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Asymptotic solutions of Hamilton-Jacobi equations with non-periodic perturbations*

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概要

We study the long time behavior of viscosity solutions to some Cauchy problem for Hamilton-Jacobi equations. We deal with Hamiltonians and initial data that consist of the principal part which is periodic and a non-periodic perturbation term. We also discuss a generalization of the results.

1 Introduction and Known results.

We are concerned with the large time behavior of continuous viscosity solutions to Hamilton-Jacobi equations of the form

$$\begin{cases} u_t + H(x, Du) = 0 & \text{in } \mathbb{R}^n \times (0, +\infty), \\ u(\cdot, 0) = u_0(\cdot) \in \mathrm{UC}(\mathbb{R}^n) & \text{on } \mathbb{R}^n, \end{cases} \tag{1}$$

where the Hamiltonian $H = H(x,p)$ is always assumed to satisfy the following:
(H1) $H \in \mathrm{BUC}(\mathbb{R}^n \times B(0,R))$ for all $R > 0$, where $B(0,R) := \{x \in \mathbb{R}^n; |x| \leq R\}$.
(H2) H is coercive, i.e., $\lim_{r\to+\infty} \inf\{H(x,p); x \in \mathbb{R}^n, |p| \geq r\} = +\infty$.
(H3) $H(x,p)$ is strictly convex in p for every $x \in \mathbb{R}^n$.
Our objective is to show that the unique continuous viscosity solution $u(x,t)$ of (1) has the asymptotic behavior of the form

$$u(x,t) + ct - \phi(x) \longrightarrow 0 \qquad \text{uniformly on compact subsets of } \mathbb{R}^n, \tag{2}$$

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where (c, ϕ) is the pair of some real number and continuous function on $\mathbb{R}^n$. Remark here that if the convergence (2) holds, then (c, ϕ) should satisfy the following time-independent Hamilton-Jacobi equation (additive eigenvalue problem):

$$H(x, D\phi) - c = 0 \qquad \text{in } \mathbb{R}^n. \tag{3}$$

In particular, the function $\phi(x) - ct$ is a viscosity solution of $\phi_t + H(x, D\phi) = 0$ in $\mathbb{R}^n \times (0, +\infty)$, and it characterizes the large time asymptotic behavior of $u(x,t)$. We shall call such function the *asymptotic solution* of the Cauchy problem (1). Unfortunately, as far as asymptotic problems in the whole Euclidean space are concerned, the above three conditions (H1)-(H3) are insufficient to guarantee such solutions. Our aim is, therefore, to find some reasonable sufficient conditions on H and u_0 for the existence of asymptotic solutions of (1).

This problem can be restated as follows. Let $M = \mathbb{T}^n$ or $\mathbb{R}^n$ and define the Lax-Oleinik semigroup $(T_t)_{t\geq 0}$ acting on $\mathrm{UC}(M)$ by

$$(T_t u_0)(x) := \inf \left\{ \int_{-t}^{0} L(\gamma(s), \dot\gamma(s))\, ds + u_0(\gamma(0)) \;\middle|\; \gamma \in \mathrm{AC}([-t,0], \mathbb{R}^n), \quad \gamma(0) = x \right\}, \tag{4}$$

where $L(x,\xi) := \sup_{p \in \mathbb{R}^n}(\xi \cdot p - H(x,p))$ and $\mathrm{AC}([-t,0],\mathbb{R}^n)$ stands for the totality of absolutely continuous functions on $[-t, 0]$ with values in $\mathbb{R}^n$. We would like to know if $T_t u_0 + ct$ has the limit in the topology of $C(M)$ as $t \to \infty$ for some $c \in \mathbb{R}$ and $u_0 \in \mathrm{UC}(M)$.

The first attempt to attack such problem in the case where $M = \mathbb{T}^n$ (or M is a smooth compact manifold) was made by Fathi [5, 6]. He proves (2) under some additional assumptions on H. Recently, basing on the so-called Aubry-Mather theory, Davini-Siconolfi [4] improve his results and show the convergence result without assuming any condition except for (H1)-(H3).

Similar results are obtained by Namah-Roquejoffre [12] and Barles-Souganidis [3]. Their proof is based on the theory of partial differential equations and viscosity solutions. It is worth noting that the latter admits some class of Hamiltonians that are not convex.

Concerning asymptotic problems in non-compact regions, Fujita-Ishii-Loreti [7] and Ishii [10] treat the case $M = \mathbb{R}^n$. The main assumption of [10] in addition to (H1)-(H3) is

(H4) $\exists \phi_i \in C^{0+1}(\mathbb{R}^n)$, $\exists \sigma_i \in C(\mathbb{R}^n)$ with $i = 0, 1$ such that for $i = 0, 1$,

$$H(x, D\phi_i(x)) \leq -\sigma_i(x) \qquad \text{a.e. } x \in \mathbb{R}^n,$$

$$\lim_{|x|\to\infty} \sigma_i(x) = \infty, \qquad \lim_{|x|\to\infty} (\phi_0 - \phi_1)(x) = \infty,$$

and the class of initial data is taken as

$$\Phi_0 := \{v \in C(\mathbb{R}^n)\,;\, \inf_{\mathbb{R}^n}(v-\phi_0) > -\infty\}.$$

The proof is based on some dynamical approach associated with the variational formula (4) as well as some techniques on the theory of viscosity solution. An important feature by virtue of assumption (A4) is that any extremal curve $\gamma(\cdot)$ in the right-hand side of (4) stays in a compact subset of $\mathbb{R}^n$ for all $t > 0$. Roughly speaking, this fact corresponds to the compactness of the Aubry set, a uniqueness set for (3).

2 Assumption and the Main theorem.

In this note, we try to find another type of conditions on the Hamiltonian so that there exist asymptotic solutions of (1) for some class of initial data. Notice that this note is based on the paper [9], and a part of the results presented in this note has been announced in [8]. We are especially interested in the case where extreme curves may diverge (i.e. $|\gamma(t)| \longrightarrow \infty$) as t goes to the infinity. Similar situations are also investigated by Barles-Roquejoffre [2].

Now, we state our standing assumption.

(A1) $H(x,p) = h(x,p) - f(x)$ for some $h \in C(\mathbb{R}^n \times \mathbb{R}^n)$ satisfying (H1)-(H3) and $f \in C(\mathbb{R}^n)$.
(A2) $h(\cdot,p)$ is $\mathbb{Z}^n$-periodic for all $p \in \mathbb{R}^n$.
(A3) $f \geq 0$ and $\mathrm{supp}(f)$ is compact.

The main theorem of this note is the following:

Theorem 2.1 (c.f. Theorem 2.4 of [8]). *Let H satisfy (A1)-(A3). Suppose moreover that additive eigenvalue problem (3) has a solution in the class* $\mathrm{BUC}(\mathbb{R}^n)$ *for some c. Then, for any initial function u_0 belonging to*

$$\Phi_0 := \big\{u_0 \in \mathrm{BUC}(\mathbb{R}^n)\,;\, \exists\, \hat{u}_0 \in C(\mathbb{R}^n) : \mathbb{Z}^n\text{-periodic such that}$$
$$\hat{u}_0 \leq u_0 \text{ in } \mathbb{R}^n \quad \text{and} \quad \lim_{|x|\to\infty}(u_0 - \hat{u}_0)(x) = 0\big\},$$

there exists an asymptotic solution $\phi(x) - c\,t$ of (1).

We give here one of the simplest but most typical examples of Hamiltonian satisfying (A1)-(A3).

Example 1. Let $n = 1$, and define $h \in C(\mathbb{R} \times \mathbb{R})$ by

$$h(x,p) := |p-1|^2 - 1 - V(x),$$

where $V \in C(\mathbb{R})$ is a non-negative and $\mathbb{Z}$-periodic function such that $\min_{\mathbb{R}} V = 0$ and $\int_0^1 \sqrt{V(x)}\,dx < 1$. Remark that the equation $h(x, D\phi) = 0$ in $\mathbb{R}$ has bounded solutions (c.f. [11]).

Now, let $f \in C(\mathbb{R})$ be such that $f \geq 0$ and $\operatorname{supp} f \subset B(0,1)$, and define H by

$$H(x,p) := h(x,p) - f(x).$$

Clearly, H satisfies (A1)-(A3). We shall prove that the equation

$$H(x, D\phi) = 0 \qquad \text{in } \mathbb{R} \tag{5}$$

has a solution in the class $\mathrm{BUC}(\mathbb{R}^n)$. Observe first that the Lagrangian L associated with H can be calculated as

$$L(x,\xi) = \frac{1}{4}|\xi + 2|^2 + f(x) \geq 0.$$

For any $x \in \mathbb{R}$, we define $\gamma_x \in \mathrm{AC}((-\infty, 0])$ by $\gamma_x(s) := x - 2s$. Then, for every $t > 0$,

$$\int_{-t}^{0} L(\gamma_x(s), \dot\gamma_x(s))\,ds = \int_{-t}^{0} f(\gamma_x(s))\,ds \leq \max f.$$

If we set

$$d(x,y) := \inf\left\{ \int_{-t}^{0} L(\gamma(s), \dot\gamma(s))\,ds \mid t > 0,\ \gamma \in \mathrm{AC}([-t,0]),\ \gamma(-t) = y,\ \gamma(0) = x \right\},$$

then, $d(\cdot, y)$ is a viscosity solution of (5) in $\mathbb{R} \setminus \{y\}$. Moreover, for any $a > 0$, we see

$$0 \leq d(x, x+a) \leq \int_{-\frac{a}{2}}^{0} L(\gamma_x(s), \dot\gamma_x(s))\,ds \leq \max f.$$

Thus, by the Ascoli-Arzela theorem, we conclude that there exists $\phi \in \mathrm{BUC}(\mathbb{R}^n)$ such that $d(x, x + j_k) \longrightarrow \phi$ in $C(\mathbb{R})$ for some diverging sequence $\{j_k\}_{k\in\mathbb{N}} \subset \mathbb{N}$ as $k \to \infty$. In view of stability, ϕ is indeed a viscosity solution of (5). Hence, Theorem 2.1 is valid with $c = 0$.

3 Proof of Theorem 2.1.

This section is devoted to the proof of Theorem 2.1. Note first that in order to prove Theorem 2.1, we need only to study the case where $c = 0$ by replacing, if necessary,

$H - c$ and $u(x,t) + ct$ with H and $u(x,t)$, respectively. Therefore, from now on, we always assume that $c = 0$.

Let us denote by $\mathcal{S}_H^-$ (resp. $\mathcal{S}_H^+$) the totality of viscosity subsolutions (resp. supersolutions) of

$$H(x, D\phi) = 0 \qquad \text{in } \mathbb{R}^n. \tag{6}$$

We set $\mathcal{S}_H := \mathcal{S}_H^- \cap \mathcal{S}_H^+$. It is known that, under (H1) and (H2), we have $\mathcal{S}_H^- \subset \mathrm{Lip}(\mathbb{R}^n)$.

Let $\phi \in \mathcal{S}_H$. Then, for any $(x,t) \in \mathbb{R}^n \times [0,\infty)$, we have

$$\phi(x) = \inf\Big\{ \int_{-t}^{0} L(\gamma(s), \dot\gamma(s))\, ds + \phi(\gamma(-t)) \;\Big|\; \gamma \in \mathrm{AC}([-t,0]),\ \gamma(0) = x \Big\}. \tag{7}$$

For a given $\rho > 0$, we denote by $\mathcal{E}_\rho((-\infty,0]; x; \phi)$ the set of curves $\gamma \in \mathrm{AC}((-\infty,0])$ satisfying $\gamma(0) = x$ and

$$\phi(x) > \int_{-t}^{0} L(\gamma(s), \dot\gamma(s))\, ds + \phi(\gamma(-t)) - \rho \qquad \text{for all } t > 0. \tag{8}$$

It is not difficult to check that $\mathcal{E}_\rho((-\infty,0]; x; \phi) \neq \emptyset$.

Lemma 3.1. *For any $\phi \in \mathcal{S}_H$, $x \in \mathbb{R}^n$, $\rho > 0$ and $\gamma \in \mathcal{E}_\rho((-\infty,0]; x; \phi)$, there exists $\lambda > 1$ and a modulus ω_1 such that for every τ and $t > 0$ satisfying $t \geq \lambda\tau$,*

$$u(x,t) - \phi(x) \leq u(\gamma(-t), \tau) - \phi(\gamma(-t)) + \rho + \frac{t\tau}{t-\tau}\omega_1\Big(\frac{\tau}{t-\tau}\Big). \tag{9}$$

Proof. This lemma has essentially been proved in [4] and [10]. So, we omit to reproduce the proof (see also [9]). □

Proof of Theorem 2.1. Let u be the unique solution of Cauchy problem (1) satisfying $u(\cdot,0) = u_0 \in \Phi_0$ (see Appendix in [9] for the solvability of Cauchy problem (1)). Let ϕ be any bounded solution of (6). Since u_0 is bounded, we can take $A > 0$ so that $\phi(x) - A \leq u_0(x) \leq \phi(x) + A$ for all $x \in \mathbb{R}^n$. Remark also that $\phi + A$ and $\phi - A$ are solutions of (1) with initial data $\phi + A$ and $\phi - A$, respectively. Then, the standard comparison theorem for (1) infers that $\phi(x) - A \leq u(x,t) \leq \phi(x) + A$ for all $(x,t) \in \mathbb{R}^n \times [0,+\infty)$. In particular, u is bounded on $\mathbb{R}^n \times [0,+\infty)$.

We next define $u^+, u^- \in \mathrm{BUC}(\mathbb{R}^n)$ by

$$u^+(x) := \limsup_{t\to+\infty} u(x,t), \qquad u^-(x) := \liminf_{t\to+\infty} u(x,t).$$

Note that from the general theory of viscosity solution, u^+ and u^- are sub- and supersolutions of (6), respectively. Moreover, the convexity of $H(x,\cdot)$ implies that u^- is a subsolution of (6) (see [1]). In particular, u^- is a bounded solution of (6).

We now show that $u^+ \leq u^-$ in $\mathbb{R}^n$. Fix any $y \in \mathbb{R}^n$ and choose a diverging sequence $\{t_j\}_{j\in\mathbb{N}} \subset (0, \infty)$ so that $u^+(y) = \lim_{j\to\infty} u(y, t_j)$. Take any $\rho > 0$, $\gamma \in \mathcal{E}_\rho((-\infty, 0]; y; u^-)$, and set $y_j = \gamma(-t_j)$ for $j \in \mathbb{N}$.

Case 1: $|y_j| \longrightarrow \infty$ as $j \to \infty$.
In this case, since $h(x,p)$ is $\mathbb{Z}^n$-periodic in x, we may assume by taking a subsequence of $\{y_j\}$ if necessary that there exists $\{\theta_j\}_{j\in\mathbb{N}} \subset [0,1)^n$ such that $y_j \equiv \theta_j$ by mod $\mathbb{Z}^d$ and $\theta_j \longrightarrow \theta$ for some $\theta \in [0,1]^n$ as $j \to \infty$. Setting $\xi_j := \theta - \theta_j$ and using (A3), we see

$$
\begin{aligned}
H(\cdot + y_j, \cdot) \longrightarrow h(\cdot + \theta, \cdot) \quad & \text{in } C(\mathbb{R}^n \times \mathbb{R}^n) \quad \text{as } j \to \infty, \\
H(x + y_j + \xi_j, p) \leq h(x + \theta, p) \quad & \text{for all } (x, p, j) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{N}.
\end{aligned}
$$

Similarly, by the definition of Φ_0, there exists a $\mathbb{Z}^n$-periodic $\hat{u}_0 \in \mathrm{BUC}(\mathbb{R}^n)$ such that

$$
\begin{aligned}
u_0(\cdot + y_j) \longrightarrow \hat{u}_0(\cdot + \theta) \quad & \text{in } C(\mathbb{R}^n) \quad \text{as } j \to \infty, \\
u_0(x + y_j + \xi_j) \geq \hat{u}_0(x + \theta) \quad & \text{for all } (x, j) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{N}.
\end{aligned}
$$

Case 2: $\sup_j |y_j| < \infty$.
In this case, there exists $z \in \mathbb{R}^n$ such that $y_j \longrightarrow z$ as $j \to \infty$. Thus, by setting $\xi_j := z - y_j$, we have

$$
\begin{aligned}
H(\cdot + y_j, \cdot) \longrightarrow H(\cdot + z, \cdot) \quad & \text{in } C(\mathbb{R}^n \times \mathbb{R}^n) \quad \text{as } j \to \infty, \\
H(x + y_j + \xi_j, p) \leq H(x + z, p) \quad & \text{for all } (x, p, j) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{N},
\end{aligned}
$$

and

$$
\begin{aligned}
u_0(\cdot + y_j) \longrightarrow u_0(\cdot + z) \quad & \text{in } C(\mathbb{R}^n) \quad \text{as } j \to \infty, \\
u_0(x + y_j + \xi_j) \geq u_0(x + z) \quad & \text{for all } (x, j) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{N}.
\end{aligned}
$$

Summarizing these two cases, we may assume that there exist functions $G \in C(\mathbb{R}^n \times \mathbb{R}^n)$, $v_0 \in \mathrm{BUC}(\mathbb{R}^n)$ and a sequence $\{\xi_j\}_{j\in\mathbb{N}} \subset \mathbb{R}^n$ converging to zero such that

$$
\begin{aligned}
H(\cdot + y_j, \cdot) \longrightarrow G \quad & \text{in } C(\mathbb{R}^n \times \mathbb{R}^n) \quad \text{as } j \to \infty, \\
H(x + y_j + \xi_j, p) \leq G(x, p) \quad & \text{for all } (x, p, j) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{N},
\end{aligned}
$$

and

$$
\begin{aligned}
u_0(\cdot + y_j + \xi_j) \longrightarrow v_0 \quad & \text{in } C(\mathbb{R}^n) \quad \text{as } j \to \infty, \\
u_0(x + y_j + \xi_j) \geq v_0(x) \quad & \text{for all } (x, j) \in \mathbb{R}^n \times \mathbb{N}.
\end{aligned}
$$

We now consider the Cauchy problem

$$\begin{cases} v_t + G(x, Dv) = 0 & \text{in } \mathbb{R}^n \times (0, +\infty), \\ v(\cdot, 0) = v_0(\cdot) & \text{on } \mathbb{R}^n, \end{cases} \tag{10}$$

and let $v(x,t)$ be the unique viscosity solution of (10). We denote by $\mathcal{S}_G^-$, $\mathcal{S}_G^+$ the set of all sub- and supersolutions of

$$G(x, D\phi) = 0 \qquad \text{in } \mathbb{R}^n,$$

respectively. Set $\mathcal{S}_G := \mathcal{S}_G^- \cap \mathcal{S}_G^+$. Then, we can check that $\emptyset \neq \mathcal{S}_G \subset \mathrm{BUC}(\mathbb{R}^n)$. In particular, the function

$$v^-(x) := \liminf_{t\to+\infty} v(x,t)$$

is well-defined and moreover $v^- \in \mathcal{S}_G$.

Now we apply Proposition 3.1 by taking $\phi := u^-$ to get

$$u(y, t_j) - u^-(y) \le u(y_j, \tau) - u^-(y_j) + \rho + \frac{t_j \tau}{t_j - \tau}\omega_1\Big(\frac{\tau}{t_j - \tau}\Big) \tag{11}$$

for every $\tau > 0$ and sufficiently large $j \in \mathbb{N}$. On the other hand, by comparison and stability, we see

$$u(x + y_j + \xi_j, t) \longrightarrow v(x,t) \quad \text{in } C(\mathbb{R}^n \times [0,\infty)) \quad \text{as } j \to \infty,$$
$$u(x + y_j + \xi_j, t) \ge v(x,t) \quad \text{for all } (x,t,j) \in \mathbb{R}^n \times [0,\infty) \times \mathbb{N}.$$

In particular, the latter implies $u^-(y_j) \ge v^-(-\xi_j)$ for all $j \in \mathbb{N}$. Thus, in view of (11), we have

$$u(y, t_j) - u^-(y) \le u(y_j, \tau) - v^-(-\xi_j) + \rho + \frac{t_j \tau}{t_j - \tau}\omega_1\Big(\frac{\tau}{t_j - \tau}\Big). \tag{12}$$

Sending $j \to \infty$ in (12), we have

$$u^+(y) - u^-(y) \le v(0,\tau) - v^-(0) + \rho.$$

Letting $\tau = \tau_j \to \infty$ along a sequence $\{\tau_j\}$ such that $\lim_{j\to\infty} v(0,\tau_j) = \liminf_{t\to\infty} v(0,t)$, we finally obtain $u^+(y) \le u^-(y) + \rho$. Since $\rho > 0$ and $y \in \mathbb{R}^n$ are arbitrary, we conclude that $u^+ \le u^-$ in $\mathbb{R}^n$, and the proof of Theorem 2.1 has been completed. □

4 Final remarks.

Theorem 1 can be generalized considerably. We first introduce the notion of semi-periodicity and obliquely semi-almost periodicity (see [9] for details).

Definition 1. A function $\phi \in C(\mathbb{R}^n)$ is called lower (resp. upper) semi-periodic if for any sequence $\{y_j\}_{j\in\mathbb{N}} \subset \mathbb{R}^n$, there exist a subsequence $\{z_j\}_{j\in\mathbb{N}}$ of $\{y_j\}$, a sequence $\{\xi_j\}_{j\in\mathbb{N}} \subset \mathbb{R}^n$ converging to zero, and a function $\psi \in C(\mathbb{R}^n)$ such that $\phi(\,\cdot\, + z_j) \longrightarrow \psi(\cdot)$ in $C(\mathbb{R}^n)$ as $j \to \infty$ and $\phi(x + z_j + \xi_j) \geq \psi(x)$ (resp. $\phi(x + z_j + \xi_j) \leq \psi(x)$) for all $(x, j) \in \mathbb{R}^n \times \mathbb{N}$.

Definition 2. A function $\phi \in C(\mathbb{R}^n)$ is called obliquely lower (resp. upper) semi-almost periodic if for any sequence $\{y_j\}_{j\in\mathbb{N}} \subset \mathbb{R}^n$ and any $\varepsilon > 0$, there exist a subsequence $\{z_j\}_{j\in\mathbb{N}}$ of $\{y_j\}$ and a function $\psi \in C(\mathbb{R}^n)$ such that $\phi(\,\cdot\, + z_j) - \phi(z_j) \longrightarrow \psi(\cdot)$ in $C(\mathbb{R}^n)$ as $j \to \infty$ and $\phi(x + z_j) - \phi(z_j) + \varepsilon > \psi(x)$ (resp. $\phi(x + z_j) - \phi(z_j) - \varepsilon < \psi(x)$) for all $(x, j) \in \mathbb{R}^n \times \mathbb{N}$.

Theorem 4.1 (Theorem 2.2 of [9]). *Let H be a Hamiltonian satisfying (H1)-(H3) and*

(H5) $H(\cdot, p)$ *is upper semi-periodic for all* $p \in \mathbb{R}^n$*,*

(H6) There exist a constant $c \in \mathbb{R}$ and functions $\phi_0 \in \mathcal{S}^-_{H-c}$ and $\psi_0 \in \mathcal{S}^+_{H-c}$ such that $\phi_0 \leq u_0 \leq \phi_0 + C_0$ for some $C_0 > 0$ and $u_0^- \leq \psi_0$, where u_0^- is defined by

$$u_0^-(x) := \sup\{\, \phi(x) \mid \phi \in \mathcal{S}^-_{H-c}, \quad \phi \leq u_0 \quad \text{in } \mathbb{R}^n\}.$$

Then, for any initial datum u_0 belonging to

$$\Phi_0 := \{v \in \mathrm{UC}(\mathbb{R}^n)\,;\, v \text{ is obliquely lower semi-almost periodic}\},$$

there exists an asymptotic solution of (1).

Remark. It seems that almost periodicity for H might be sufficient to guarantee the existence of asymptotic solutions. In fact, there are a few examples that answer this question affirmatively. However, a complete research will be left in future investigation.

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