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Singular perturbation problem for nonlinear-diffusive logistic equations

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1 Introduction

In this paper we consider the following boundary value problem

$$\begin{cases} -\varepsilon\Delta_p u = f(x,u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega \end{cases} \tag{P}$$

for small $\varepsilon > 0$. Here Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 2)$ with $C^{2,\omega}$-boundary $\partial\Omega$ $(0 < \omega < 1)$, Δ_p is the p-Laplace operator $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ $(p > 1)$, and f is assumed to satisfy the following conditions:

(F1) $f(x,u) \in C(\overline{\Omega} \times [0,\infty))$;

(F2) There exist $\xi > 0$ and $\sigma > 0$ such that $f(x,0) = 0$, the map $u \mapsto f(x,u)$ is nondecreasing in $[0,\xi]$ for all $x \in \Omega$ and $\liminf_{u\to+0} f(x,u)/u^{p-1} > \sigma$ uniformly in Ω;

(F3) There exists a positive function $a(x) \in C(\overline{\Omega})$ such that

$$f(x,u)\begin{cases} >0 & \text{when } 0<u<a(x),\\ <0 & \text{when } u>a(x); \end{cases}$$

(F4) There exists a strictly increasing function $g(x) \in C([0,\infty))$ such that $g(0) = 0$ and the map $u \mapsto f(x,u) + g(u)$ is nondecreasing in $[0,\infty)$ for all $x \in \Omega$.

We shall describe the case where $f(x,u)$ is independent of x, e.g., $f(x,u) = f(u) = u^{p-1}|1-u|^{q-1}(1-u)$ $(p>1,\ q>0)$, which satisfies (F3) with $a(x) \equiv 1$. For $\varepsilon > 0$ small enough, the boundary value problem

$$\begin{cases} -\varepsilon\Delta_p u = f(u), & x \in \Omega,\\ u = 0, & x \in \partial\Omega \end{cases}$$

has the unique positive solution u_ε, which converges the value 1 uniformly in any compact subset of Ω as $\varepsilon \to 0$. Guedda and Véron [5] in 1-dimensional case and Kamin and Véron [7] in N-dimensional case investigated that when $q < p-1$ and ε is sufficiently small, the coincidence set of u_ε with the value 1, or the flat core of u_ε, defined by

$$\mathcal{O}_\varepsilon = \{x \in \Omega \,|\, u_\varepsilon(x) = 1\}$$

is not empty and that there exists a constant $C > 0$ such that

$$\{x \in \Omega \,|\, \mathrm{dist}(x, \partial\Omega) > C\varepsilon^{1/p}\} \subset \mathcal{O}_\varepsilon.$$

If $q \geq p-1$, then $\mathcal{O}_\varepsilon$ is empty for any ε because u_ε is strictly less than 1 by the strong maximum principle of Vázquez [9]. After their works, García-Melián and Sabina de Lis [4] gave the precise speed of expansion of $\mathcal{O}_\varepsilon$ as $\varepsilon \to 0$, namely, the estimate of width of the boundary layer of u_ε. In the results above, they all assume that $f(u)/u^{p-1}$ is decreasing in order to assure the uniqueness of positive solutions. Guo

[6] eliminated this assumption and showed that the positive solution is nevertheless unique for small ε (cf. Theorem 2 of Dancer [3] for $p = 2$).

This paper deals with the case where $f(x, u)$ depends on x, particularly, $a(x)$ is not constant. In the semilinear case $p = 2$, Angenent [1] described that for small $\varepsilon > 0$, the positive solution of (P) is unique and converges to $a(x)$ uniformly in any compact subset of Ω as $\varepsilon \to 0$. For the quasilinear case $p \neq 2$, however, there is no preceding study on singular perturbation problems for (P). Our purpose is to extend the results of Angenent [1] and of Kamin and Véron [7], respectively, to the x-dependent case: we give the proof that any positive solution of (P) converges to $a(x)$ uniformly in any compact subset of Ω as $\varepsilon \to 0$, and we show that for $\varepsilon > 0$ small enough, the solutions coincide with $a(x)$ on the domain where $a(x)$ is constant and $f(x, u)$ tends to zero as $u \to a(x)$ with the order less than $p - 1$.

To state theorems, we give the following notation, which will be in force through the paper:

$$A = \max\{a(x) \mid x \in \overline{\Omega}\},$$
$$\alpha = \min\{a(x) \mid x \in \overline{\Omega}\},$$
$$D(\Omega, R) = \{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) > R\}.$$

Theorem 1.1. *Suppose* (F1)–(F4). *All nontrivial nonnegative solutions are positive in Ω. Moreover, for sufficiently small $\varepsilon > 0$, there exists a positive solution $u \in C^{1,\tilde{\omega}}(\overline{\Omega})$ of* (P) *with some $\tilde{\omega} \in (0, 1)$.*

Theorem 1.2. *Suppose* (F1)–(F4). *For any $\delta \in (0, \alpha)$, there exist $K > 0$ and $\varepsilon_* > 0$ such that $D(\Omega, K\varepsilon_*^{1/p})$ is not empty and that if $\varepsilon \in (0, \varepsilon_*)$ then every positive solution u_ε of* (P) *satisfies*

$$|u_\varepsilon(x) - a(x)| < \delta \quad \textit{for all } x \in D(\Omega, K\varepsilon^{1/p}).$$

Theorem 1.3. *Suppose* (F1)–(F4) *and*

(F5) $a(x) \equiv a$ *for some* $a \in [\alpha, A]$ *in a nonempty subdomain* Ω_0 *of* Ω *and there exist* $q \in (0, p-1)$ *and* $\lambda > 0$ *such that*

$$\limsup_{u \to a} \frac{f(x,u) - f(x,a)}{|u-a|^{q-1}(u-a)} < -\lambda \quad \textit{uniformly in } \Omega_0. \tag{1.1}$$

Then, for sufficiently small $\eta > 0$, *there exists* $\varepsilon_0 \in (0, \varepsilon_*)$ *such that if* $\varepsilon \in (0, \varepsilon_0)$ *then every positive solution* u_ε *of* (P) *satisfies*

$$u_\varepsilon(x) = a = a(x) \quad \textit{for all } x \in D(\Omega_0, \eta).$$

Sections 2 and 3 are devoted to proofs of the theorems. In Section 4, we shall announce that when $p = 2$, the condition (F5) in Theorem 1.3 can be weaker. In Section 5, we give a few remark on the theorems.

2 Preliminaries

In this section, we shall define solutions, super- and subsolutions of (P), and show a weak comparison principle for the p-Laplace operator with monotone perturbation. We also acquaint the reader with an existence result given by Cañada-Drábek-Gámez [2] and the strong maximum principle given by Vázquez [9]. Finally, we prove a generalization of Serrin's sweeping principle to the p-Laplace operator.

Definition 2.1. A function $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ is called a *solution of* (P) when

$$\varepsilon \int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi \, dx = \int_\Omega f(x,u) \varphi \, dx$$

for any $\varphi \in W_0^{1,p}(\Omega)$.

For any function v, define the positive part v_+ of v by $v_+ = \max\{v, 0\}$. We say that a function $v \in W^{1,p}(\Omega)$ is less than or equal to $w \in W^{1,p}(\Omega)$ on $\partial\Omega$ if $(v-w)_+ \in W_0^{1,p}(\Omega)$, which is denoted by $v \leq w$ on $\partial\Omega$.

Definition 2.2. A function $\underline{u} \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ is called a *subsolution of* (P) when $u \leq 0$ on $\partial\Omega$ and $-\varepsilon\Delta_p \underline{u} \leq f(x, \underline{u})$ in Ω, i.e.,

$$\varepsilon \int_\Omega |\nabla \underline{u}|^{p-2} \nabla \underline{u} \cdot \nabla \varphi \, dx \leq \int_\Omega f(x, \underline{u}) \varphi \, dx$$

for any $\varphi \in W_0^{1,p}(\Omega)$ with $\varphi \geq 0$ a.e. in Ω. In the same way, a function $\overline{u} \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ is called a *supersolution of* (P) when $u \geq 0$ on $\partial\Omega$ and $-\varepsilon\Delta_p \overline{u} \geq f(x, \overline{u})$ in Ω, i.e.,

$$\varepsilon \int_\Omega |\nabla \overline{u}|^{p-2} \nabla \overline{u} \cdot \nabla \varphi \, dx \geq \int_\Omega f(x, \overline{u}) \varphi \, dx$$

for any $\varphi \in W_0^{1,p}(\Omega)$ with $\varphi \geq 0$ a.e. in Ω.

Lemma 2.1. *Let h be a strictly increasing continuous function, and assume that functions $\underline{u}$, $\overline{u} \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ satisfy*

$$\begin{cases} -\Delta_p \underline{u} + h(\underline{u}) \leq -\Delta_p \overline{u} + h(\overline{u}) \ \textit{in } \Omega, \\ \underline{u} \leq \overline{u} \ \textit{on } \partial\Omega. \end{cases}$$

Then $\underline{u} \leq \overline{u}$ a.e. in Ω.

Proof. We use an inequality for a, $b \in \mathbb{R}^N$: There exist positive numbers C_1 and C_2 such that

$$(|a|^{p-2}a - |b|^{p-2}b) \cdot (a - b) \geq \begin{cases} C_1 |a-b|^p \quad (p \geq 2), \\ C_2 \dfrac{|a-b|^2}{(|a|+|b|)^{2-p}} \quad (1 < p < 2). \end{cases}$$

Choosing $\varphi = (\underline{u} - \overline{u})_+ \in W_0^{1,p}(\Omega)$, we have

$$\begin{aligned} 0 &\geq \int_\Omega (|\nabla \underline{u}|^{p-2} \nabla \underline{u} - |\nabla \overline{u}|^{p-2} \nabla \overline{u}) \cdot \nabla (\underline{u} - \overline{u})_+ \, dx + \int_\Omega (h(\underline{u}) - h(\overline{u}))(\underline{u} - \overline{u})_+ \, dx \\ &\geq \int_\Omega (h(\underline{u}) - h(\overline{u}))(\underline{u} - \overline{u})_+ \, dx + \begin{cases} C_1 \displaystyle\int_\Omega |\nabla (\underline{u} - \overline{u})_+|^p \, dx \quad (p \geq 2), \\ C_2 \displaystyle\int_{\{|\nabla \underline{u}| + |\nabla \overline{u}| \neq 0\}} \dfrac{|\nabla (\underline{u} - \overline{u})_+|^2}{(|\nabla \underline{u}| + |\nabla \overline{u}|)^{2-p}} \, dx \quad (1 < p < 2) \end{cases} \\ &\geq \int_\Omega (h(\underline{u}) - h(\overline{u}))(\underline{u} - \overline{u})_+ \, dx. \end{aligned}$$

The last expression is nonnegative, and hence $(\underline{u} - \overline{u})_+ = 0$ a.e. in Ω. Thus $\underline{u} \leq \overline{u}$ a.e. in Ω. $\square$

Lemma 2.2 **([2])**. *Suppose* (F1) *and* (F4). *Let* $\underline{u}$, $\overline{u} \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ *be, respectively, a subsolution and a supersolution of* (P), *with* $\underline{u} \leq \overline{u}$ *a.e. in* Ω. *Then there exists a minimal (resp. a maximal) solution* u_* *(resp.* u^**) for* (P) *in the interval*

$$[\underline{u}, \overline{u}] = \{u \in L^\infty(\Omega) \,|\, \underline{u}(x) \leq u(x) \leq \overline{u}(x) \text{ a.e. in } \Omega\}.$$

In particular, every solution $u \in [\underline{u}, \overline{u}]$ *of* (P) *satisfies also* $u_*(x) \leq u(x) \leq u^*(x)$ *a.e. in* Ω.

Lemma 2.3 **([9])**. *Let* $u \in C^1(\Omega)$ *be such that* $\Delta_p u \in L^2_{\mathrm{loc}}(\Omega)$, $u \geq 0$ *a.e. in* Ω, $-\Delta_p u + \beta(u) \geq 0$ *a.e. in* Ω *with* $\beta : [0, \infty) \to \mathbb{R}$ *continuous, nondecreasing,* $\beta(0) = 0$ *and either* $\beta(s) = 0$ *for some* $s > 0$ *or* $\beta(s) > 0$ *for all* $s > 0$ *but* $\int_0^1 (s\beta(s))^{-1/p}\, ds = +\infty$. *Then if* u *does not vanish identically on* Ω, *then it is positive everywhere in* Ω. *Moreover, if* $u \in C^1(\Omega \cup \{x_0\})$ *for an* $x_0 \in \partial\Omega$ *that satisfies an interior sphere condition and* $u(x_0) = 0$, *then* $\frac{\partial u}{\partial \mathbf{n}}(x_0) < 0$, *where* $\mathbf{n}$ *is the outer normal unit vector to* $\partial\Omega$ *at* x_0.

Finally in this section, we generalize Serrin's sweeping principle for uniformly elliptic operators to the p-Laplace operator. When $f(x, u)$ is independent of x, a generalized principle has been already given by Guo [6].

Proposition 2.1. *Let* (F1), (F4), $I = [a, b]$ $(a < b)$ *and* $u \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$ *be a solution of* $-\Delta_p u = f(x, u)$. *Suppose that a family of functions* $\{v_t \in W^{1,p}(\Omega) \cap C(\overline{\Omega}) \,|\, t \in I\}$ *satisfies* $v_t < u$ *on* $\partial\Omega$ *and that there exists* $c > 0$ *such that* $-\Delta_p v_t \leq f(x, v_t) - c$ *for all* $t \in I$. *If the map* $t \mapsto v_t$ *is continuous with respect to the topology of* $C(\overline{\Omega})$ *and* $v_a \leq u$ *in* $\overline{\Omega}$, *then* $v_t < u$ *in* $\overline{\Omega}$ *for all* $t \in I$.

Proof. Set $E = \{t \in I \,|\, u \geq v_t \text{ in } \overline{\Omega}\}$. By the assumption of the proposition, E is nonempty and closed. It suffices to show that E is also open in I, which means that $E = I$.

Fix $t \in E$. Since $u > v_t$ on $\partial\Omega$, there exists a neighborhood Γ of $\partial\Omega$ such that $u > v_t$ on Γ. Let Ω_* be a subset of Ω with $\partial\Omega_* \subset \Gamma$. Then $u > v_t$ on $\partial\Omega_*$. There exists $\tau_* > 0$ such that if $0 < \tau < \tau_*$, then $g(u) - g(u-\tau) < c$, and we choose $\tau \in (0, \tau_*)$ such that $u - \tau > v_t$ on $\partial\Omega_*$. From (F4), we have

$$\begin{aligned}
-\Delta_p(u-\tau) + g(u-\tau) &= f(x,u) + g(u-\tau) \\
&> f(x,u) + g(u) - c \\
&\geq f(x,v_t) + g(v_t) - c \\
&\geq -\Delta_p v_t + g(v_t)
\end{aligned}$$

in Ω_*. It follows from Lemma 2.1 that $u > u - \tau \geq v_t$ in Ω_*. Since so is in Γ, we conclude that $u > v_t$ in $\overline{\Omega}$, and hence E is open. $\square$

Remark 2.1. Suppose that a family of functions $\{w_t \in W^{1,p}(\Omega) \cap C(\overline{\Omega}) \,|\, t \in I\}$ satisfies $w_t > u$ on $\partial\Omega$ and that there exists $c > 0$ such that $-\Delta_p w_t \geq f(x, w_t) + c$ for all $t \in I$. In the same way, we can prove that if the map $t \mapsto w_t$ is continuous with respect to the topology of $C(\overline{\Omega})$ and $w_a \geq u$ in $\overline{\Omega}$, then $w_t > u$ in $\overline{\Omega}$ for all $t \in I$.

3 Proofs

We devote the rest of this paper to the proofs of Theorems 1.1, 1.2 and 1.3. Along the way, we prepare Lemma 3.1, which is needed for proving Theorem 1.3.

Proof of Theorem 1.1. First we shall show the existence of nonnegative solutions of

(P). Since $\overline{u} = A$ is a supersolution, by Lemma 2.2, it suffices to show the existence of a subsolution $\underline{u}$ which is less than or equal to $\overline{u}$.

Take any $x_0 \in \Omega$. From (F2), there exist $r > 0$ and $\delta \in (0, A)$ such that $f(x,u) > \sigma u^{p-1}$ if $|x - x_0| < r$ and $0 < u < \delta$. Let λ_0 be the principal eigenvalue of $-\Delta_p$ with Dirichlet boundary condition on the unit ball $B(0,1)$ in $\mathbb{R}^N$ and ϕ_0 the principal eigenfunction corresponding to λ_0 such that $\max\{\phi_0(x) \mid x \in B(0,1)\} = 1$:

$$\begin{cases} -\Delta_p \phi_0 = \lambda_0 |\phi_0|^{p-2}\phi_0, & x \in B(0,1), \\ \phi_0 = 0, & x \in \partial B(0,1). \end{cases}$$

It is well-known that $\lambda_0 > 0$ and ϕ_0 is positive. Then we can show that the following function is a subsolution of (P) for $\varepsilon < \sigma/\lambda_0$:

$$\underline{u}(x) = \begin{cases} \gamma\phi_0\left(\frac{x-x_0}{r}\right) & \text{in } B, \\ 0 & \text{in } \Omega \setminus B, \end{cases}$$

where $\gamma \in (0, \delta)$ and $B = B(x_0, r)$ is the ball in $\mathbb{R}^N$ with center x_0 and radius r. Indeed, for any $\varphi \in W_0^{1,p}(\Omega_0)$ with $\varphi \geq 0$

$$\begin{aligned} \varepsilon \int_\Omega |\nabla \underline{u}|^{p-2}\nabla \underline{u} \cdot \nabla\varphi\, dx &= \varepsilon\gamma^{p-1} \int_B |\nabla\phi_0|^{p-2}\nabla\phi_0 \cdot \nabla\varphi\, dx \\ &= \varepsilon\gamma^{p-1} \int_{\partial B} |\nabla\phi_0|^{p-2}\frac{\partial\phi_0}{\partial\mathbf{n}}\varphi\, ds - \varepsilon\gamma^{p-1}\int_B \Delta_p\phi_0\varphi\, dx \\ &\leq \lambda_0\varepsilon\gamma^{p-1}\int_B \phi_0^{p-1}\varphi\, dx \\ &< \frac{\lambda_0\varepsilon}{\sigma}\int_B f(x, \gamma\phi_0)\varphi, dx \\ &< \int_\Omega f(x, \underline{u})\varphi\, dx, \end{aligned}$$

where $\mathbf{n}$ is the outer normal unit vector to ∂B at s. Since $\underline{u} \leq \delta < A = \overline{u}$, it follows from (F4) and Lemma 2.2 that there exists a nonnegative solution $u \in [\underline{u}, \overline{u}]$ of (P).

By virtue of Theorem 1 of Lieberman [8] combined with the use of boundedness of the solution, we have that $u \in C^{1,\tilde{\omega}}(\overline{\Omega})$ for some $\tilde{\omega} \in (0,1)$.

Next we shall show the positivity of nonnegative solutions of (P). From (F2), there exists $\xi > 0$ such that the map $u \mapsto f(x,u)$ is nondecreasing in $[0,\xi]$ for all $x \in \Omega$. Let

$$\beta(u) = \begin{cases} 0, & u \in [0,\xi], \\ g(u) - g(\xi), & u \in (\xi, \infty), \end{cases}$$

where g is an increasing continuous function in (F4). Then, by (F5), β is nondecreasing and $-\varepsilon\Delta_p u + \beta(u) = f(x,u) + \beta(u) \geq f(x,0) + \beta(0) = 0$ for any nonnegative solution u of (P). By Lemma 2.3 with $\beta(\xi) = 0$, we conclude that u is positive in Ω. $\square$

Remark 3.1. For the positivity, we assumed in (F2) that $f(x,u)$ is nondecreasing in $[0,\xi]$ for some $\xi > 0$. Or alternatively, we may assume in (F4) that g satisfies $\int_0^1 (sg(s))^{-1/p}\,ds = +\infty$, for Lemma 2.3.

Proof of Theorem 1.2. Let λ_0 and ϕ_0 be the same as those in the proof of Theorem 1.1. From (F2) and (F3), there exist $r > 0$ and $\underline{\sigma} \in (0,\sigma)$ such that for any $x_0 \in \Omega$, we have that $f(x,u) \geq \underline{\sigma} u^{p-1}$ for all $x \in B(x_0,r) \cap \Omega$ and all $u \in [0, a(x_0) - \delta)$. Let $\underline{K}$ be a constant satisfying $\underline{K}^p > \lambda_0/(c_p\underline{\sigma})$ with $c_p = \min\{2^{p-2}, 1\}$.

Let $\underline{\varepsilon_*} > 0$ be a number such that $D(\Omega, \underline{K}\underline{\varepsilon_*}^{1/p}) \neq \emptyset$, $\underline{K}\underline{\varepsilon_*}^{1/p} < r$ and that Problem (P) with $\varepsilon = \underline{\varepsilon_*}$ has a positive solution. Take any $\varepsilon < \underline{\varepsilon_*}$ and any $x_0 \in D(\Omega, \underline{K}\varepsilon^{1/p})$. Changing scaling as $\underline{\phi}(x) = \phi_0((x - x_0)/(\underline{K}\varepsilon^{1/p}))$, we have

$$\begin{cases} -\varepsilon\Delta_p\underline{\phi} = \dfrac{\lambda_0}{\underline{K}^p}\underline{\phi}^{p-1}, & x \in B(x_0, \underline{K}\varepsilon^{1/p}), \\ \underline{\phi} = 0, & x \in \partial B(x_0, \underline{K}\varepsilon^{1/p}). \end{cases}$$

Taking a constant $\underline{\eta} \in (0, a(x_0) - \delta)$ so that $\underline{\eta} < \min\{u_\varepsilon(x) \,|\, x \in B(x_0, \underline{K}\varepsilon^{1/p})\}$, we shall show that the family of functions $\{t\underline{\phi}(x) + \underline{\eta} \,|\, 0 \leq t \leq a(x_0) - \delta - \underline{\eta}\}$ satisfies the assumption for v_t of Proposition 2.1. Indeed, set $v_t = t\underline{\phi} + \underline{\eta}$. Then $v_t = \underline{\eta} < u_\varepsilon$ on $\partial B(x_0, \underline{K}\varepsilon^{1/p})$, and since $0 \le v_t \le a(x_0) - \delta$ in $B(x_0, \underline{K}\varepsilon^{1/p})$, we have

$$\begin{aligned}
-\varepsilon\Delta_p v_t - f(x, v_t) &\le -\varepsilon t^{p-1}\Delta_p\underline{\phi} - \underline{\sigma}(t\underline{\phi} + \underline{\eta})^{p-1} \\
&\le t^{p-1}\frac{\lambda_0}{\underline{K}^p}\underline{\phi}^{p-1} - c_p\underline{\sigma}(t^{p-1}\underline{\phi}^{p-1} + \underline{\eta}^{p-1}) \\
&= \left(\frac{\lambda_0}{\underline{K}^p} - c_p\underline{\sigma}\right) t^{p-1}\underline{\phi}^{p-1} - c_p\underline{\sigma}\underline{\eta}^{p-1} \\
&\le -c_p\underline{\sigma}\underline{\eta}^{p-1}
\end{aligned}$$

in $B(x_0, \underline{K}\varepsilon^{1/p})$ for all $t \in [0, a(x_0) - \delta - \underline{\eta}]$. Since $v_0 \le u_\varepsilon$ in $B(x_0, \underline{K}\varepsilon^{1/p})$, it follows from Proposition 2.1 that $u_\varepsilon(x) > (a(x_0) - \delta - \underline{\eta})\underline{\phi}(x) + \underline{\eta}$ in $B(x_0, \underline{K}\varepsilon^{1/p})$. Thus $u_\varepsilon(x_0) > a(x_0) - \delta$ for all $\varepsilon < \underline{\varepsilon_*}$ and all $x_0 \in D(\Omega, \underline{K}\varepsilon^{1/p})$.

Next we show the inverse inequality in a similar way. From (F3), there exist $r > 0$ and $\overline{\sigma} > 0$ such that for any $x_0 \in \Omega$, we have that $f(x, u) \le -\overline{\sigma}(3A - u)^{p-1}$ for all $x \in B(x_0, r) \cap \Omega$ and all $u \in (a(x_0) + \delta, 3A]$. Let $\overline{K}$ be a constant satisfying $\overline{K}^p > \lambda_0/(c_p\overline{\sigma})$.

Let $\overline{\varepsilon_*} > 0$ be a number such that $D(\Omega, \overline{K}\overline{\varepsilon_*}) \ne \emptyset$, $\overline{K}\overline{\varepsilon_*}^{1/p} < r$ and that (P) with $\varepsilon = \overline{\varepsilon_*}$ has a positive solution. Take any $\varepsilon < \overline{\varepsilon_*}$ and any $x_0 \in D(\Omega, \overline{K}\varepsilon^{1/p})$. Changing scaling as $\overline{\phi}(x) = \phi_0((x - x_0)/(\overline{K}\varepsilon^{1/p}))$, we have

$$\begin{cases}
-\varepsilon\Delta_p\overline{\phi} = \dfrac{\lambda_0}{\overline{K}^p}\overline{\phi}^{p-1}, & x \in B(x_0, \overline{K}\varepsilon^{1/p}), \\
\overline{\phi} = 0, & x \in \partial B(x_0, \overline{K}\varepsilon^{1/p}).
\end{cases}$$

Taking a constant $\overline{\eta} \in (0, 2A - a(x_0) - \delta)$, we shall show that the family of functions $\{3A - t\overline{\phi}(x) - \overline{\eta} \mid 0 \le t \le 3A - a(x_0) - \delta - \overline{\eta}\}$ satisfies the assumption for w_t of Remark 2.1. Note that A is a supersolution of (P) and that Lemma 2.1 with (F3) and (F4) gives $u_\varepsilon \le A$ in Ω. Set $w_t = 3A - t\overline{\phi}(x) - \overline{\eta}$. Then $w_t = 3A - \overline{\eta} > A + a(x_0) + \delta >$

$A \geq u_\varepsilon$ on $\partial B(x_0, \overline{K}\varepsilon^{1/p})$, and since $a(x_0) + \delta \leq w_t \leq 3A$ in $B(x_0, \overline{K}\varepsilon^{1/p})$, we have

$$\begin{aligned}
-\varepsilon\Delta_p w_t - f(x, w_t) &\geq \varepsilon t^{p-1}\Delta_p\overline{\phi} + \overline{\sigma}(t\overline{\phi} + \overline{\eta})^{p-1} \\
&\geq -t^{p-1}\frac{\lambda_0}{\overline{K}^p}\overline{\phi}^{p-1} + c_p\overline{\sigma}(t^{p-1}\overline{\phi}^{p-1} + \overline{\eta}^{p-1}) \\
&= \left(c_p\overline{\sigma} - \frac{\lambda_0}{\overline{K}^p}\right) t^{p-1}\overline{\phi}^{p-1} + c_p\overline{\sigma}\overline{\eta}^{p-1} \\
&\geq c_p\overline{\sigma}\overline{\eta}^{p-1}
\end{aligned}$$

in $B(x_0, \overline{K}\varepsilon^{1/p})$ for all $t \in [0, 3A - a(x_0) - \delta - \overline{\eta}]$. Since $w_0 > A \geq u_\varepsilon$ in $B(x_0, \overline{K}\varepsilon^{1/p})$, it follows from Remark 2.1 that $u_\varepsilon(x) < 3A - (3A - a(x_0) - \delta - \overline{\eta})\overline{\phi}(x) - \overline{\eta}$ in $B(x_0, \overline{K}\varepsilon^{1/p})$. Thus $u_\varepsilon(x_0) < a(x_0) + \delta$ for all $\varepsilon < \overline{\varepsilon_*}$ and all $x_0 \in D(\Omega, \overline{K}\varepsilon^{1/p})$. Setting $\varepsilon_* = \min\{\underline{\varepsilon_*}, \overline{\varepsilon_*}\}$ and $K = \min\{\underline{K}, \overline{K}\}$, we conclude that $|u_\varepsilon(x_0) - a(x_0)| < \delta$ when $\varepsilon < \varepsilon_*$ and $x_0 \in D(\Omega, K\varepsilon^{1/p})$. $\square$

To show Theorem 1.3, we prepare

Lemma 3.1. *Let λ, q, R and δ be positive constants and h the unique solution of*

$$\begin{cases} -\varepsilon\Delta_p h + \lambda h^q = 0, & x \in B(0, R), \\ h = \delta, & x \in \partial B(0, R). \end{cases} \tag{3.2}$$

If $q < p - 1$ and

$$0 < \varepsilon < \frac{\lambda\theta^p R^p}{(pq + N\theta)p^{p-1}\delta^\theta} \quad (\theta := p - 1 - q > 0), \tag{3.3}$$

then $h(0) = 0$.

Proof. Due to the uniqueness of the solution of (3.2), it is easy to see that the solution h must be radially symmetric. Writing $h = h(r)$ with $r = |x|$, we have

$$\begin{cases} -\dfrac{\varepsilon}{r^{N-1}}(r^{N-1}|h_r|^{p-2}h_r)_r + \lambda h^q = 0, & r \in (0, R), \\ h'(0) = 0, \quad h(R) = \delta. \end{cases} \tag{3.4}$$

It follows from direct computation that the following function satisfies the equation of (3.4) when $q < p - 1$:

$$\overline{h}(r) = Cr^{p/\theta},$$

where $\theta = p - 1 - q > 0$ and

$$C = \left(\frac{\lambda \theta^p}{\varepsilon (pq + N\theta) p^{p-1}} \right)^{1/\theta}.$$

Since (3.3) implies $\delta < \overline{h}(R) = CR^{p/\theta}$, Lemma 2.1 gives $0 \le h(r) \le \overline{h}(r)$ for $r \in [0, R]$. Since $\overline{h}(0) = 0$, we conclude that $h(0) = 0$. □

Proof of Theorem 1.3. By (F5), there exists $\delta_0 \in (0, \alpha)$ such that for any $v \in [0, \delta_0)$

$$f(x, a + v) \le -\lambda v^q \quad \text{for all } x \in \Omega_0, \tag{3.5}$$

$$f(x, a - v) \ge \lambda v^q \quad \text{for all } x \in \Omega_0. \tag{3.6}$$

Since the function $v = u_\varepsilon - a$ satisfies $-\varepsilon \Delta_p v = f(x, a + v)$ a.e. in Ω_0, the positive part $v_+ \in W^{1,p}(\Omega_0)$ of v satisfies

$$-\varepsilon \Delta_p v_+ \le f(x, a + v_+) \quad \text{in } \Omega_0. \tag{3.7}$$

Indeed, for any $\varphi \in W_0^{1,p}(\Omega_0)$ with $\varphi \ge 0$

$$\begin{aligned}
\varepsilon \int_{\Omega_0} |\nabla v_+|^{p-2} \nabla v_+ \cdot \nabla \varphi \, dx &= \varepsilon \int_{\{v>0\} \cap \Omega_0} |\nabla v|^{p-2} \nabla v \cdot \nabla \varphi \, dx \\
&= \varepsilon \int_{\partial(\{v>0\} \cap \Omega_0)} |\nabla v|^{p-2} \frac{\partial v}{\partial \mathbf{n}} \varphi \, ds - \varepsilon \int_{\{v>0\} \cap \Omega_0} \Delta_p v \varphi \, dx \\
&\le - \int_{\{v>0\} \cap \Omega_0} f(x, a + v) \varphi \, dx \\
&= - \int_{\Omega_0} f(x, a + v_+) \varphi \, dx,
\end{aligned}$$

where $\mathbf{n}$ is the outer normal unit vector to $\partial(\{v > 0\} \cap \Omega_0)$ at s. In a similar way, $v_- = (a - u_\varepsilon)_+$ satisfies

$$-\varepsilon \Delta_p v_- \le -f(x, a - v_-) \quad \text{in } \Omega_0. \tag{3.8}$$

Take so small $\eta > 0$ that $D(\Omega_0, \eta) \neq \emptyset$. Fix $\delta \in (0, \delta_0)$. By Theorem 1.2, there exists $\varepsilon(\delta) > 0$ such that for any $\varepsilon \in (0, \varepsilon(\delta))$, $\max\{v_\pm(x) \,|\, x \in D(\Omega_0, \eta/2)\} < \delta$. Applying (3.5) and (3.6) to $v = v_\pm \in [0, \delta)$, we have, respectively,

$$f(x, a + v_+) \leq -\lambda v_+^q \quad \text{for all } x \in D(\Omega_0, \eta/2), \tag{3.9}$$

$$f(x, a - v_-) \geq \lambda v_-^q \quad \text{for all } x \in D(\Omega_0, \eta/2). \tag{3.10}$$

Combining these inequalities (3.7)–(3.10), we obtain

$$-\varepsilon \Delta_p v_\pm + \lambda v_\pm^q \leq 0 \quad \text{in } D(\Omega_0, \eta/2). \tag{3.11}$$

Let $\varepsilon_0 \in (0, \varepsilon(\delta))$ be an ε satisfying (3.3) with $R = \eta/2$. Then, by Lemma 3.1 with $q \in (0, p-1)$, for any $\varepsilon \in (0, \varepsilon_0)$, the unique solution of the boundary value problem

$$\begin{cases} -\varepsilon \Delta_p h + \lambda h^q = 0, & x \in B(0, \eta/2), \\ h = \delta, & x \in \partial B(0, \eta/2) \end{cases} \tag{3.12}$$

satisfies $h(0) = 0$.

Take any $x_0 \in D(\Omega_0, \eta)$. It follows from (3.11), (3.12) and Lemma 2.1 that $v_\pm(x) \leq h(x - x_0)$ for all $x \in B(x_0, \eta/2)$. Since $h(0) = 0$, we have $v_\pm(x_0) = 0$, and hence $u_\varepsilon(x) = a$ for all $x \in D(\Omega_0, \eta)$. □

4 Announcement: Semilinear Case

Theorem 1.3 says that if ε is sufficiently small, then the coincidence set $\mathcal{O}_\varepsilon = \{x \in \Omega \mid u_\varepsilon(x) = a(x)\}$ has an interior point in a subdomain where $a(x)$ is constant. However, if we assume that $\mathcal{O}_\varepsilon$ has an interior point, then $a(x)$ also satisfies the equation in (P) on the interior of $\mathcal{O}_\varepsilon$, and hence $a(x)$ has to be p-harmonic. Thus it is natural to expect that the coincidence set has an interior point if and only if $a(x)$ is p-harmonic.

We shall announce that Theorem 1.3 with $p = 2$ will be extended to the case where $a(x)$ is harmonic on a subdomain.

Theorem 4.1. *Let* $p = 2$. *Suppose* (F1)–(F4) *and*

(F6) $a(x) \in C(\overline{\Omega}) \cap H^2(\Omega)$, $\Delta a(x) = 0$ *a.e. in a nonempty subdomain* Ω_0 *of* Ω *and there exist* $q \in (0,1)$ *and* $\lambda > 0$ *such that*

$$\limsup_{u \to a(x)} \frac{f(x,u) - f(x,a(x))}{|u - a(x)|^{q-1}(u - a(x))} < -\lambda \quad \textit{uniformly in } \Omega_0. \tag{4.13}$$

Then, for sufficiently small $\eta > 0$, *there exists* $\varepsilon_0 \in (0, \varepsilon_*)$ *such that if* $\varepsilon \in (0, \varepsilon_0)$ *then every positive solution* u_ε *of* (P) *satisfies*

$$u_\varepsilon(x) = a(x) \quad \textit{for all } x \in D(\Omega_0, \eta).$$

Theorem 4.1 can be shown in the similar way as Theorem 1.3. The corresponding result to the p-Laplace operator has not been obtained because the proof strongly relies on the linearity of Laplace operator.

5 Remarks

We give a few remark on the theorems.

(1) For all the results of this paper, it is sufficient for (F4) to be assumed only in the interval $[\xi, A]$ with $g(\xi) = 0$, instead of $[0, \infty)$ with $g(0) = 0$.

(2) It is easy to extend Theorem 1.3 to the case where $a(x)$ is constant on more than one subdomain.

(3) Theorem 1.1 does not assure the uniqueness of positive solutions. It is an interesting problem whether the positive solutions will be unique (when ε is sufficiently small). We have positive answers under some cases: $p = 2$ by Angenent [1]; $p > 1$ and $f(x,u) = f(u)$ by Guo [6].

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