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Duality of Fractional Integral Programming with Generalized Invexity*)

Hang-Chin Lai

Chung Yuan Christian University and
National Tsing Hua University, Taiwan

Abstract

A new dual type for ratio of integral variational programming is constructed by mixing the Wolfe type dual and Mond-Weir type dual problem. The existence theorem for optimal solution for the mixed programming problem is then established from necessary optimality conditions by using extra assumptions of generalized invexity. Finally we prove that the weak, strong, and strict converse duality theorems in the mixed framework.

1 Introduction and Preliminaries

Consider a fractional programming of variational problem as the following form.

$$(P) \qquad \operatorname*{Min}_{x} \left\{ \max_{1\le i\le p} \frac{\int_a^b f^i(t,x,\dot{x})\,dt}{\int_a^b g^i(t,x,\dot{x})\,dt} \right\}$$
subject to $x \in PS(T,R^n)$, $x(a)=\alpha$, $x(b)=\beta$, *and*
$$\int_a^b h^j(t,x,\dot{x})\,dt \le 0,\ \ j\in \underline{m} \equiv \{1,2,\cdots,m\},$$

where functions f^i, g^i, $i \in \underline{p}$ and h^j, $j \in \underline{m}$ are continuous in t and have continuous partial derivatives with respect to x and $\dot{x}$; $T=[a,b]$ denotes the time space, and $PS(T,R^n)$ stands for the state space of all piecewise smooth functions $x: T \to R^n$ with norm defined by $\|x\| = \|x\|_\infty + \|Dx\|_\infty$ and D is the differential operator on $PS(T,R^n)$ defined by

$$y = Dx \quad \text{if and only if} \quad x(t) = x(a) + \int_a^t y(s)\,ds.$$

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Thus $D = d/dt$ except at the point of discontinuity. Without loss of generality, we may assume throughout that

$$\int_a^b g^i(t,x,\dot{x})\,dt > 0, \quad \int_a^b f^i(t,x,\dot{x})\,dt \geq 0 \qquad \text{for each } i \in \underline{p}.$$

Denote by $\mathcal{F}_P$ the set of all feasible solutions of (P).

In order to simplify the symbols in problem (P), as in [7], we scalarize functionals as the following:

$$\begin{aligned}
\Phi(x,y) &= \langle y, F(x)\rangle = \sum_{i=1}^p y_i F_i(x) = \sum_{i=1}^p y_i \int_a^b f^i(t,x,\dot{x})\,dt \\
\Psi(x,y) &= \langle y, G(x)\rangle = \sum_{i=1}^p y_i G_i(x) = \sum_{i=1}^p y_i \int_a^b g^i(t,x,\dot{x})\,dt \\
\Omega(x,z) &= \langle z, H(x)\rangle = \sum_{j=1}^m z_j H_j(x) = \sum_{j=1}^m z_j \int_a^b h^j(t,x,\dot{x})\,dt
\end{aligned}$$

where $y \in I = \{y \in R_+^p \mid \sum_{i=1}^p y_i = 1\}$ and $z \in R_+^m$. Then for any feasible solution x of (P), the objective fractional function can be represented by

$$\varphi(x) = \max_{i \in \underline{p}} \frac{F_i(x)}{G_i(x)} = \max_{y \in I} \frac{\langle y, F(x)\rangle}{\langle y, G(x)\rangle} = \max_{y \in I} \frac{\Phi(y,x)}{\Psi(y,x)}. \tag{1.1}$$

The problem (P) is equivalent to

$$\begin{aligned}
(\widetilde{P}) \qquad & \operatorname*{Min}_{x \in PS} \max_{y \in I} \frac{\Phi(y,x)}{\Psi(y,x)} = \operatorname*{Min}_{x} \left(\max_{i \in \underline{p}} \frac{F_i(x)}{G_i(x)} \right) \\
& \textit{subject to } x \in PS(T,R^n),\ x(a) = \alpha,\ x(b) = \beta \\
& \qquad \textit{and } H(x) \leq 0.
\end{aligned}$$

It is equivalent to the parametric minimization problem:

$$\begin{aligned}
(EP_\nu) \qquad & \text{Minimize } q(\nu) \\
& \textit{subject to } F_i(x) - \nu G_i(x) \leq q,\ i \in \underline{p}, \\
& \qquad \textit{and } H_j(x) \leq 0, \quad j \in \underline{m},\ x \in PS(T,R^n), \\
& \qquad x(a) = \alpha,\ x(b) = \beta.
\end{aligned}$$

From (EP_ν), one can reduce to the optimal solution x^* for (P) with its optimal value ν^* which is given by

$$\nu^* = \varphi(x^*) = \frac{\Phi(x^*,y^*)}{\Psi(x^*,y^*)}. \tag{1.2}$$

The concept used here for solution of (EP_ν) coincides with finding the minimax solution (x^*,y^*) of the Lagrangian

$$\begin{aligned}
L(x,y;x,z) &= \langle y, F(x)\rangle - \nu\langle y, G(x)\rangle + \langle z, H(x)\rangle \\
&= \Phi(x,y) - \nu\,\Psi(x,y) + \Omega(x,z)
\end{aligned} \tag{1.3}$$

with multipliers $\nu^* \in R_+$ and $z^* \in R^m_+$. The minimax solution (x^*, y^*) is given in the equation

$$L'_x(x,y)\xi = 0 \quad \text{for all } \xi \in C(T, R^n).$$

Hence the necessary optimality conditions for problem (P) can be stated as the following.

Theorem 1.1: *Let x^* be an optimal solution of (P). Then there exist Lagrangian multipliers $y^* \in I \subset R^p_+$ and $z^* \in R^m_+$ such that the Kuhn-Tucker type conditions hold for the Lagrangian* (1.3) :

$$L'_x(x^*, y^*; \nu^*, z^*) = 0 \tag{1.4}$$

$$\Omega(x^*, z^*) = 0 \tag{1.5}$$

where $\nu^* = \dfrac{\Phi(x^*, y^*)}{\Psi(x^*, y^*)}$ *given in* (1.2) *is the optimal value of problem* (P), *and the equation* $L'_x(\cdot,\cdot) = 0$ *is then expressed by*

$$\Psi(x^*, y^*)\Big[\Phi'(x^*, y^*) + \Omega'(x^*, z^*)\Big] - \Phi(x^*, y^*)\Psi'(x^*, y^*) = 0. \tag{1.6}$$

□

2 Sufficient Optimality Conditions

The existence theorems of optimal solutions for problem (P) can be considered as the converses of necessary optimality conditions (in Theorem 1.1) with some extra assumptions. Thus the sufficient theorem for problem (P) usually would not be unique. Many authors have searched for sufficient conditions, and employed the sufficiency for optimal solutions to study the duality problem. In [7], Lai and Liu established the sufficient optimality conditions under generalized invexity, and employed the result to construct the Wolfe type dual and Mond-Weir type dual, respectively, as the following forms.

$$
\begin{array}{lll}
(WD) & \text{Maximize} & \dfrac{\Phi(u,y) + \Omega(u,z)}{\Psi(u,y)} \\
& \textit{subject to} & (u,z) \in PS(T, R^n) \times R^m_+ \\
& & u(a) = \alpha, \ u(b) = \beta, \ y \in I \subset R^p_+, \text{ and} \\
& & \Psi(u,y)\Big[\Phi'(u,y) + \Omega'(u,z)\Big] \\
& & \qquad -\Psi'(u,y)\Big[\Phi(u,y) + \Omega(u,z)\Big] = 0 \\
& \textit{where} & \Phi(u,y) + \Omega(u,z) \geq 0 \text{ and } \Psi(u,y) > 0;
\end{array}
$$

$$
\begin{array}{lll}
(MWD) & \text{Maximize} & \dfrac{\Phi(u,y)}{\Psi(u,y)} \\
& \textit{subject to} & (u,y) \in PS(T,R^n) \times I, \\
& & u(a) = \alpha, \;\; u(b) = \beta \text{ and} \\
& & \Psi(u,y)\Big[\Phi'(u,y) + \Omega'(u,z)\Big] \\
& & \qquad -\Phi(u,y)\Psi'(u,y) = 0 \\
& & \Omega(u,z) \geq 0, \;\; z \in R^m_+.
\end{array}
$$

In [7], the duality theorems are established for the problems (WD) and (MWD) under generalized invexity, and in [8] the parameter-free dual is also studied for problem (P). There are many authors who investigated the duality programming by invexity as well as generalized invexity for other kinds of fractional or nonfractional problems (Cf, the cited papers in the References). Now a question arises that whether we could combine the two dual problems (WD) and (MWD) in [7] as a new type dual (MD) to problem (P) in which the problems (WD) and (MWD) become the special cases of the new type dual. To do this, we will consider a part of constrained inequality to add into the numerator of the fractional objective of the primary variational problem (P), and maximize the corresponding objective fractional functional to satisfy the necessary conditions where new mixed dual problem is stated as following.

$$
\begin{array}{lll}
(MD) & \text{Maximize} & \dfrac{\Phi(u,y) + \sum_{j\in M_0} z_j H_j(u)}{\Psi(u,y)} \\
& \textit{subject to} & (u,y) \in PS(T,R^n) \times I, \\
& & u(a) = \alpha, u(b) = \beta \textit{ and } z \in R^m_+; \\
& & \Psi(u,y)\Big[\Phi'(u,y) + \sum_{\alpha=0}^{k} z^{\mathsf{T}}_{M_\alpha} H'_{M_0}(u)\Big] \\
& & \quad -\Big[\Phi(u,y) + \sum_{j\in M_0} z_j H_j(u)\Big]\Psi'(u,y) \geq 0, \qquad (2.1) \\
& & \sum_{\alpha=0}^{k} z^{\mathsf{T}}_{M_\alpha} H_{M_\alpha} \geq 0, \qquad (2.2)
\end{array}
$$

where the index sets $M_\alpha \subseteq M$, $\alpha = 0, 1, 2, \cdots, k$ are mutually disjoint, that is

$$
M_\alpha \cap M_\beta = \varnothing \quad \text{if } \alpha \neq \beta \quad \text{and} \quad \bigcup_{\alpha=0}^{k} M_\alpha = M.
$$

Denote the set of all feasible solutions of (MD) by

$$
\mathcal{F}_{MD} = \{(u,u,z) \in PS(T,R^n) \times I \times R^m_+\}.
$$

Note that the constrained functional

$$\Omega(u,z)=\sum_{\alpha=0}^{k} z_{M_\alpha}^{\top} H_{M_\alpha}(u)=\sum_{j\in M} z_j H_j(u).$$

For convenience, we recall briefly the following definitions for generalized invexity (cf. Lai and Liu [7]).

For any $u \in PS(T,R^n)$, a differentiable function J is said to be **invex** at u w.r.t. η, a vector function defined by $\eta : PS(T,R^n)\times(T,R^n)\to C(T,R^n)$, $\big(\eta(x,u)=0$ only if $x=u\big)$, if

$$J(x)-J(u)\geq J'(u)\eta(x,u).$$

J is said to be **pseudoinvex** at u w.r.t η if

$$J'(u)\eta(x,u)\geq 0 \quad\Rightarrow\quad J(x)\geq J(u).$$

Or equivalently

$$J(x)<J(u) \quad\Rightarrow\quad J'(u)\eta(x,u)<0.$$

J is said to be **strictly pseudoinvex** at u w.r.t η if

$$J'(u)\eta(x,u)\geq 0 \quad\Rightarrow\quad J(x)>J(u).$$

$$\text{or } J(x)\leq J(u) \quad\Rightarrow\quad J'(u)\eta(x,u)<0.$$

J is said to be **quasiinvex** at u w.r.t. η if

$$J(x)\leq J(u) \quad\Rightarrow\quad J'(u)\eta(x,u)\leq 0;$$

Or equivalently

$$J'(u)\eta(x,u)>0 \quad\Rightarrow\quad J(x)>J(u).$$

Sufficiency for optimality solution of (P) was stated in [7; Theorem 3.1]. Under invexity assumptions proposed in the dual problem (MD), we will establish the weak, strong, and strict converse duality relations between the mixed type dual problem (MD) and the primary problem (P) respectively. Furthermore, under generalized invexity assumptions, we can deduce that there are no duality gap between (MD) and (P).

3 The Mixed Type Dual Problem

In view of the problem (MD), if the index set M of the constrained inequalities of problem (P) is divided into two disjoint parts M_0 and M_1 in problem (MD), that is $M_0\cup M_1=M$, then the (MD) is reduced to (WD) and (MWD), respectively as the following:

(i) if $M_1 = \varnothing$, $M_0 = M$, then $(MD) = (WD)$,

(ii) if $M_0 = \varnothing$, $M_1 = M$, then $(MD) = (WD)$.

Therefore the work on the (MD) type dual is indeed an extended work in the paper [7]. To this purpose,some sufficient optimality conditions are available to establish the weak, strong, and strict converse duality theorems. At first we state the weak duality theorem as follows.

Theorem 3.1: (Weak Duality). *Let $x \in \mathcal{F}_P$ and $(u, y, z) \in \mathcal{F}_{MD}$ be any feasible solutions of the problem (P) and the dual problem (MD), respectively. Define a functional $A(\cdot)$ on $PS(T, R^n)$ by*

$$A(\cdot) = \Psi(u, y)\left[\Phi(\cdot, y) + \sum_{\alpha=0}^{k} z_{M_\alpha}^{\top} H_{M_\alpha}(\cdot)\right] - \Psi(\cdot, y)\left[\Phi(u, y) + \sum_{j=0} z_j H_j(u)\right].$$

If for each y and z, either one of the following conditions (a) and (b) holds:

(a) for $y \in R_+^p$ and $z \in R_+^m$, $(u, y, z) \in \mathcal{F}_{MD}$, the functions $\Phi(\cdot, y)$, $-\Psi(\cdot, y)$ and $\sum_{\alpha=0}^{k} z_{M_\alpha}^{\top} H_{M_\alpha}(\cdot)$ are invex at u w.r.t. the function $\eta(x, u)$.

(b) the function $A(\cdot)$ is pseudoinvex at u w.r.t. η.

Then

$$\varphi(x) \geq \frac{\Phi(u, y) + \sum_{j \in M_0} z_j H_j(u)}{\Psi(u, y)} \tag{3.1}$$

where $\varphi(x)$ is, defined by (1.1), the objective function of the minimization problem (P). □

Now if $x^* \in \mathcal{F}_P$ is an optimal solution of (P), then from Theorem 1.1, there exist $y^* \in I$ and $z^* \in R_+^m$ such that (1.6) holds. Hence if $(u, y^*, z^*) \in \mathcal{F}_{MD}$, a feasible solution of the duality problem (MD) satisfying the conditions (a) and (b) of Theorem 3.1, then one can get

$$\max_{(u,y^*,z^*) \in \mathcal{F}_{MD}} \frac{\Phi(u, y^*) + \sum_{j \in M_0} z_j^* H_j(u)}{\Psi(u, y^*)} = \varphi(x^*) = \min_{x \in \mathcal{F}_P} \varphi(x),$$

and (x^*, y^*, z^*) is an optimal solution of (MD). Hence we get the following strong duality theorem:

Theorem 3.2: (Strong Duality). *Let x^* be an optimal solution of (P) corresponding $y^* \in I$ and $z^* \in R_+^m$ such that the feasible solution (u, y^*, z^*)*

of (MD) satisfying the conditions (a) and (b) of Theorem 3.1. Then the feasible solution (u, y^*, z^*) of (MD) is optimal if and only if $u = x^*$ and the two optimal values of (P) and (MD) are equal. That is $\min(P) = \max(MD)$ □.

Next if we assume that x_1 and (x^0, y^0, z^0) are optimal solutions of problem (P) and the dual problem (MD), respectively, then one can ask whether $x_1 = x^0$ and $\min(P) = \max(MD)$? The following theorem explore these properties whish will hold under some extra conditions for invexity.

Theorem 3.3: (Strict Converse Duality). *Let x_1 and (x^0, y^0, z^0) be optimal solutions of problem (P) and the dual problem (MD), respectively. Further, assume that the conditions of Theorem 3.2 are fulfilled, and if the functional $A(\cdot)$ defined on $PS(T, R^n)$ in Theorem 3.1 is strictly pseudoinvex. Then $x_1 = x^0$ is an optimal solution of (P), and* $\min(P) = \max(MD)$. *That is, the maximum value*

$$\varphi(x_1) = \frac{\Phi(x^0, y^0) + \sum_{j \in M_0} z_j^0 H_j(x^0)}{\Psi(x^0, y^0)}. \qquad \square$$

Remark. *It is remarkable to observe that by using different suitable combinations of invexity, quasiinvexity, pseudoinvexity, as well as strictly pseudoinvexity etc., then one can also get some different conditions for (MD) to establish the duality theorems.*

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