

Bendings and tuckings in planar portraits of manifolds

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1 Introduction

A *planar portrait* of a smooth manifold is the image of the manifold through a stable map into the plane, paired with the curve of critical values. In this article, we present characterisations of manifold pieces over certain subpieces of planar portraits, which will be applied to reductions of planar portraits to simpler ones. By their geometric nature in 2-dimensional case, a series of these characterisations are called the *bending and tucking lemma* here.

The lemma is based on another characterisation for a special subpiece of a planar portrait named the *cusped fan* given in [K]. We state that and give another application of that to obtain planar portraits of manifolds in explicit styles.

2 Planar portrait

Let $f : M^n \rightarrow \mathbf{R}^2$ be a smooth stable map of a smooth closed manifold M of dimension n . Denote by S_f the singular points of f .

Definition The pair $P = (f(M), f(S_f))$ up to self-diffeomorphism of \mathbf{R}^2 is called a *planar portrait* of M through f .

In Figure 1, we show an example of a planar portrait.

Remark A singular point $p \in S_f$ is either a cusp or a fold, as is well known.

A planar portrait of a manifold is a geometric representation of a manifold. One can find topological properties of a manifold M in its planar portraits. For example,

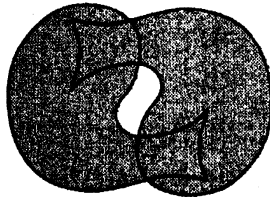


Figure 1: A planar portrait of a manifold

the number of cusps have the same parity as the Euler characteristic $\chi(M)$, by classical result of Thom. In case the indices of folds are known, one has a chance to know $\chi(M)$ absolutely, as pointed out in [L]. But it is not clear yet that what kind of properties can one read from a planar portrait. In the latter half of this article, we show a looseness of planar portraits to manifolds; spininess of a 4-manifold, for example, are not carried to the portraits. On the other hand, we show soon later that some planar portraits strongly regulate the class of admissible source manifolds.

3 Bending and tucking lemma

Let X be a compact manifold with boundary. We say that $h : X \rightarrow \mathbf{R}^2$ is a *fibrewise cut of a stable map* ([K]) if there exist an open manifold \hat{X} which contains X as a proper submanifold, and a stable map $f : \hat{X} \rightarrow \mathbf{R}^2$ such that the three conditions bellow are satisfied:

1. The restriction $f|_X$ coincides with h .
2. There exists a finite collection $\lambda_i, i = 1, \dots, m$, of smooth plane arcs such that the intersections of λ_i with other λ_j and with $f(S_f)$ are transverse.
3. X is obtained from \hat{X} by cutting it along $f^{-1}(\lambda_1) \cup \dots \cup f^{-1}(\lambda_m)$.

For a fibrewise cut of a stable map h , we define the set of singularities by $S_h = S_f \cap X$, and call the pair $(h(X), h(S_h))$ the planar portrait of X , as before.

Let $P = (h(X), h(S_h))$ be a planar portrait in the figure bellow, where $h : X \rightarrow \mathbf{R}^2$ is a fibrewise cut of a stable map.

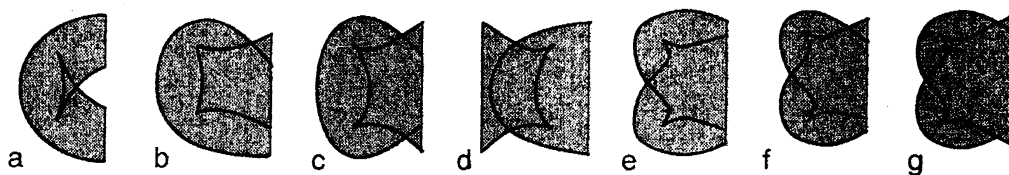


Figure 2: Bendings and tuckings in planar portraits

Theorem (Bending and Tucking lemma)

The source manifold X of a planar portrait in Figure 2 is diffeomorphic to

- $\Sigma^{n-1} \times D^1$ (in case of a, b, c, d), or
- $D^{n-1} \times S^1$ or $D^{n-1} \bar{\times} S^1$ (the non-trivial disc bundle) (in case of e), or
- $D^1 \times S^1$ or $D^1 \bar{\times} S^1$ (in case of f, g),

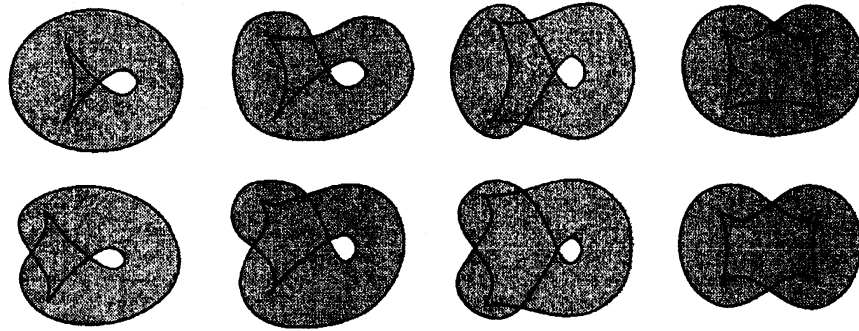
where Σ^{n-1} is a homotopy $(n-1)$ -sphere of the form $D^{n-1} \cup_{\varphi} D^{n-1}$ with $\varphi : \partial D^{n-1} \rightarrow \partial D^{n-1}$ a self-diffeomorphism.

Conversely, every diffeomorphism types of X listed above can be realised.

Remark It is known that any homotopy $(n-1)$ -sphere is of the form Σ^{n-1} as above if $n \geq 8$, whereas in $n \leq 7$, any of Σ^{n-1} above is the standard sphere S^{n-1} .

In $\dim X = 2$, the theorem says that the planar portraits a, b, c represent bendings of a tube, and d does tucking of a tube. The planar portraits in Figure 2 are the list of configurations of a connected *depth 2* region with two cusps of the condition that at least one edge abutting them bounds the image. up to certain cancelling of crossings. Here a region of regular values in a planar portrait is of depth 2 if there exists an arc connecting a point in that region to the outside of the planar portrait which meets the critical value set transversely at two points.

As an easy application, we can see that the source manifolds of the planar portraits in figure below, for example, are all diffeomorphic to the total space of a Σ^{n-1} bundle over S^1 .

Figure 3: Planar portraits of Σ^{n-1} bundles over S^1

4 A key lemma

To prove the B-T (bending and tucking) lemma, we prepare a lemma for a special subpiece of a planar portrait. We call the pair of a compact region and plane curves with a cuspidal point as shown in Figure 4 the *cusped fan*.



Figure 4: The cusped fan

For a stable cut f over the cusped fan, we have the lemma bellow:

Lemma [K]

Let $f : X \rightarrow \mathbf{R}^2$ be a stable cut over the cusped fan. Then f is right-left equivalent to the map $h : D^{a+1} \times D^{b+1} \rightarrow \mathbf{R}^2$ defined by $h(x, y) = (|x|^2 + 2\epsilon y_0(1 - |x|^2), |y|^2 + 2\epsilon x_0(1 - |y|^2))$ for some non-negative integers a, b and for any positive small constant ϵ , where $D^{a+1} = \{|x|^2 \leq 1\}$, $D^{b+1} = \{|y|^2 \leq 1\}$, $x = (x_0, x_1, \dots, x_a)$ and $y = (y_0, y_1, \dots, y_b)$.

See [K] for the proof and its applications. We note here only that h above is a

perturbation of the twice folding projection $(|x|^2, |y|^2) : D^p \times D^q \rightarrow I \times I \subset \mathbb{R}^2$. In the 2-dimensional case $n = 2$, what it stands for is easily understood (see the figure below).

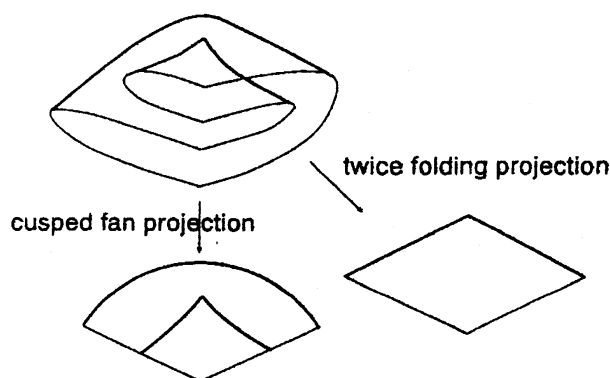


Figure 5: Cusped fan and the twice folding projection

Remark The cusp of h has the absolute index of $\max\{a, b\}$, as an easy calculation shows. The four folds on the edge of the fan has the indices $0, a, a + 1, n - 1$, lined in this order, for an orientation of the edge (or, $0, b, b + 1, n - 1$, in the reversed orientation).

We give here a sketch of the proof for the Bending and Tucking lemma. First we note that any crossings are derived from indefinite folds $(u, |x|^2)$, and hence fibres are disconnected near the crossings. This implies that: any of the portrait through a to d is moved to the union of two cusped fans attached along a boundary, by moving the projection h of the manifold X up to homotopy. In actual, for a to c, one can remove the crossings by homotopy to obtain the required style of planar portrait. For d, we can achieve the move by a twisting (see Figure 6). Similarly, the planar portraits f and g are moved to e, by homotopy move of h . Therefore we are enough to study the source manifold X for the two cases.

For the first case, note that X is the union of two $D^{a+1} \times D^{b+1}$ attached along $\partial D^{a+1} \times D^{b+1}$, by the key lemma. On the other hand, Remark to the key lemma shows that $b = 0$, because the crossing is caused by definite folds. The diffeomorphism type of X is hence as in the statement.

For the second case, or e, note that e is the union of two cusped fans but with a degeneration. Since the degeneration is caused by a definite fold, it is easy to move

the projection to the standard cusped fan projection similarly as before. Hence the source manifold X is the union of two $D^1 \times D^{n-1}$ attached along $S^0 \times D^{n-1}$. The diffeomorphism type of X is hence as required. Note that for f and g , the dimension n must be 2, since both side of a cusp are definite folds.

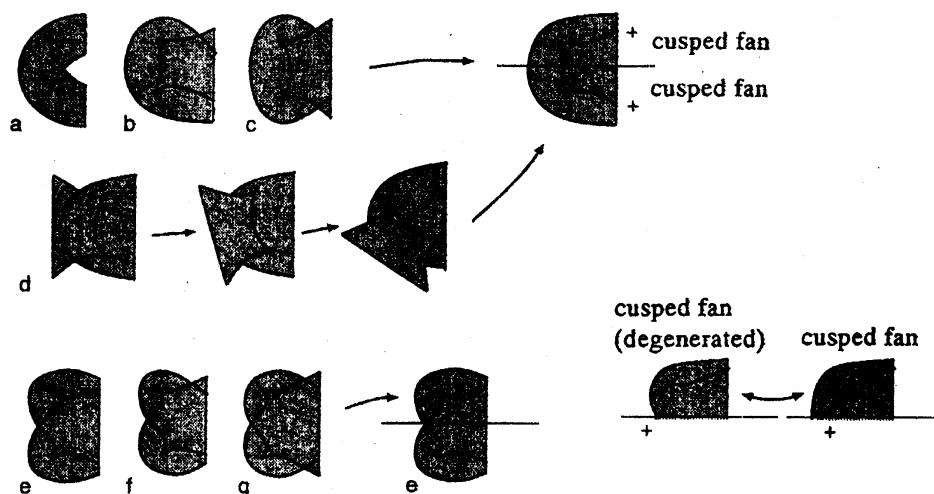


Figure 6: Proof of the B-T lemma

5 Another application of the key lemma

As another application of the key lemma, we present here constructions of planar portraits in two different manners. See [K] for detail.

1. Lifting Morse functions

The first manner is the construction by lifting Morse functions. The basic idea of this is as follows: Note that the twice folding projection $(|x|^2, |y|^2)$ is a lift into \mathbf{R}^2 of a Morse type critical point, as seen by the factorisation below:

$$(x, y) \mapsto (|x|^2, |y|^2) \mapsto |x|^2 \pm |y|^2, \quad x \in D^p, y \in D^q$$

The stable cut over a cusped fan is still so (see Figure 7).

We can apply this to take planar portraits.

Example [K]: *Lifting of a Morse function of kP^2 , $k = \mathbf{R}, \mathbf{C}$ or \mathbf{H} .*

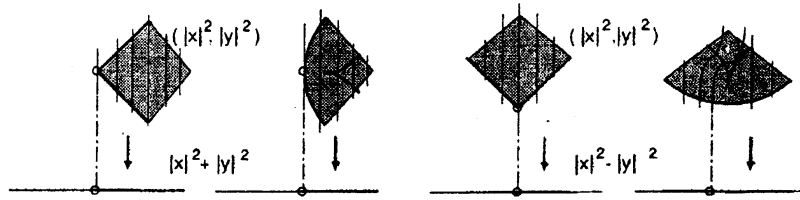


Figure 7: Cusped fans and lifts of Morse critical points

Let $g : kP^2 \rightarrow \mathbf{R}$ be the Morse function defined by $g([x : y : z]) = \frac{a|x|^2 + b|y|^2 + c|z|^2}{|x|^2 + |y|^2 + |z|^2}$, where $0 < a < b < c$. It has three critical points $[1 : 0 : 0]$, $[0 : 1 : 0]$ and $[0 : 0 : 1]$. We can lift g as follows.

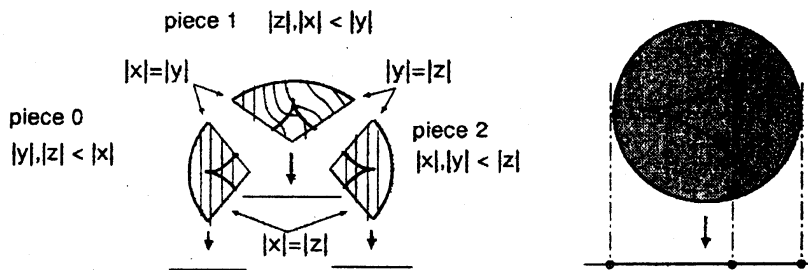


Figure 8: A lift of a Morse function of the projective plane

Take a decomposition $kP^2 = H_0 \cup H_1 \cup H_2$, where H_0, H_1, H_2 are neighbourhoods of $[1 : 0 : 0]$, $[0 : 1 : 0]$ and $[0 : 0 : 1]$ defined by $\{|y|, |z| \leq |x|\}$, $\{|z|, |x| \leq |y|\}$ and $\{|x|, |y| \leq |z|\}$, respectively. On these pieces, g has the form $|y|^2/|x|^2 + |z|^2/|x|^2$, $|z|^2/|y|^2 - |x|^2/|y|^2$ and $-|x|^2/|z|^2 - |y|^2/|z|^2$, respectively.

We take f in the key lemma on each of the three pieces and place the cusped fan produced by f so that it gives a local lift of g in each piece with respect to a submersion $\mathbf{R}^2 \rightarrow \mathbf{R}$ (refer to Figure 8). It is not difficult to check that the three copies of f can be glued together consistently. Hence we obtain a stable map \tilde{g} of kP^2 into \mathbf{R}^2 which is a lift of g , and whose planar portrait is the union of three $2\pi/3$ -angled cusped fans pasted together to form a disc.

2. Perturbation of a Moment map

The stable cut over the cusped fan can be also regarded a perturbation of the quotient map of the linear product actions of $O(p) \oplus O(q)$ to $D^p \times D^q$. We can apply

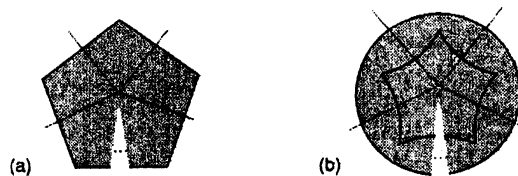


Figure 9: (a) image of the moment map, and (b) the planar portrait of a regular toric surface

this to take planar portraits of a manifold with suitable action of $O(p) \oplus O(q)$.

Example [K]: *Stable perturbations of the moment maps of regular toric surfaces*

A regular complex toric surface M is obtained by putting k -copies of $D^2 \times D^2$ together by the diffeomorphisms $\tilde{\gamma}_i(z, w) = (\bar{w}, w^{\gamma_i} z)$, $i = 1, \dots, k$ for some $\gamma_1, \dots, \gamma_k \in \mathbf{Z}$ (we remark that k equals to $\chi(M)$). Note that the twice folding projection $(|z|^2, |w|^2)$ on each $D^2 \times D^2$ gives a moment map of the toric action (see Figure 9, left). Now take f of the key lemma on each piece. The k copies of f are well-pasted by $\tilde{\gamma}_i$, $i = 1, \dots, k$ so that they define a global perturbation of the moment map into a stable map. The planar portrait thus obtained is the $2\pi/k$ -angled cusped fans pasted together to form a disc (see Figure 9, right).

Remark For the complex projective plane, the two maps constructed in the two examples are the same one.

By applying the second construction for Hirzebruch surfaces, we obtain a corollary on the projection-to-portrait correspondance as follows. The proof of this is given in [K].

Corollary [K]

Let M be either $S^2 \times S^2$ or $S^2 \tilde{\times} S^2$, where the latter denotes the total space of the non-trivial S^2 bundle over S^2 . Then there exist stable maps $f_i : M \rightarrow \mathbf{R}^2$, $i \in \mathbf{Z}_+$ with the properties that:

- 1 They have the same planar portrait as drawn in Figure 10 in common.
 - 2 Any pair of maps f_i and f_j are not right-left equivalent, if $i \neq j$.
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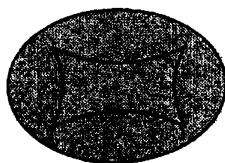


Figure 10: A common planar portrait of Hirzebruch surfaces

Based on the same idea, we can generalise the above corollary as bellow:

Corollary [K]

Any S^p bundle over S^q ($p, q \geq 1$) that admits a cross-section has the planar portrait in Figure 10 in common.

The corollary above shows a looseness of planar portraits to manifolds. We remark that the possible source manifolds for the planar portrait in Figure 10 spread outside that class of sphere bundles. In actual, $P^2 \# P^2$ admits it as a planar portrait, as it is realised by coupling two copies of one in Figure 8 and by eliminating a pair of cusps.

References

- [L] H. Levine, *Mappings of manifolds into the plane*, Amer. J. Math. **88** (1966), 357–365.
- [K] M. Kobayashi, *On the cusped fan in a planar portrait of a manifold*, preprint, Hokkaido Univ. preprint series in math. Series #816, November (2006).