2 (Miller and Mocanu [4]) If $\mu+1 \geqq \sqrt{1+(n-1)^2}$, then the solution $u(z)$ of Sonine's differential equation (3.14) is univalent in U.

<Proof of Theorem 3> If we put

(3.16) $$p(z)=\frac{1+\frac{zu''(z)}{u'(z)}-\alpha}{1-\alpha} \qquad (z\in U),$$

then $p(z)$ is analytic in U with $p(0)=1$.

By differentiating (3.16), we have

(3.17) $$zp'(z)+(1-\alpha)[p(z)]^2+(\mu-1+2\alpha-z)p(z)+\frac{n-\alpha}{1-\alpha}z-(\mu+1)=0.$$

If we let

$$H(w_1,w_2;z)=w_2+(1-\alpha)w_1^2+(\mu-1+2\alpha-z)w_1+\frac{n-\alpha}{1-\alpha}z-\mu-1$$

and $E=\{0\}$, then (3.17) can be written as $H(p(z),zp'(z);z)\in E$. We will use (3.15) and Lemma 1 to prove that $\operatorname{Re}\{p(z)\}>0$.

Letting $z=x+iy$, we obtain

(3.18) $$\operatorname{Re}H(is,t;z)=t-(1-\alpha)s^2+ys+\frac{n-\alpha}{1-\alpha}x-\mu-1$$
$$\leqq -\left(\frac{3}{2}-\alpha\right)s^2+ys+\frac{n-\alpha}{1-\alpha}-\frac{3}{2}-\mu\equiv Q(s),$$

for $t\leqq -\frac{1}{2}(1+s^2)$.

We next show that $Q(s)<0$ for all real s and $x^2+y^2<1$. The discriminant D of $Q(s)$ satisfies

$$D=y^2+4\left(\frac{3}{2}-\alpha\right)\left(\frac{n-\alpha}{1-\alpha}x-\frac{3}{2}-\mu\right)$$
$$<1-x^2+(6-4\alpha)\left(\frac{n-\alpha}{1-\alpha}x-\frac{3}{2}-\mu\right)\equiv h(x).$$

Then $h(1) = (6-4\alpha)\left(\frac{n-\alpha}{1-\alpha} - \frac{3}{2} - \mu\right) \leqq 0$, therefore $\mu \geqq \frac{2n+\alpha-3}{2(1-\alpha)}$. From (3.18), we deduce $\operatorname{Re} H(is, t; z) < 0$ for $z \in U$ and all real s, t with $t \leqq -\frac{1}{2}(1+s^2)$. By Lemma 1, we conclude that $\operatorname{Re}\{p(z)\} > 0$, that is, $\operatorname{Re}\left\{1 + \frac{zu''(z)}{u'(z)}\right\} > \alpha$, which shows that $u(z)$ is convex of order α in U $(0 \leqq \alpha < 1)$.

Q.E.D.

Example 4 Sonine's polynomials are given by

$$(3.19) \qquad S_n^{\mu}(z) = \frac{\Gamma(n+\mu-1)}{\Gamma(\mu+1)\Gamma(n+1)} F(-n; \mu+1; z)$$
$$= \sum_{k=0}^{n} \binom{n+\mu}{n-k} \frac{(-z)^k}{k!}.$$

If $n=2$ and $\mu=2$ are satisfy the conditions in Theorem 3 in the case of $\alpha = \frac{2}{3}$. We put

$$u(z) = S_2^2(z) = \frac{z^2}{2} - 4z + 6,$$

then we have $\operatorname{Re}\left\{1 + \frac{zu''(z)}{u'(z)}\right\} > \frac{2}{3}$.

Next, we show the following theorem.

Theorem 4. Let $v(z)$ be analytic in U and be the solution of Bessel's differential equation

$$(3.20)\qquad z^2v''(z)+z\,v'(z)+(z^2-\nu^2)v(z)=0$$

with $v(0)=0$ and $v'(0)\neq 0$. If $\mathrm{Re}(\nu^2)\geqq\frac{1}{2}$, then the solution $v(z)$ is starlike in U.

<Proof> If we put

$$(3.21)\qquad p(z)=\frac{zv'(z)}{v(z)}\qquad (z\in U),$$

then $p(z)$ is analytic in U with $p(0)=1$. Since $v(z)$ satisfies the differential equation (3.20), using this transformation in (3.20) results in

$$(3.22)\qquad zp'(z)+[p(z)]^2+z^2-\nu^2=0 .$$

If we let $H(w_1,w_2;z)=w_2+w_1^2+z^2-\nu^2$ and $E=\{0\}$, then (3.22) can be written as $H(p(z),zp'(z);z)\in E$. We will use (3.15) and Lemma 1 to prove that $\mathrm{Re}\{p(z)\}>0$. Letting $z=x+iy$, we obtain

$$(3.23)\qquad \mathrm{Re}\,H(is,t;z)=t^2-s^2+x^2-y^2-\mathrm{Re}(\nu^2)$$
$$\leqq -\frac{1}{2}s^2+x^2-y^2-\mathrm{Re}(\nu^2)-\frac{1}{2}$$
$$<-\frac{1}{2}s^2-2y^2-\mathrm{Re}(\nu^2)-\frac{1}{2} .$$

From assumption, $\mathrm{Re}(\nu^2)\geqq\frac{1}{2}$, then

$\mathrm{Re}\, H(is, t; z) < 0 \ (z \in U)$. By Lemma 1, we have

$$\mathrm{Re}\{p(z)\} = \mathrm{Re}\left\{\frac{z v'(z)}{v(z)}\right\} > 0 \quad (z \in U),$$

that is, $v(z)$ is starlike.

Q.E.D.

Remark. Bessel function of the first kind is given by

$$J_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{z}{2}\right)^{\nu+2k}}{k!\, \Gamma(\nu+k+1)} \quad (3.24)$$

Example 5 In the case of $\nu = 1$ in (3.24),

$$J_1(z) = \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{z}{2}\right)^{2k+1}}{k!\, \Gamma(k+2)}$$ is starlike.

References

[1] P. L. Duren, Univalent Functions, Springer-Verlag, New York, (1983).

[2] E. Hille, Ordinary Differential Equations in the complex plane, Wiley, New York, (1976).

[3] S. S. Miller, A class of differential inequalities implying boundedness, Illinois J. Math. 20 (1976), 647-649.

[4] S.S. Miller and P.T. Mocanu, Univalence of Gaussian and confluent hypergeometric functions, Proc. Amer. Math. Soc. 110 (1990), 333-342.

[5] S. Owa, H. Saitoh, H.M. Srivastava and R. Yamakawa, Geometric properties of solutions of a class of differential equations, Computers and Math. Appl. 47 (2004), 1689-1696.

[6] Ch. Pommerenke, Univalent Functions, Vanderhoeck and Ruprecht, Göttingen, (1975).

[7] H. Saitoh, Univalence and starlikeness of solutions of $W'' + aW' + bW = 0$, Ann. Univ. Mariae Curie-Skłodowska, Section A 53 (1999), 209-216.