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AN APPLICATION OF AUBRY–MATHER THEORY IN LORENTZIAN GEOMETRY

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1. Overview

The results presented in this note have been obtained in the framework of a research in progress with Albert Fathi, see [4].

We give an application of the Aubry–Mather theory for non regular Hamiltonian, recently developed in [6], see also [5], [3] in Lorentzian geometry. Namely, we provide an alternative proof, based on Aubry–Mather theory, of the existence of a smooth time function for Lorentzian manifolds enjoying suitable assumptions. Our approach is related to that of [9], where Aubry–Mather theory is exploited to construct a Lyapunov function for a multivalued dynamics.

We make in this way a connection which is, at a first sight, quite surprising since the Aubry–Mather theory concerns the qualitative analysis of Hamilton–Jacobi equations at the critical value, i.e. when subsolutions, but not *strict* subsolutions, do exist, while, in the time function problem, no Hamiltonians are involved. In what follows we try to explain how to bridge this gap.

2. Aubry set

In terms of Hamilton–Jacobi equations the issue we are interested on can be described as follows:

Given an Hamiltonian $H(x,p)$ defined on a cotangent bundle of a (smooth, boundaryless, connected, paracompact, Hausdorff) manifold M, we look for a smooth (C^∞) strict subsolution of the equation

$$H(x, Du) = 0, \tag{1}$$

or at least for a strict subsolution which is smooth in a open subset of M as large as possible. Here and throughout the paper *subsolution* means locally Lipschitz–continuous a.e. subsolution. The Hamiltonian H is required to be continuous in both arguments, convex and coercive in p. Clearly, to make the problem sensible, we also have to assume that some subsolutions to (1) do exist.

We then consider the points y of M possessing a neighborhood where some subsolution u to (1) satisfies

$$H(x, Du(x)) \leq -\delta \quad \text{a.e. for some } \delta > 0, \tag{2}$$

and denote by M_0 the (possibly empty) open subset made up by such points. We have:

Theorem 2.1. *There exists a subsolution to (1), which is smooth and strict in M_0.*

The proof of the Theorem can be found in [5], [6]. It is basically structured in two steps. First, by exploiting the convex character of H, it is constructed, through an infinite convex combination, a strict subsolution in M_0. This function is then regularized, in order to satisfy the statement, by using a partition of unity technique and a mollification procedure.

The *bad points* are then those around which no subsolution u satisfies (2). This (possibly empty) set is named after Aubry. An expert of the field knows that it can be actually nonempty only if the value 0 is *critical* for H. We skip here this kind of results since we do not need them in what follows. It immediately comes from Theorem 2.1:

Theorem 2.2. *There exists a strict smooth subsolution to (1) if and only if the Aubry set $M \setminus M_0$ is empty.*

It is crucial for the subsequent analysis to provide a metric characterization of the Aubry set. We consider the set–valued covector field

$$x \mapsto Z_x = \{p \,:\, H(x,p) \leq 0\} \subset T^*_M(x),$$

where $T^*_M(x)$ indicates the cotangent space to M at x. According to the assumptions made on H, we see that Z is compact convex valued and continuous with respect to the Fell topology , at least at the points x where the interior of Z_x is nonempty, which is actually the case we will be interested on in what follows. We recall that a sequence of closed subsets K_n of a topological space converges to a closed subset K in the Fell sense provided that

$$\begin{array}{rl} \text{for any subsequence } n_i & p_{n_i} \in K_{n_i} \text{ and } p_{n_i} \to p \Rightarrow p \in K \\ \text{for any } p \in K & \text{there exists } p_n \in K_n \text{ with } p_n \to p. \end{array}$$

We also consider the support function

$$(x,q) \mapsto \sigma_Z(x,q) = \max\{\langle p,q\rangle_x : p \in Z_x\},$$

where q is a tangent vector to M at x ($q \in T_M(x)$), and $\langle\cdot,\cdot\rangle_x$ denotes the pairing in $T^*_M(x) \times T_M(x)$. We see that σ_Z is continuous in both variables and convex positively homogeneous in q. We define for any continuous piecewise C^1 curve ξ defined on some interval $[a,b]$ the *intrinsic* length ℓ_Z via the formula

$$\ell_Z(\xi) = \int_a^b \sigma_Z(\xi,\dot{\xi})\,dt, \tag{3}$$

note that, thanks to positive homogeneity of σ_Z, the value of this integral is not affected by orientation–preserving change of parameter. We make precise that any curve considered in the remainder of the paper will be intended to be continuous piecewise C^1.

It is not difficult to prove, see [6], that there are subsolutions to (1) if and only if the intrinsic length of any cycle (i.e. closed curve) is nonnegative. In the following crucial characterization, see [6], we use the fact that the manifold M, being paracompact, can be endowed with a (complete) Riemannian metric.

Theorem 2.3. *Let $\tilde{g}$ be any Riemannian metric on M, and denote by $\ell^{\tilde{g}}$ the associated length functional defined on the curves of M.*

A point y belongs to the Aubry set $M \setminus M_0$ if and only if there exists a sequence ξ_n of cycles based on y such that

$$\lim_n \ell_Z(\xi_n) = 0 \quad \textit{and} \quad \inf_n \ell^{\tilde{g}}(\xi_n) > 0.$$

An important remark is now in order. In the previous analysis we have never make direct use of the Hamiltonian H, all the notions introduced being instead based on the set–valued map Z. In particular the property of being subsolution to (1) for a Lipschitz–continuous function u can be equivalently expressed by requiring u to solve the partial differential inclusion

$$Du(x) \in Z_x \quad \text{a.e. in } M. \tag{4}$$

Similarly a smooth function φ is a strict subsolution of (1) if and only if

$$D\varphi(x) \in \overset{\circ}{Z}_x \quad \text{for } x \in M,$$

where $\overset{\circ}{Z}_x$ is the interior of Z_x. Looking at the issue from this angle, we furthermore understand that the assumption of compactness on the images of Z can be eliminated. The possibility of finding smooth functions with differential contained in the interior of Z_x becomes in fact greater the larger is the size of Z_x.

To make more precise this intuition, we consider, without reference to any initial Hamiltonian, a continuous closed convex valued covector field Z, and a Riemannian metric g on M, then we cut Z_x with the closed Riemannian cotangent unitary balls having center at 0, denoted by $B^g_x(0,1) \subset T^*_M(x)$. We define in this way a convex compact valued map

$$x \mapsto Z_x \cap B^g_x(0,1) \quad \text{for } x \in M, \tag{5}$$

which is still continuous, provided its values possess nonempty interior for any x. Note that (5) depends on the Riemannian metric g, by varying it we accordingly modify the set–valued map as well as the associated intrinsic length and Aubry set. We can, in conclusion, rephrase part of the previous results and state them in the more convenient form for later use as follows:

Theorem 2.4. *Let Z be a continuous (in the Fell sense) closed convex - valued covector field of M. If there is a a Riemannian metric g on M such that*

i. *$Z_x \cap B^g_x(0,1)$ has nonempty interior for any x,*

ii. *the inclusion (4) with $Z_x \cap B^g_x(0,1)$ in place of Z_x admits a solution,*

iii. *the Aubry set associated to $Z_x \cap B^g_x(0,1)$ is empty,*

then there exists a smooth function φ with

$$D\varphi(x) \in \mathring{Z}_x \quad \textit{for any } x.$$

Remark 2.5. Taking into account the Theorem 2.3, we note that, given g, to prove the emptiness of the associated Aubry set it is enough to exhibit a Riemannian metric $\tilde{g}$ such that any sequence of cycles with infinitesimal intrinsic length (associated to $Z_x \cap B^g_x(0,1)$) also have infinitesimal $\tilde{g}$–Riemannian length. We will use this principle in Section 5.

The Riemannian metrics g, $\tilde{g}$ play a quite different role in this context: g provides the cutting balls $B^g_x(0,1)$, while $\tilde{g}$ is involved in the fact the associated length functional appears in the characterization of the Aubry set given by Theorem 2.3.

3. Lorentzian setting

We recall some basic material of Lorentzian geometry, following [8], [1]. Unfortunately, to explain our setting, we are forced to introduce not few definitions, however the terminology is nice, and a gross understanding of underlying physical model is not difficult.

As a starting point, we consider the classical 4–dimensional Minkowski manifold $\mathbb{R}^4$, where the general point (x, y, z, t) is interpreted as representing an event (the first three components are spatial and the last is relative to the time), together with the indefinite nondegenerate symmetric bilinear form g given by the diagonal matrix diag $(1, 1, 1, -1)$, mathematically:

$$g(x, y, z, t) = x^2 + y^2 + z^2 - t^2 \quad \text{for any (x,y,z,t).}$$

The crucial role of such form is to select the physically admissible displacements. We think the time as running at the light speed and, in accord with the relativity theory which predicts that no physical particle can move faster than light, we qualify as admissible a displacement vector $v = (v_x, v_y, v_z, v_t)$ if

$$|v_t| \geq \sqrt{v_x^2 + v_y^2 + v_z^2} \quad \text{or equivalently} \quad g(v) \leq 0.$$

We call *causal* the vectors satisfying the previous inequality; among them we further distinguish the *timelike* and *lightlike* vectors at which g is strictly negative and vanishing, respectively. We refer to the remaining vectors (with g strictly positive) as *spacelike.*

This terminology applies to curves and smooth hypersufaces of $\mathbb{R}^4$ as well, looking at the tangent vectors. A timelike curve is then the path of an observer traveling less than the speed of light, a spacelike curve can have a geometric interest but does not describe a physically admissible motion.

The causal vectors make up a closed nonconvex cone with vertex at 0. We determine a *time–orientation* of R^4 by choosing one of the two convex cones composing it, say the one with nonnegative time component. This is called *future directed* causal cone. It is *salient* in the sense that it does not contain any pair of opposite nonzero vectors.

More generally a future–oriented Lorentzian manifold M, or in short a *space–time*, of dimension N is a manifold with the continuous assignment to each tangent space of an indefinite nondegenerate symmetric bilinear form with signature

$$(+, +, \cdots, +, -),$$

i.e. with all the eigenvalues positive, except one, which is negative, and the continuous choice of a future–directed causal cone, that will be denoted, from now on, by $C_x \subset T_M(x)$. We will be particulary interested in what follows by the salient closed cone–valued vector field $x \mapsto C_x$, which is continuous in the Fell topology. Note that any tangent space to M is an N–dimensional Minkowski manifold, up to the choice of a basis.

We refer to any curve ξ satisfying the inclusion $\dot{\xi}(t) \in C_{\xi(t)}$ (resp. $\dot{\xi}(t) \in \overset{\circ}{C}_{\xi(t)}$) for a.e. t, as future–directed causal (resp. timelike). Given x in M, we denote by $J^+(x)$ (resp. $J^-(x)$) the set of points y contained in some future–directed causal curve starting at x (resp. ending at x).

4. Time functions

A classical subject of investigation in General Relativity is to look for surfaces of simultaneity in a general space–time M, where time and space variables are not separated any more, playing the same role as the levels $t = constant$ in a Minkowski space.

The sets we are interested on should at least be *achronal*, in the sense that no points of them can be connected by future–directed timelike curves. One should also more stringently ask that any *inextendible* timelike curve intersects it exactly once, where a curve is qualified as inextendible if the limits at the endpoints of its interval of definition do not exist. A surface of this type is termed after *Cauchy*, since it is the natural region to pose initial conditions for hyperbolic equations, as Einstein's one.

These issues are strictly related to the existence of *time functions*, i.e. functions strictly increasing along each future–directed causal curve, being any level set of such a function achronal. If, in addition, the time function runs from $-\infty$ to $+\infty$ along any inextendible future–directed causal curve, then every level set is also a Cauchy surface.

Quite understandingly, it turns out that suitable assumptions on the causal structure of the space–time are needed for the existence of such objects. The basic one is called *weakly causality*, and requires no time-like curves to be a cycle, or equivalently the inclusion $\dot{\xi}(t) \in \overset{\circ}{C}_{\xi(t)}$ not to have closed curves as solutions. The violation of it would apparently yield causality breakdowns, in that one could travel into one's past. Note that this condition rules out compact space–times since it can be proved (see [1], Proposition 3.10) that compactness and weak causality are incompatible.

The previous notion can be strengthen by requiring some stability with respect to small perturbations of the Lorentzian structure. We first define the order relation $\succ$ in the family of convex cones with vertex 0 contained in $T^*_M(x)$, for some x, as follows:

$$D_0 \succ C_0 \quad \text{if } \overset{\circ}{D}_0 \supset C_0 \setminus \{0\}.$$

We say that M is *stably causal* if there is a continuous closed cone valued covector field D, with $D_x \succ C_x$ for any x, such that the inclusion $\dot{\xi}(t) \in \overset{\circ}{D}_{\xi(t)}$ is not solved by any closed curve. If, in addition, the set $J^+(x) \cap J^-(y)$ is compact for any x, y, M will be called *globally hyperbolic.* The role played by cycles in the definition of (weak, stable) causality casts some light on the possible connection between the topic we are treating here and the results exposed in Section 2. Recall, in fact, that the definition of Aubry set has been also given in term of cycles, see Theorem 2.3.

The stable causality and the global hyperbolicity result to be equivalent to the existence of a time function and of a Cauchy surface, respectively, see [8], [1]. However these results are, in their classical formulation, of topological nature. More precisely Geroch establishes in his seminal paper [7] the equivalence for an N–dimensional space–time M between global hyperbolicity and the existence of a topological Cauchy $(N-1)$–dimensional surface, i.e. a topological subspace of M where any point possess a neighborhood homeomorphic to an open subset of $\mathbb{R}^{N-1}$. The derived splitting Theorem states that a global hyperbolic M is homeomorphic to $\mathbb{R} \times S$, where S is a topological hypersurface of M. A modification of Geroch's technique allows to show the already announced equivalence between stable causality and the existence of a continuous time function.

The possibility of smoothing up these results has remained as an open question for a long time, and, after some fallacious attempts, it has been only recently solved by Bernal and Sánchez in [2]. They assume the global hyperbolicity of the space–time, and starting from Geroch's splitting theorem, which is in turn based on suitable probability measures defined on M, they prove the existence of a smooth Cauchy surface, and from this they construct a smooth time function whose level sets are Cauchy surfaces.

We look at the subject from a new angle and propose a new deduction of these results by using Aubry–Mather theory.

5. Main result

To keep our treatment as light as possible, we restrict ourselves to explain how Theorem 2.4 can be applied to prove:

Theorem 5.1. *Any stably causal space–time M admits a smooth time function.*

We sacrifice precision for the sake of clarity and brevity. We consider the dual cone

$$C_x^* = \{p \in T_M^*(x) \ : \ \langle p, q \rangle_x \leq 0 \ \text{ for any } q \in C_x\}.$$

The time function evoked in the statement of the theorem is nothing but a smooth solution of the partial differential inclusion

$$-Du(x) \in \overset{\circ}{C}{}_x^*. \tag{6}$$

We therefore proceed to check if the assumptions of Theorem 2.4 are satisfied by the set–valued map $x \mapsto C_x^*$. We see that it inherits from C the property of being continuous in the Fell sense, moreover the interior of C_x^* is nonempty for every x, since C_x does not contain any linear subspace. This implies that the interior of $C_x^* \cap B_x^g(0,1)$ is also nonempty for every Riemannian metric g on M. Finally the null function is a solution of the partial differential inclusion (4) with $C_x^* \cap B_x^g(0,1)$ in place of Z_x. In order to get existence of solutions to (6), it is thus left to prove the emptiness of the associated Aubry set for a suitable choice of g.

Given a Riemannian metric g, we define an intrinsic length for curves, denoted by ℓ_C^g, through the integral formula (3) with the support function σ_C^g of $C_x^* \cap B_x^g(0,1)$ as integrand, and look at the intrinsic length of the cycles, in accord with Theorem 2.3. Note that the intrinsic length of any curve is nonnegative since $0 \in C_x$ for any x, in addition

$$\sigma_C^g(x,q) = 0 \quad \text{for all } x \in M, \ q \in C_x,$$

so that the intrinsic length of any causal curve is zero, and consequently, in the light of Theorem 2.3, any causal nonconstant cycle is contained in the Aubry set, regardless to the choice of g. The stable causality rules out the possibility of having such cycles. We are going furthermore to prove that this condition allows a choice of g yielding an empty Aubry set, which implies Theorem 5.1. For this it will be enough to show the following:

Proposition 5.2. *There exist two Riemannian metrics g and $\tilde{g}$ such that for every nonconstant curve $\xi : [a,b] \to M$ with*

$$\ell^g_C(\xi) < \min\{\ell^{\tilde{g}}(\xi)\ ,\ 1/2\}, \tag{7}$$

we can find a curve γ, with of $\dot{\gamma}(t) \in \overset{\circ}{D}_{\gamma(t)}$, joining $\xi(a)$ to $\xi(b)$.

In the statement D denotes the cone valued map, with $D_x \succ C_x$ for any x, satisfying the causality condition. We indeed deduce from the stable causality that $\xi(a) \neq \xi(b)$, and so any ξ satisfying (7) cannot be a cycle. Consequently every sequence of cycles with infinitesimal intrinsic length ℓ^g_C, must also have infinitesimal Riemannian length $\ell^{\tilde{g}}$. This shows that the Aubry set corresponding to ℓ^g_C is empty, thanks to Remark 2.5.

The relation $D_x \succ C_x$ for any x implies that the values of such cone maps are locally separated by pairs of constant cones. More precisely it can be proved that there exists a countable atlas of smooth charts

$$\varphi_n : U_n \subset M \to V_n \subset \mathbb{R}^N$$

and two sequences of cones C^n_0, D^n_0 of $\mathbb{R}^N$ such that

$$D\varphi_n(\varphi_n^{-1}(y))D_{\varphi_n^{-1}(y)} \succ D^n_0 \succ C^n_0 \succ D\varphi_n(\varphi_n^{-1}(y))C_{\varphi_n^{-1}(y)},$$

for each n, and $y \in V_n \subset \mathbb{R}^N$.

A crucial step is then to establish a simple property linking the order relation $\succ$ on the family of cones, natural and intrinsic length, in the Euclidean space $\mathbb{R}^N$ for a pair of constant cone maps. The symbols $|\cdot|$, ℓ, $B(0,1)$, $\langle\cdot,\cdot\rangle$ will denote Euclidean norm, length of curves, unitary dual closed ball and pairing in $(\mathbb{R}^N)^* \times \mathbb{R}^N$, respectively.

Lemma 5.3. *Given two convex cones C_0 and D_0 of $\mathbb{R}^N$ with $D_0 \succ C_0$, we indicate by $\sigma_1(\cdot)$, ℓ_1 the support function of $C_0 \cap B(0,1)$ and the length defined by formula (3) with σ_1 as integrand, respectively.*

There exists a positive δ such that for any curve ξ, defined in some interval $[a,b]$, satisfying $\ell_1(\xi) < \delta\ell(\xi)$, one has $\xi(b) - \xi(a) \in \overset{\circ}{D}_0$.

Proof. Define

$$S = \{p \in D^*_0\ :\ |p| = 1/2\}.$$

Since $D_0 \succ C_0$, there exists $1/4 > \delta > 0$ such that

$$B(p, 2\delta) \subset C^*_0 \cap B(0,1) \quad \text{for any } p \in S.$$

This implies

$$\sigma_1(q) > \langle p, q\rangle + 2\,\delta|q| \quad \text{for any } q \in \mathbb{R}^N,\, p \in S.$$

From this we deduce

$$\ell_1(\xi) \geq \langle p, \xi(b) - \xi(a)\rangle + 2\,\delta\,\ell(\xi),$$

and we exploit the inequality $\ell_1(\xi) < \delta\ell(\xi)$ to get

$$\langle p, \xi(b) - \xi(a)\rangle < -\delta\ell(\xi) < 0,$$

which implies the assertion. □

Note that the preceding result can be equivalently expressed by saying that the Euclidean segment $\gamma(t) = (1-t)\,\xi(a) + t\,\xi(b)$, $t \in [0,1]$, satisfies $\dot\gamma(t) \in \overset{\circ}{D}_0$ for any t.

We can determine a compact cover $\{K_n\}$ of M, with $K_n \subset U_n$, and, in correspondence, a Riemannian metric $\tilde{g}$ such that any point at $\tilde{g}$–Riemannian distance less than or equal 1 from a fixed point of K_n lies in U_n, for any n. The previous lemma and the local separation property for C and D imply:

Lemma 5.4. *For any n we can find $\delta_n > 0$ such that if ζ is a curve defined on an interval $[a,b]$, with $\zeta(a) \in K_n$, $\zeta([a,b] \subset U_n$, satisfying*

$$\ell_C^{\tilde{g}}(\zeta) < \delta_n \ell^{\tilde{g}}(\zeta),$$

then we can find a curve γ, with $\gamma(t) \in \overset{\circ}{D}_{\gamma(t)}$ for a.e. t, joining $\xi(a)$ to $\xi(b)$.

We can construct, by using a smooth partition of unity subordinated to the cover $\{U_n\}$, a smooth function $\alpha : M \to]0, +\infty[$ with $\alpha > 1/\delta_n$ on U_n, for each $n \in \mathbb{N}$. We define a new Riemannian metric g on M by setting $g = \alpha^2\,\tilde{g}$. Therefore

$$|p|_x^g = 1/\alpha(x)|p|_x^{\tilde{g}} \quad \text{for any } x \in M,\, p \in T_M^*(x),$$

where $|\cdot|_x^g$ (resp. $|\cdot|_x^{\tilde{g}}$) indicates the dual g–Riemannian (resp. $\tilde{g}$–Riemannian) norm on $T_M^*(x)$. This implies

$$\sigma_C^g(x,q) = \alpha(x)\,\sigma_C^{\tilde{g}}(x,q) \quad \text{for any } x \in M,\, q \in T_M(x),$$

and consequently

$$\ell_C^g(\zeta) > 1/\delta_n\,\ell_C^{\tilde{g}}(\zeta) \quad \text{for any } n \text{ and any curve } \zeta \text{ contained in } U_n. \qquad (8)$$

Sketch of the proof of Proposition 5.2. We choose the previously introduced Riemannian metrics $\tilde{g}$ and g. The idea is to find a sequence of times $t_0 = a, t_1, \cdots, t_{j-1}, t_j = b$ such that the Lemma 5.4 can be applied to $\xi\big|_{[t_i,t_i+1]}$, $i = 1, \cdots, j$, for some K_n depending on i.

We define t_1 as the first time such that $\xi(t_1)$ is at $\tilde{g}$–Riemannian distance 1 from $\xi(a)$. We can assume without loss of generality that $\gamma(a) \in K_1$, so that $\gamma([a, t_1]) \subset U_1$.

If the $\tilde{g}$–Riemannian distance between $\xi(a)$ and $\xi(t_1)$ is less than 1 then $t_1 = b$; by (8) and the assumption we get

$$\frac{1}{\delta_1}\,\ell_C^{\tilde{g}}(\xi) < \ell_C^{g}(\xi) \leq \ell^{\bar{g}}(\xi).$$

If instead $\xi(a)$ and $\xi(t_1))$ are at $\tilde{g}$–Riemannian distance 1 then by (8), the assumption and since the intrinsic length is nonnegative we have

$$\frac{1}{\delta_1}\,\ell_C^{\tilde{g}}(\xi\big|_{[a,t_1]}) < \ell_C^{g}(\xi\big|_{[a,t_1]}) \leq \ell_C^{g}(\xi) \leq 1/2 \leq \ell^{\bar{g}}(\gamma|[a, t_1]).$$

In both cases there is a curve γ_1, with $\dot{\gamma}_1(t) \in \overset{\circ}{D}_{\gamma_1(t)}$, joining $\xi(a)$ to $\xi(t_1)$, in view of Lemma 5.4. This completes the proof in the case where $t_1 = b$.

If $t_1 < b$, we select as above a time t_2, and construct a curve γ_2, with $\dot{\gamma}_2(t) \in \overset{\circ}{D}_{\gamma_2(t)}$, joining $\xi(t_1)$ to $\xi(t_2)$. We then get, by juxtaposition, a curve of the same type connecting $\xi(a)$ to $\xi(t_2)$. Through iteration of this procedure we obtain the assertion. Some additional precautions must actually be taken at the last step, but we skip this part of the proof.

□

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