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Iterative methods for infinite families of nonexpansive mappings in Banach spaces

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1 Introduction

Throughout this paper, let E be a real Banach space with norm $\|\cdot\|$ and let $\mathbb{N}$ be the set of all positive integers. Let C be a nonempty closed convex subset of E. Then, a mapping $T: C\to C$ is called nonexpansive if

$$\|Tx-Ty\|\leq\|x-y\|$$

for all $x, y\in C$. Browder [4] considered a sequence $\{x_{n}\}$ as follows:

$$x\in C, \quad x_{n}=\alpha_{n}x+(1-\alpha_{n})Tx_{n} \quad (\forall n\in \mathbb{N}), \tag{1.1}$$

where $\{\alpha_{n}\}\subset(0,1)$ and he proved the first strong convergence theorem in the framework of a Hilbert space. Later, Reich [29], Takahashi and Ueda [51], Shioji and Takahashi [39], Nakajo [21] and others also proved strong convergence theorems of Browder's type in Hilbert spases or Banach spaces. On the other hand, Halpern [9] considered the following process: $x_{1}=x\in C$ and

$$x_{n+1}=\alpha_{n}x+(1-\alpha_{n})Tx_{n} \quad (\forall n\in \mathbb{N}), \tag{1.2}$$

where $\{\alpha_{n}\}\subset[0,1)$. Wittmann [52] proved a strong convergence theorem of Halpern's type in the framework of a Hilbert space and then, several authors [2, 10, 11, 12, 13, 14, 17, 21, 33, 35, 38, 39, 40, 50] proved strong convergence theorems of Halpern's type in Hilbert spaces or Banach spaces. Recently, Moudafi [20] and Xu [53] considered the following process by the viscosity approximation method: $x_{1}=x\in C$ and

$$x_{n+1}=\alpha_{n}f(x_{n})+(1-\alpha_{n})Tx_{n} \quad (\forall n\in \mathbb{N}), \tag{1.3}$$

where $\{\alpha_{n}\}\subset[0,1)$ and $f: C\to C$ is a contraction.

In this article, for an infinite family $\{T_{n}\}$ of nonexpansive mappings of C into itself such that $\emptyset\neq\cap_{n=1}^{\infty}F(T_{n})$, we consider a sequence $\{x_{n}\}$ generated by

$$x_{n}=\alpha_{n}f(x_{n})+(1-\alpha_{n})T_{n}x_{n} \quad (\forall n\in \mathbb{N}),$$

where $\{\alpha_n\} \subset (0,1)$ and $f : C \to C$ is a contraction. Then, we give the conditions of $\{\alpha_n\}$ and $\{T_n\}$ under which $\{x_n\}$ converges strongly to a common fixed point of $\cap_{n=1}^{\infty} F(T_n)$. We also consider a sequence $\{x_n\}$ generated by

$$x_1 = x \in C, \quad x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)T_n x_n \quad (\forall n \in \mathbb{N}),$$

where $\{\alpha_n\} \subset [0,1)$ and $f : C \to C$ is a contraction. Then, we give the conditions of $\{\alpha_n\}$ and $\{T_n\}$ under which $\{x_n\}$ converges strongly to a common fixed point of $\cap_{n=1}^{\infty} F(T_n)$. Using these results, we improve and extend well-known strong convergence theorems.

2 Preliminaries

Let E be a real Banach space with norm $\|\cdot\|$ and let E^* denote the dual of E. We denote the value of $y^* \in E^*$ at $x \in E$ by $\langle x, y^* \rangle$. The duality mapping J from E into 2^{E^*} is defined by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}$$

for every $x \in E$. Let $U = \{x \in E : \|x\| = 1\}$. The norm of E is said to be Gâteaux differentiable if for each $x, y \in U$, the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$

exists. In the case, E is called smooth. The norm of E is said to be uniformly Gâteaux differentiable if for each $y \in U$, the limit (2.1) is attained uniformly for $x \in U$. We know that if E is smooth, then the duality mapping J is single valued. Further, if the norm of E is uniformly Gâteaux differentiable, then J is norm to weak* uniformly continuous on each bounded subset of E. Let C be a closed convex subset of E. A mapping $T : C \to C$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. We denote by $F(T)$ the set of all fixed points of T. Let I denote the identity operator on E. An operator $A \subset E \times E$ with domain $D(A) = \{x \in E : Az \neq \emptyset\}$ and range $R(A) = \bigcup\{Az : z \in D(A)\}$ is said to be accretive if for each $x_i \in D(A)$ and $y_i \in Ax_i, i = 1, 2$, there exists $j \in J(x_1 - x_2)$ such that $\langle y_1 - y_2, j \rangle \geq 0$. If A is accretive, then we have

$$\|x_1 - x_2\| \leq \|x_1 - x_2 + r(y_1 - y_2)\|$$

for all $r > 0$ and $y_i \in Ax_i, i = 1, 2$. If A is accretive, then we can define, for each $r > 0$, a nonexpansive single valued mapping $J_r : R(I + rA) \to D(A)$ by $J_r = (I + rA)^{-1}$. It is called the resolvent of A. We also define the Yosida approximation A_r by $A_r = (I - J_r)/r$. We know that $A_r x \in AJ_r x$ for all $x \in R(I + rA)$ and $\|A_r x\| \leq \inf\{\|y\| : y \in Ax\}$ for all $x \in D(A) \cap R(I + rA)$. We also know that for an accretive operator A, $A^{-1}0 = F(J_r)$ for all $r > 0$, where $A^{-1}0 = \{u \in E : 0 \in Au\}$. An accretive operator A is said to be m-accretive if $R(I + rA) = E$ for all $r > 0$. A closed convex subset C of a Banach space E is said to have normal structure if for each bounded closed convex subset of K of C which contains at least two points, there exists an element x of K which is not a diametral point of K, i.e.,

$$\sup\{\|x - y\| : y \in K\} < \delta(K),$$

where $\delta(K)$ is the diameter of K. It is well known that a closed convex subset of a uniformly convex Banach space has normal structure and a compact convex subset of a Banach space has normal structure; see [44] for more details. The following result was proved by Kirk [18].

Theorem 2.1 (Kirk [18]). *Let E be a reflexive Banach space and let C be a nonempty bounded closed convex subset of E which has normal structure. Let T be a nonexpansive mapping of C into itself. Then $F(T)$ is nonempty.*

A closed convex subset C of a Banach space E is said to have the fixed point property for nonexpansive mappings if every nonexpansive mapping of a nonempty bounded closed convex subset of K of C into itself has a fixed point in K. If C is a closed convex subset of a reflexive Banach space which has normal structure, from Theorem 2.1, C has the fixed point property for nonexpansive mappings.

We denote by $\mathbb{N}$ the set of all natural numbers and let μ be a mean on $\mathbb{N}$, i.e., a continuous linear functional on ℓ^∞ satisfying $\|\mu\| = 1 = \mu(1)$. We know that μ is a mean on $\mathbb{N}$ if and only if

$$\inf_{n\in\mathbb{N}} a_n \le \mu(f) \le \sup_{n\in\mathbb{N}} a_n$$

for each $f = (a_1, a_2, \dots) \in \ell^\infty$. Occasionally, we use $\mu_n(a_n)$ instead of $\mu(f)$. Let $f = (a_1, a_2, \dots) \in \ell^\infty$ with $a_n \to a$ and let μ be a Banach limit on $\mathbb{N}$. Then, $\mu(f) = \mu_n(a_n) = a$; see [44] for more details. Further, we know the following result [51].

Theorem 2.2 (Takahashi and Ueda [51]). *Let C be a nonempty closed convex subset of a Banach space E with a uniformly Gâteaux differntiable norm, let $\{x_n\}$ be a bounded sequence of E and let μ be a mean on $\mathbb{N}$. Let $z \in C$. Then*

$$\mu_n\|x_n - z\|^2 = \min_{y\in C} \mu_n\|x_n - y\|^2$$

if and only if $\mu_n \langle y - z, J(x_n - z)\rangle \le 0$ for all $y \in C$, where J is the duality mapping of E.

Let C be a nonempty subset of a Banach space E. Let D be a subset of C and let P be a retraction of C onto D, i.e., $Px = x$ for each $x \in D$. Then P is said to be sunny [28] if for each $x \in C$ and $t \ge 0$ with $Px + t(x - Px) \in C$,

$$P(Px + t(x - Px)) = Px.$$

A subset D of C is said to be a sunny nonexpansive retract of C if there exists a sunny nonexpansive retraction P of C onto D. We know that if E is smooth and P is a retraction of C onto D, then P is sunny and nonexpansive if and only if for each $x \in C$ and $z \in D$,

$$\langle x - Px, J(z - Px)\rangle \le 0. \tag{2.2}$$

For more details, see [44].

3 Conditions for infinite families

Let E be a Banach space and let C be a nonempty closed convex subset of E. Let $\{T_n\}$ and $\mathcal{T}$ be families of nonexpansive mappings of C into itself such that $\emptyset \ne F(\mathcal{T}) = \cap_{n=1}^{\infty} F(T_n)$, where $F(T_n)$ is the set of all fixed points of T_n and $F(\mathcal{T})$ is the set of all common fixed points of $\mathcal{T}$. Then, $\{T_n\}$ is said to satisfy the condition (I) with $\mathcal{T}$ if for each bounded sequence $\{z_n\}$ in C,

$$\lim_{n\to\infty} \|z_n - T_n z_n\| = 0$$

implies that $\lim_{n\to\infty}\|z_n - Tz_n\| = 0$ for all $T \in \mathcal{T}$. In particular, if $\mathcal{T} = \{T\}$, i.e., $\mathcal{T}$ consists of one mapping T, then $\{T_n\}$ is said to satisfy the condition (I) with T. $\{T_n\}$ is said to satisfy the condition (II) if for each bounded sequence $\{z_n\} \subset C$,

$$\lim_{n\to\infty}\|z_{n+1} - T_n z_n\| = 0$$

implies that $\lim_{n\to\infty}\|z_n - T_m z_n\| = 0$ for all $m \in \mathbb{N}$. $\{T_n\}$ is said to satisfy the condition (III) if for every bounded subset B of C,

$$\sum_{n=1}^{\infty}\sup\{\|T_n x - T_{n+1}x\| : x \in B\} < \infty.$$

Proposition 3.1. *Let C be a nonempty closed convex subset of a Banach space E and let T be a nonexpansive mapping of C into itself with $F(T) \neq \emptyset$. Then, $\{T_n\}$ with $T_n = T$ for all $n \in \mathbb{N}$ satisfies the condition (I) with T and the condition (III).*

Proof. Put $T_n = T$ for all $n \in \mathbb{N}$. Then, it is obvious that $\{T_n\}$ satisfies the condition (I) with T and the condition (III). □

Theorem 3.2 ([24]). *Let C be a nonempty closed convex subset of a uniformly convex Banach space E and let S and T be nonexpansive mappings of C into itself such that $F(S)\cap F(T) \neq \emptyset$. Let $\{\gamma_n\} \subset [a,b]$ for some $a,b \in (0,1)$ with $a \leq b$. Then, $\{T_n\}$ with $T_n = \gamma_n S + (1-\gamma_n)T$ for all $n \in \mathbb{N}$ satisfies the condition (I) with $\frac{S+T}{2}$. Further, $\{T_n\}$ with $T_n = \gamma_n S + (1-\gamma_n)T$ for all $n \in \mathbb{N}$ such that $\sum_{n=1}^{\infty}|\gamma_n - \gamma_{n+1}| < \infty$ satisfies the condition (I) with $\frac{S+T}{2}$ and the condition (III).*

The following lemma is related to Edelstein and O'Brien [6, Theorem 1].

Lemma 3.3 ([48]). *Let C be a nonempty closed convex subset of a Banach space E and let T be a nonexpansive mapping of C into itself with $F(T) \neq \emptyset$. Let $\{\beta_n\}$ be a sequence of real numbers with $0 < a \leq \beta_n \leq b < 1$ and let B be a nonempty bounded subset of C. Define a mapping S_n of C into itself by*

$$S_n x = S(\beta_n)x = (1-\beta_n)x + \beta_n Tx \quad \text{for all } x \in C$$

and put $a_n = \sup_{x\in B}\|TS^n x - S^n x\|$ for all $n \in \mathbb{N}$, where $S^n = S_n S_{n-1}\cdots S_1$. Then, $a_n \to 0$. In particular, for any $m \in \mathbb{N}$,

$$\lim_{n\to\infty}\sup_{x\in B}\|S_m S^n x - S^n x\| = 0.$$

The following lemma was also proved by Takahashi [48].

Lemma 3.4 ([48]). *Let C be a nonempty closed convex subset of a Banach space E and let T be a nonexpansive mapping of C into itself with $F(T) \neq \emptyset$. For a nonempty bounded subset B of C and $n \in \mathbb{N}$, define a mapping f_n of $[0,1]^n$ into $(-\infty,\infty)$ by*

$$f_n(\beta_n,\beta_{n-1},\ldots,\beta_1) = \sup_{x\in B}\|TU^n x - U^n x\|$$

for all $(\beta_n,\beta_{n-1},\ldots,\beta_1) \in [0,1]^n$, where $U^n = S(\beta_n)S(\beta_{n-1})\cdots S(\beta_1)$ and

$$S(\beta_k)x = (1-\beta_k)x + \beta_k Tx$$

for all $x \in C$ and $k \in \{1,2,\ldots,n\}$. Then, f_n is continuous.

Using Lemmas 3.3 and 3.4 we obtain the following theorem.

Theorem 3.5 ([48]). *Let C be a nonempty closed convex subset of a Banach space E and let T be a nonexpansive mapping of C into itself such that $F(T)$ is nonempty. For any $n \in \mathbb{N}$ and $\beta_n \in \mathbb{R}$ with $0 < a \leq \beta_n \leq b < 1$, define $S_n : C \to C$ as follows:*

$$S_n x = (1 - \beta_n)x + \beta_n Tx \quad \text{for all } x \in C.$$

Then, $\{S_n\}$ satisfies the condition (I) with T and the condition (II).

We know the following lemma for resolvents of accretive operators; see [44].

Lemma 3.6. *Let E be a Banach space and let $A \subset E \times E$ be an accretive operator. Let $r, \lambda > 0$ and $D(A) \subset R(I + \lambda A)$. Then,*

$$\frac{1}{\lambda}\|J_r x - J_\lambda J_r x\| \leq \frac{1}{r}\|x - J_r x\|$$

for every $x \in R(I + rA)$.

Using Lemma 3.6, we also have the following theorem.

Theorem 3.7 ([48]). *Let C be a nonempty closed convex subset of a Banach space E and let $A \subset E \times E$ be an accretive operator such that*

$$\overline{D(A)} \subset C \subset \bigcap_{\lambda > 0} R(I + \lambda A)$$

and $A^{-1}0 \neq \emptyset$. Let $\{\lambda_n\}$ be a sequence of real numbers such that $\lambda_n \in (0, \infty)$ and $\lim_{n\to\infty} \lambda_n = \infty$. Define $S_n = J_{\lambda_n}$ for any $n \in \mathbb{N}$. Then, $\{S_n\}$ satisfies the condition (I) with J_1 and the condition (II), where $J_1 = (I + A)^{-1}$. Moreover, $\{T_n\}$ with $T_n = J_{\lambda_n}$ ($\forall n \in \mathbb{N}$) such that $\{\lambda_n\} \subset (0, \infty)$, $\liminf_{n\to\infty} \lambda_n > 0$ and $\sum_{n=1}^{\infty} |\lambda_n - \lambda_{n+1}| < \infty$ satisfies the condition (I) with $\{J_1\}$ and the condition (III).

Let C be a nonempty closed convex subset of E. Let $S_1, S_2, \ldots$ be infinite nonexpansive mappings of C into itself and let $\beta_1, \beta_2, \ldots$ be real numbers such that $0 \leq \beta_i \leq 1$ for every $i \in \mathbb{N}$. Then, for any $n \in \mathbb{N}$, Takahashi [43] (see also [34, 45, 49]) introduced a mapping W_n of C into itself as follows:

$$\begin{aligned}
U_{n,n+1} &= I,\\
U_{n,n} &= \beta_n S_n U_{n,n+1} + (1 - \beta_n)I,\\
U_{n,n-1} &= \beta_{n-1} S_{n-1} U_{n,n} + (1 - \beta_{n-1})I,\\
&\vdots\\
U_{n,k} &= \beta_k S_k U_{n,k+1} + (1 - \beta_k)I,\\
&\vdots\\
U_{n,2} &= \beta_2 S_2 U_{n,3} + (1 - \beta_2)I,\\
W_n = U_{n,1} &= \beta_1 S_1 U_{n,2} + (1 - \beta_1)I.
\end{aligned}$$

Such a mapping W_n is called the W-mapping generated by $S_n, S_{n-1}, \ldots, S_1$ and $\beta_n, \beta_{n-1}, \ldots, \beta_1$. We know that if E is strictly convex, $\cap_{i=1}^n F(S_i) \neq \emptyset$, $0 < \beta_i < 1$ for every $i = 2, 3, \ldots, n$ and $0 < \beta_1 \leq 1$, then, $F(W_n) = \cap_{i=1}^n F(S_i)$; see [45, 49]. We also have that if E is strictly convex, $\cap_{n=1}^{\infty} F(S_n) \neq \emptyset$ and $0 < \beta_i \leq b < 1$ for every $i \in \mathbb{N}$ for some $b \in (0,1)$, then, $\lim_{n\to\infty} U_{n,k}x$ exists for every $x \in C$ and $k \in \mathbb{N}$; see [34]. So, we can define a mapping W of C into itself as follows:

$$Wx = \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}x$$

for every $x \in C$. Such a W is called the W-mapping generated by $S_1, S_2, \ldots$ and $\beta_1, \beta_2, \ldots$. We have that if E is strictly convex, $\cap_{i=1}^{\infty} F(S_i) \neq \emptyset$ and $0 < \beta_i \leq b < 1$ for every $i \in \mathbb{N}$ for some $b \in (0,1)$, then, $F(W) = \cap_{i=1}^{\infty} F(S_i)$; see [34]. We know the following result for the W-mappings.

Theorem 3.8 ([24]). *Let C be a nonempty closed convex subset of a strictly convex Banach space E. Let $S_1, S_2, \ldots$ be infinite nonexpansive mappings of C into itself with $\cap_{n=1}^{\infty} F(S_n) \neq \emptyset$ and let $\beta_1, \beta_2, \ldots$ be real numbers with $0 < \beta_i \leq b < 1$ for every $i \in \mathbb{N}$ for some $b \in (0,1)$. Let W_n be the W-mapping generated by $S_n, S_{n-1}, \ldots, S_1$ and $\beta_n, \beta_{n-1}, \ldots, \beta_1$ for every $n \in \mathbb{N}$ and let W be the W-mapping generated by $S_1, S_2, \ldots$ and $\beta_1, \beta_2, \ldots$. Then, $\{T_n\}$ with $T_n = W_n$ $(\forall n \in \mathbb{N})$ satisfies the condition (I) with W and the condition (III).*

4 Strong convergence theorem of Browder's type

We can prove a strong convergence theorem of Browder's type for a countable family of nonexpansive mappings in a Banach space.

Theorem 4.1 ([48]). *Let E be a reflexive Banach space with a uniformly Gâteaux differentiable norm and let C be a nonempty closed convex subset of E which has the fixed point property for nonexpansive mappings. Let T be a nonexpansive mapping of C into itself and let $\{T_n\}$ be a family of nonexpansive mappings of C into itself which satisfies $\emptyset \neq F(T) = \bigcap_{n=1}^{\infty} F(T_n)$. Further, suppose that $\{T_n\}$ satisfies the codition (I) with T. Define a sequence $\{x_n\}$ in C as follows:*

$$x_n = \alpha_n f(x_n) + (1-\alpha_n)T_n x_n, \quad n = 1, 2, 3, \ldots,$$

where $\{\alpha_n\} \subset (0,1)$ satisfies $\lim_{n\to\infty} \alpha_n = 0$ and f is a contraction of C into itself. Then, $\{x_n\}$ converges strongly to $u \in F(T)$, where $u = P_{F(T)}f(u)$ and $P_{F(T)}$ is a sunny nonexpansive retraction of C onto $F(T)$.

We have the following result for nonexpansive mappings by Proposition 3.1 and Theorem 4.1.

Theorem 4.2. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E whose norm is uniformly Gâteaux differentiable and let T be a nonexpansive mapping of C into itself such that $F(T) \neq \emptyset$. Let $x \in C$ and $\{x_n\}$ be a sequence by $x_n = \alpha_n x + (1-\alpha_n)Tx_n$ $(\forall n \in \mathbb{N})$, where $\{\alpha_n\} \subset (0,1)$ with $\lim_{n\to\infty} \alpha_n = 0$. Then, $\{x_n\}$ converges strongly to $P_{F(T)}x$, where $P_{F(T)}$ is a sunny nonexpansive retraction of C onto $F(T)$.*

We also get the following result for nonexpansive mappings by Theorems 3.2 and 4.1.

Theorem 4.3. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E whose norm is uniformly Gâteaux differentiable and let S and T be nonexpansive mappings of C into itself such that $F(S) \cap F(T) \neq \emptyset$. Let $x \in C$ and $\{x_n\}$ be a sequence by $x_n = \alpha_n x + (1-\alpha_n)(\gamma_n Sx_n + (1-\gamma_n)Tx_n)$ $(\forall n \in \mathbb{N})$, where $\{\alpha_n\} \subset (0,1)$ with $\lim_{n\to\infty} \alpha_n = 0$ and $\{\gamma_n\} \subset [a,b]$ for some $a, b \in (0,1)$ with $a \leq b$. Then, $\{x_n\}$ converges strongly to $P_{F(S)\cap F(T)}x$, where $P_{F(S)\cap F(T)}$ is a sunny nonexpansive retraction of C onto $F(S) \cap F(T)$.*

We have the following result for accretive operators from Theorems 3.7 and 4.1.

Theorem 4.4. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E whose norm is uniformly Gâteaux differentiable and let $A \subset E \times E$ be an accretive operator with $\overline{D(A)} \subset C \subset \cap_{\lambda>0} R(I + \lambda A)$ and $A^{-1}0 \neq \emptyset$. Let $x \in C$ and $\{x_n\}$ be a sequence by $x_n = \alpha_n x + (1-\alpha_n) J_{\lambda_n} x_n$ $(\forall n \in \mathbb{N})$, where $\{\lambda_n\} \subset (0, \infty)$ and $\{\alpha_n\} \subset (0,1)$ with $\lim_{n\to\infty} \alpha_n = 0$. If $\lim_{n\to\infty} \lambda_n = \infty$, $\{x_n\}$ converges strongly to $P_{A^{-1}0}x$, where $P_{A^{-1}0}$ is a sunny nonexpansive retraction of C onto $A^{-1}0$.*

We get the following result for the W-mappings from Theorems 3.8 and 4.1.

Theorem 4.5. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E whose norm is uniformly Gâteaux differentiable. Let $S_1, S_2, \ldots$ be infinite nonexpansive mappings of C into itself with $F := \cap_{n=1}^{\infty} F(S_n) \neq \emptyset$ and let $\beta_1, \beta_2, \ldots$ be real numbers with $0 < \beta_i \leq b < 1$ for every $i \in \mathbb{N}$ for some $b \in (0,1)$. Let W_n be the W-mapping generated by $S_n, S_{n-1}, \ldots, S_1$ and $\beta_n, \beta_{n-1}, \ldots, \beta_1$ for every $n \in \mathbb{N}$. Let $x \in C$ and $\{x_n\}$ be a sequence by $x_n = \alpha_n x + (1-\alpha_n) W_n x_n$ $(\forall n \in \mathbb{N})$, where $\{\alpha_n\} \subset (0,1)$ with $\lim_{n\to\infty} \alpha_n = 0$. Then, $\{x_n\}$ converges strongly to $P_F x$, where P_F is a sunny nonexpansive retraction of C onto F.*

5 Strong convergence theorem of Halpern's type

In this section, we prove two strong convergence theorems of Halpern's type for a countable family of nonexpansive mappings in a Banach space.

Theorem 5.1 ([48]). *Let E be a reflexive Banach space with a uniformly Gâteaux differentiable norm and let C be a nonempty closed convex subset of E which has the fixed point property for nonexpansive mappings. Let T be a nonexpansive mapping of C into itself and let $\{T_n\}$ be a family of nonexpansive mappings of C into itself which satisfy $\emptyset \neq F(T) = \bigcap_{n=1}^{\infty} F(T_n)$. Further, suppose that $\{T_n\}$ satisfies the condition (I) with T and the condition (II). Let $\{x_n\}$ be a sequence in C as follows: $x_1 = x \in C$ and*

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) T_n x_n, \quad n = 1, 2, 3, \ldots,$$

where $\{\alpha_n\} \subset [0,1)$ satisfies $\lim_{n\to\infty} \alpha_n = 0$, and $\sum_{n=1}^{\infty} \alpha_n = \infty$, and f is a contraction of C into itself. Then, $\{x_n\}$ converges strongly to $u \in F(T) = \bigcap_{n=1}^{\infty} F(T_n)$, where $u = Pf(u)$ and P is a sunny nonexpansive retraction of C onto $F(T)$.

Using Theorems 3.5 and 5.1, we obtain the following result:

Theorem 5.2 ([48]). *Let E be a reflexive Banach space with a uniformly Gatêaux differentiable norm. Let C be a nonempty closed convex subset of E which has the fixed point property for nonexpansive mappings and let $T : C \to C$ be a nonexpansive mapping such that $F(T)$ is nonempty and let f be a contraction of C into itself. Define a sequence $\{x_n\}$ of C as follows: $x_1 = x \in C$ and*

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)((1-\beta_n)x_n + \beta_n T x_n) \quad \textit{for all } n \in \mathbb{N},$$

where $\{\alpha_n\} \subset (0,1)$ and $\{\beta_n\} \subset (0,1)$ satisfy the following conditions:

$$\alpha_n \to 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty \quad \textit{and} \quad 0 < a \leq \beta_n \leq b < 1.$$

Then, the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Theorem 5.2 improves and extends Suzuki's result [42]. Using Theorems 3.7 and 5.1, we also obtain the following result which was proved by Takahashi [47].

Theorem 5.3 ([47]). *Let E be a reflexive Banach space with a uniformly Gatêaux differentiable norm and let C be a nonempty closed convex subset of E which has he fixed point property for nonexpansive mappings. Let $A \subset E \times E$ be an accretive operator with $A^{-1}0 \neq \emptyset$ satisfying*

$$\overline{D(A)} \subset C \subset \bigcap_{t>0} R(I + tA),$$

where $\overline{D(A)}$ is the closure of $D(A)$ and let f be a contraction of C into itself. Let $\{x_n\}$ be a sequence of C defined by $x_1 = x \in C$ and

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) J_{t_n} x_n \quad \text{for all } n \in \mathbb{N},$$

where $\{\alpha_n\} \subset (0,1)$ and $\{t_n\} \subset (0, \infty)$ satisfy the following conditions:

$$\alpha_n \to 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty \quad \text{and} \quad t_n \to \infty.$$

Then, the sequence $\{x_n\}$ converges strongly to $u \in A^{-1}0$, where $u = Pf(u)$ and P is a sunny nonexpansive retraction of C onto $A^{-1}0$.

Theorem 5.4 ([24]). *Let C be a nonempty closed convex subset of a uniformly convex Banach space E whose norm is uniformly Gâteaux differentiable and let $\{T_n\}$ and $\mathcal{T}$ be families of nonexpansive mappings of C into itself which satisfy $\emptyset \neq F(\mathcal{T}) = \cap_{n=1}^{\infty} F(T_n)$. Further, suppose that $\{T_n\}$ satisfies the condition (I) with $\mathcal{T}$ and the condition (III). Let $\{x_n\}$ be a sequence generated as follows: $x_1 = x \in C$ and*

$$x_{n+1} = \alpha_n x + (1 - \alpha_n) T_n(\beta_n x + (1 - \beta_n) x_n) \quad (\forall n \in \mathbb{N}),$$

where $\{\alpha_n\} \subset [0,1)$ and $\{\beta_n\} \subset [0,1)$ satisfy $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = 0$ and $\prod_{n=1}^{\infty}(1 - \alpha_n)(1 - \beta_n) = 0$. If $\sum_{n=1}^{\infty}(|\alpha_n - \alpha_{n+1}| + |\beta_n - \beta_{n+1}|) < \infty$, then $\{x_n\}$ converges strongly to $P_{F(\mathcal{T})}x$, where $P_{F(\mathcal{T})}$ is a sunny nonexpansive retraction of C onto $F(\mathcal{T})$.

Using Proposition 3.1 and Theorem 5.4, we obtain the following theorem:

Theorem 5.5. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E whose norm is uniformly Gâteaux differentiable and let T be a nonexpansive mapping of C into itself with $F(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated as follows: $x_1 = x \in C$ and*

$$x_{n+1} = \alpha_n x + (1 - \alpha_n) T(\beta_n x + (1 - \beta_n) x_n) \quad (\forall n \in \mathbb{N}),$$

where $\{\alpha_n\} \subset [0,1)$ and $\{\beta_n\} \subset [0,1)$ satisfy $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = 0$, $\prod_{n=1}^{\infty}(1-\alpha_n)(1-\beta_n) = 0$ and $\sum_{n=1}^{\infty}(|\alpha_n - \alpha_{n+1}| + |\beta_n - \beta_{n+1}|) < \infty$. Then, $\{x_n\}$ converges strongly to $P_{F(T)}x$, where $P_{F(T)}$ is a sunny nonexpansive retraction of C onto $F(T)$.

We have the following result [17] for nonexpansive mappings by Theorems 3.2 and 5.4.

Theorem 5.6. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E whose norm is uniformly Gâteaux differentiable and let S and T be nonexpansive mappings of C into itself with $F(S) \cap F(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated as follows: $x_1 = x \in C$ and*

$$x_{n+1} = \alpha_n x + (1 - \alpha_n)(\gamma_n S + (1 - \gamma_n) T)(\beta_n x + (1 - \beta_n) x_n) \quad (\forall n \in \mathbb{N}),$$

where $\{\alpha_n\} \subset [0,1)$ *and* $\{\beta_n\} \subset [0,1)$ *satisfy* $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = 0$, $\prod_{n=1}^{\infty}(1-\alpha_n)(1-\beta_n) = 0$ *and* $\sum_{n=1}^{\infty}(|\alpha_n - \alpha_{n+1}| + |\beta_n - \beta_{n+1}|) < \infty$ *and* $\{\gamma_n\} \subset [a,b]$ *for some* $a, b \in (0,1)$ *with* $a \leq b$ *satisfies* $\sum_{n=1}^{\infty} |\gamma_n - \gamma_{n+1}| < \infty$. *Then,* $\{x_n\}$ *converges strongly to* $P_{F(S)\cap F(T)}x$, *where* $P_{F(S)\cap F(T)}$ *is a sunny nonexpansive retraction of* C *onto* $F(S) \cap F(T)$.

We have the following result [21] for accretive operators from Theorems 3.7 and 5.4.

Theorem 5.7. *Let* C *be a nonempty closed convex subset of a uniformly convex Banach space* E *whose norm is uniformly Gâteaux differentiable and let* $A \subset E \times E$ *be an accretive operator with* $\overline{D(A)} \subset C \subset \cap_{\lambda>0} R(I+\lambda A)$ *and* $A^{-1}0 \neq \emptyset$. *Let* $\{x_n\}$ *be a sequence generated as follows:* $x_1 = x \in C$ *and*

$$x_{n+1} = \alpha_n x + (1-\alpha_n) J_{\lambda_n}(\beta_n x + (1-\beta_n)x_n) \quad (\forall n \in \mathbb{N}),$$

where $\{\alpha_n\} \subset [0,1)$ *and* $\{\beta_n\} \subset [0,1)$ *satisfy* $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = 0$, $\prod_{n=1}^{\infty}(1-\alpha_n)(1-\beta_n) = 0$ *and* $\sum_{n=1}^{\infty}(|\alpha_n - \alpha_{n+1}| + |\beta_n - \beta_{n+1}|) < \infty$ *and* $\{\lambda_n\} \subset (0,\infty)$ *satisfies* $\liminf_{n\to\infty} \lambda_n > 0$ *and* $\sum_{n=1}^{\infty} |\lambda_n - \lambda_{n+1}| < \infty$. *Then,* $\{x_n\}$ *converges strongly to* $P_{A^{-1}0}x$, *where* $P_{A^{-1}0}$ *is a sunny nonexpansive retraction of* C *onto* $A^{-1}0$.

We get the following result [34] for W-mappings by Theorems 3.8 and 5.4.

Theorem 5.8. *Let* C *be a nonempty closed convex subset of a uniformly convex Banach space* E *whose norm is uniformly Gâteaux differentiable. Let* $S_1, S_2, \ldots$ *be infinite nonexpansive mappings of* C *into itself with* $F := \cap_{n=1}^{\infty} F(S_n) \neq \emptyset$ *and let* $\beta_1, \beta_2, \ldots$ *be real numbers with* $0 < \beta_i \leq b < 1$ *for every* $i \in N$ *for some* $b \in (0,1)$. *Let* W_n *be the* W*-mapping generated by* $S_n, S_{n-1}, \ldots, S_1$ *and* $\beta_n, \beta_{n-1}, \ldots, \beta_1$ *for every* $n \in N$. *Let* $\{x_n\}$ *be a sequence generated as follows:* $x_1 = x \in C$ *and*

$$x_{n+1} = \alpha_n x + (1-\alpha_n) W_n(\gamma_n x + (1-\gamma_n)x_n) \quad (\forall n \in \mathbb{N}),$$

where $\{\alpha_n\} \subset [0,1)$ *and* $\{\gamma_n\} \subset [0,1)$ *satisfy* $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \gamma_n = 0$, $\prod_{n=1}^{\infty}(1-\alpha_n)(1-\gamma_n) = 0$ *and* $\sum_{n=1}^{\infty}(|\alpha_n - \alpha_{n+1}| + |\gamma_n - \gamma_{n+1}|) < \infty$. *Then,* $\{x_n\}$ *converges strongly to* $P_F x$, *where* P_F *is a sunny nonexpansive retraction of* C *onto* F.

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