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Strong Convergence of Approximating Fixed Point Sequences for Nonlinear Mappings

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Abstract

Some iteration algorithms to prove strong convergence of approximating fixed point sequences for nonlinear mappings are introduced in Hilbert spaces or Banach spaces. Also, we propose a modified iteration algorithm for Xu's iteration process [*Bull. Austral. Math. Soc.*, **74** (2006), 143-151] for nonexpansive mappings and establish strong convergence of such an iteration for asymptotically nonexpansive mappings in smooth and uniformly convex Banach spaces.

Keywords: Strong convergence, approximating fixed point sequence, iterative algorithm, nonexpansive mapping, asymptotically nonexpansive mapping.

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1 Introduction

Let C be a nonempty closed convex subset of a real Banach space X and let $T : C \to C$ be a mapping. Then T is said to be a ***Lipschitzian*** mapping if, for each $n \geq 1$, there exists a constant $k_n > 0$ such that $\|T^n x - T^n y\| \leq k_n \|x - y\|$ for all $x, y \in C$ (we may assume that all $k_n \geq 1$). A Lipschitzian mapping T is called ***uniformly k-Lipschitzian*** if $k_n = k$ for all $n \geq 1$, ***nonexpansive*** if $k_n = 1$ for all $n \geq 1$, and ***asymptotically nonexpansive*** [9] if $\lim_{n\to\infty} k_n = 1$, respectively. A point $x \in C$ is a ***fixed point*** of T provided $Tx = x$. Denote by $F(T)$ the set of fixed points of T; that is, $F(T) = \{x \in C : Tx = x\}$. A point p in C is said to be an ***asymptotic fixed point*** of T [25] if C contains a sequence $\{x_n\}$ which converges weakly to p such that the strong $\lim_{n\to\infty}(x_n - Tx_n) = 0$. The set of asymptotic fixed points of T will be denoted by $\hat{F}(T)$. We say that a sequence $\{x_n\}$ in C is said to be an ***approximating fixed point*** sequence for T if $\|x_n - Tx_n\| \to 0$.

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Let X be a smooth Banach space and let X^* be the dual of X. The function $\phi : X \times X \to \mathbb{R}$ is defined by

$$\phi(y,x) = \|y\|^2 - 2\langle y, Jx\rangle + \|x\|^2$$

for all $x, y \in X$, where J is the normalized duality mapping from X to X^*. We say that a mapping $T : C \to C$ is *relatively asymptotically nonexpansive* [15] if $F(T)$ is nonempty, $\hat{F}(T) = F(T)$ and, for each $n \geq 1$ there exists a constant $k_n > 0$ such that $\phi(p, T^n x) \leq k_n^2 \phi(p, x)$ for $x \in C$ and $p \in F(T)$, where $\lim_{n\to\infty} k_n = 1$. In particular, T is called *relatively nonexpansive* [19] if $k_n = 1$ for all n; see also [3,4,5].

The purpose of this paper is to introduce some recent results and open questions relating to strong convergence for modified Mann (or Ishikawa) iteration processes. Firstly, in section 2, we introduce three famous iteration processes introduced by Halpern [10], Mann [17], and Ishikawa [11], respectively. Next, in section 3, we give some properties of generalized projection relating to the above function $\phi : X \times X \to \mathbb{R}$, and furthermore, in section 4, we give some recent results and open questions for strong convergence of approximating fixed point sequences in Hilbert spaces or general Banach spaces. Finally, in section 5, we give a positive answer for Question3, that is, we modify Xu's iteration (4.12) and prove strong convergence for such a modified iteration for asymptotically nonexpansive mappings in smooth and uniformly convex Banach spaces.

2 Three iteration algorithms

Construction of approximating fixed points of nonexpansive mappings is an important subject in the theory of nonexpansive mappings and its applications in a number of applied areas, in particular, in image recovery and signal processing. However, the sequence $\{T^n x\}$ of iterates of the mapping T at a point $x \in C$ may not converge even in the weak topology. Thus three averaged iteration methods often prevail to approximate a fixed point of a nonexpansive mapping T. The first one is introduced by Halpern [10] and is defined as follows: Take an initial guess $x_0 \in C$ arbitrarily and define $\{x_n\}$ recursively by

$$x_{n+1} = t_n x_0 + (1 - t_n) T x_n, \quad n \geq 0, \tag{2.1}$$

where $\{t_n\}$ is a sequence in the interval $[0,1]$.

The second iteration process is now known as Mann's iteration process [17] which is defined as

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T x_n, \quad n \geq 0, \tag{2.2}$$

where the initial guess x_0 is taken in C arbitrarily and the sequence $\{\alpha_n\}$ is in the interval $[0,1]$.

The third iteration process is referred to as Ishikawa's iteration process [11] which is defined recursively by

$$\begin{cases} y_n = \beta_n x_n + (1 - \beta_n) T x_n, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T y_n, \end{cases} \quad n \geq 0, \tag{2.3}$$

where the initial guess x_0 is taken in C arbitrarily and $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in the interval $[0,1]$. By taking $\beta_n = 1$ for all $n \geq 0$ in (2.3), Ishikawa's iteration process reduces to the Mann's iteration process (2.2). It is known in [6] that the process (2.2) may fail to converge while the process (2.3) can still converge for a Lipschitz pseudo-contractive mapping in a Hilbert space.

In general, the iteration process (2.1) has been proved to be strongly convergent in both Hilbert spaces [10, 16, 30] and uniformly smooth Banach spaces [23, 26, 32], while Mann's iteration (2.2) has only weak convergence even in a Hilbert space [8].

3 Some properties of generalized projections

Let X be a real Banach space with norm $\|\cdot\|$ and let X^* be the dual of X. Denote by $\langle\cdot,\cdot\rangle$ the duality product. When $\{x_n\}$ is a sequence in X, we denote the strong convergence of $\{x_n\}$ to $x \in X$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$. We also denote the weak ω-limit set of $\{x_n\}$ by $\omega_w(x_n) = \{x : \exists x_{n_j} \rightharpoonup x\}$. The normalized duality mapping J from X to X^* is defined by

$$Jx = \{x^* \in X^* : \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2\}$$

for $x \in X$.

A Banach space X is said to be *strictly convex* if $\|(x+y)/2\| < 1$ for all x, $y \in X$ with $\|x\| = \|y\| = 1$ and $x \neq y$. It is also said to be *uniformly convex* if $\|x_n - y_n\| \to 0$ for any two sequences $\{x_n\}$, $\{y_n\}$ in X such that $\|x_n\| = \|y_n\| = 1$ and $\|(x_n + y_n)/2\| \to 1$.

Let $U = \{x \in X : \|x\| = 1\}$ be the unit sphere of X. Then the Banach space X is said to be *smooth* provided

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t} \tag{3.1}$$

exists for each x, $y \in U$. It is also known that if X is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of X. Some properties of the duality mapping have been given in [7, 24, 28]. A Banach space X is said to have the *Kadec-Klee* property if a sequence $\{x_n\}$ of X satisfying that $x_n \rightharpoonup x \in X$ and $\|x_n\| \to \|x\|$, then $x_n \to x$. It is known that if X is uniformly convex, then X has the Kadec-Klee property; see [7, 28] for more details.

Let X be a smooth Banach space. Recall that the function $\phi : X \times X \to \mathbb{R}$ is defined by

$$\phi(y,x) = \|y\|^2 - 2\langle y, Jx\rangle + \|x\|^2$$

for all $x, y \in X$. It is obvious from the definition of ϕ that

$$(\|y\| - \|x\|)^2 \leq \phi(y,x) \leq (\|y\| + \|x\|)^2 \tag{3.2}$$

for all $x, y \in X$. Further, we have that for any $x, y, z \in X$,

$$\phi(x,y) = \phi(x,z) + \phi(z,y) + 2\langle x - z, J(z) - J(y)\rangle.$$

In particular, it is easy to see that if X is strictly convex, for $x, y \in X$, $\phi(y,x) = 0$ if and only if $y = x$ (see, for example, Remark 2.1 of [19]).

Let X be a reflexive, strictly convex and smooth Banach space and let C be a nonempty closed convex subset of X. Then, for any $x \in X$, there exists a unique element $\tilde{x} \in C$ such that

$$\phi(\tilde{x}, x) = \inf_{z \in C} \phi(z, x).$$

Then a mapping $Q_C : X \to C$ defined by $Q_C x = \tilde{x}$ is called the *generalized projection* (see [1, 2, 12]). In Hilbert spaces, notice that the generalized projection is clearly coincident with the metric projection.

The following result is well known (see, for example, [1, 2, 12]).

Proposition 3.1. ([1, 2, 12]) *Let K be a nonempty closed convex subset of a real Banach space X and let $x \in X$.*

(a) *If X is smooth, then, $\tilde{x} = Q_K x$ if and only if $\langle \tilde{x} - y, Jx - J\tilde{x} \rangle \geq 0$ for $y \in K$.*

(b) *If X is reflexive, strictly convex and smooth, then $\phi(y, Q_K x) + \phi(Q_K x, x) \leq \phi(y, x)$ for all $y \in K$.*

The following subsequent two lemmas are motivated by Lemmas 1.3 and 1.5 of Martinez-Yanes and Xu [18] in Hilbert spaces, respectively; for detailed proofs, see [13].

Lemma 3.2. ([13]) *Let C be a nonempty closed convex subset of a smooth Banach space X, $x, y, z \in X$ and $\lambda \in [0,1]$. Given also a real number $a \in \mathbb{R}$, the set*

$$D := \{v \in C : \phi(v, z) \leq \lambda\phi(v, x) + (1 - \lambda)\phi(v, y) + a\}$$

is closed and convex.

Lemma 3.3. ([13]) *Let X be a reflexive, strictly convex and smooth Banach space with the Kadec-Klee property, and let K be a nonempty closed convex subset of X. Let $x_0 \in X$ and $q := Q_K x_0$, where Q_K denotes the generalized projection from X onto K. If $\{x_n\}$ is a sequence in X such that $\omega_w(x_n) \subset K$ and satisfies the condition*

$$\phi(x_n, x_0) \leq \phi(q, x_0)$$

for all n. Then $x_n \to q \,(= Q_K x_0)$.

Recently, Kamimura and Takahashi [12] proved the following result, which plays a crucial role in our discussion.

Proposition 3.4. ([12]) *Let X be a uniformly convex and smooth Banach space and let $\{y_n\}, \{z_n\}$ be two sequences of X. If $\phi(y_n, z_n) \to 0$ and either $\{y_n\}$ or $\{z_n\}$ is bounded, then $y_n - z_n \to 0$.*

Finally, concerning the set of fixed points of a relatively asymptotically nonexpansive mapping, we know the following result.

Proposition 3.5. ([15]) *Let X be a reflexive, strictly convex and smooth Banach space with the Kadec-Klee property, let C be a nonempty closed convex subset of X, and let $T : C \to C$ be a continuous mapping which is relatively asymptotically nonexpansive. Then $F(T)$ is closed and convex.*

Remark 3.6. Note that if T is relatively nonexpansive, the hypothesis of continuity of T in Proposition 2.5 is abundant. Also, $F(T)$ is closed and convex in strictly convex and smooth Banach spaces; see Proposition 2.4 of [19].

4 Strong convergence for approximating fixed point sequences

Let C be a nonempty closed convex subset of a real Banach space X and let $T : C \to C$ be a mapping with $F(T) \neq \emptyset$. Recalling that a sequence $\{x_n\}$ in C is said to be an *approximating fixed point* sequence for T if $\|x_n - Tx_n\| \to 0$, there are several ways to construct an approximating fixed point sequences for a nonexpansive mapping T. We now introduce two cases mentioned in Xu [33]. Firstly we can use Banach's contraction principle to obtain a sequence $\{x_n\}$ in C such that

$$x_n = t_n x_0 + (1 - t_n)Tx_n, \qquad n \geq 1$$

where the initial guess x_0 is taken arbitrarily in C and $\{t_n\}$ is a sequence in the interval $(0, 1)$ such that $t_n \to 0$ as $n \to \infty$, which is called as a Halpern's iteration process (2.1). Due to the assumption that $F(T) \neq \emptyset$, this sequence $\{x_n\}$ is bounded (indeed $\|x_n - p\| \leq \|x_0 - p\|$ for all $p \in F(T)$). Hence

$$\|x_n - Tx_n\| = t_n\|x_0 - Tx_n\| \to 0$$

and $\{x_n\}$ is an approximating fixed point sequence for T.

Secondly, we recall a sequence $\{x_n\}$ in C generated by Mann's iteration process (2.2) in a recursive way. This sequence $\{x_n\}$ is bounded since, for any $p \in F(T)$, we have

$$\|x_{n+1} - p\| \leq \alpha_n\|x_n - p\| + (1 - \alpha_n)\|Tx_n - p\| \leq \|x_n - p\|.$$

That is, $\{|x_n - p\|\}$ is a nonincreasing sequence. Moreover, since

$$\begin{aligned}
\|x_{n+1} - Tx_{n+1}\| &= \|\alpha_n x_n + (1 - \alpha_n)Tx_n - Tx_{n+1}\| \\
&= \|\alpha_n(x_n - Tx_n) + (Tx_n - Tx_{n+1})\| \\
&\leq \alpha_n\|x_n - Tx_n\| + \|x_n - x_{n+1}\| = \|x_n - Tx_n\|,
\end{aligned}$$

the sequence $\{|x_n - Tx_n\|\}$ is also nonincreasing and hence $\lim_{n\to\infty} \|x_n - Tx_n\|$ exists.

However, it is not known whether this sequence $\{x_n\}$ is always an approximating fixed point sequence for T. Only partial answers have been obtained. Indeed, if the space X is uniformly convex and if the control sequence $\{\alpha_n\}$ satisfies the condition

$$\sum_{n=0}^{\infty} \alpha_n(1 - \alpha_n) = \infty,$$

then Reich [22] showed that the sequence $\{x_n\}$ generated by Mann's iteration process (2.2) is an approximating fixed point sequence for T. For the sake of completeness, we include a brief proof to this fact. Let δ_X be the modulus of convexity of X. Pick a $p \in F(T)$. Assuming $\|x_n - p\| > 0$ and noticing $\|Tx_n - p\| \leq \|x_n - p\|$, we deduce that

$$\|x_{n+1} - p\| \leq \|x_n - p\| \left[1 - 2\alpha_n(1-\alpha_n)\delta_X\left(\frac{\|x_n - Tx_n\|}{\|x_n - p\|}\right)\right].$$

Hence

$$\sum_{n=0}^{\infty} \alpha_n(1-\alpha_n)\|x_n - p\|\delta_X\left(\frac{\|x_n - Tx_n\|}{\|x_n - p\|}\right) \leq \|x_0 - p\| < \infty. \tag{4.1}$$

Put $\|x_n - p\| \to r$. If $r = 0$, we are done. So assume $r > 0$. If $\sum_{n=0}^{\infty} \alpha_n(1-\alpha_n) = \infty$, we obtain from (3.1) that $\delta_X(\|x_n - Tx_n\|/r) \to 0$. This implies that $\|x_n - Tx_n\| \to 0$ and $\{x_n\}$ is an approximating sequence for T.

Recently, numerous attempts to modify the Mann iteration method (2.2) or the Ishikawa iteration method (2.3) so that strong convergence is guaranteed have recently been made.

Firstly, motivated by Solodov and Svaiter [27], Nakajo and Takahashi [21] proposed the following modification of Mann's iteration process (2.2) for a single nonexpansive mapping T with $F(T) \neq \emptyset$ and also proved the existence of an approximating fixed point sequence for T and strong convergence of such a sequence as follows.

Theorem NT. ([21]) *Let H be a real Hilbert space, let C be a nonempty closed convex subset of H and let $T : C \to C$ be a nonexpansive mapping. Assume that $F(T)$ is nonempty. Define a sequence $\{x_n\}$ in C by the algorithm:*

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ y_n = \alpha_n x_n + (1-\alpha_n)Tx_n, \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{cases} \tag{4.2}$$

where P_K denotes the metric projection from H onto a closed convex subset K of H. If the sequence $\{\alpha_n\}$ is bounded above from one, then $\{x_n\}$ generated by (4.2) is an approximating fixed point sequence for T and strongly convergent to $P_{F(T)}x_0$.

As a special case, taking $\alpha_n = 0$ for all n in Theorem NT, the above iteration scheme (4.2) reduces to the following:

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ C_n = \{z \in C : \|Tx_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0. \end{cases} \tag{4.3}$$

Recently, Kim and Xu [14] generalized Nakajo and Takahashi's iteration process (4.2) to the following iteration process for an asymptotically nonexpansive mapping T in a Hilbert space, under the hypothesis of boundedness of C.

Theorem KX. ([14]) *Let C be a nonempty bounded closed convex subset of a Hilbert space H and let $T : C \to C$ be an asymptotically nonexpansive mapping. Assume that $\{\alpha_n\}$ is a sequence in $(0,1)$ such that $\alpha_n \le a$ for some $0 < a < 1$. Define a sequence $\{x_n\}$ in C by the following algorithm:*

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ y_n = \alpha_n x_n + (1-\alpha_n)T^n x_n, \\ C_n = \{z \in C : \|y_n - z\|^2 \le \|x_n - z\|^2 + \theta_n\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{cases} \tag{4.4}$$

where

$$\theta_n = (1-\alpha_n)(k_n^2 - 1)(\operatorname{diam} C)^2 \to 0 \quad \text{as } n \to \infty. \tag{4.5}$$

Then $\{x_n\}$ is an approximating fixed point sequence for T and strongly convergent to $P_{F(T)}x_0$.

Very recently, Martinez-Yanez and Xu [18] generalized Nakajo and Takahashi's iteration process (4.2) to the following modification of Ishikawa's iteration process (2.3) for a nonexpansive mapping $T : C \to C$ with $F(T) \neq \emptyset$ in a Hilbert space H:

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ y_n = \alpha_n x_n + (1-\alpha_n)Tz_n, \\ z_n = \beta_n x_n + (1-\beta_n)Tx_n, \\ C_n = \{v \in C : \|y_n - v\|^2 \le \|x_n - v\|^2 \\ \qquad\qquad +(1-\alpha_n)(\|z_n\|^2 - \|x_n\|^2 + 2\langle x_n - z_n, v\rangle)\}, \\ Q_n = \{v \in C : \langle x_n - v, x_n - x_0\rangle \le 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{cases} \tag{4.6}$$

and proved that the sequence $\{x_n\}$ generated by (4.6) converges strongly to $P_{F(T)}x_0$ provided the sequence $\{\alpha_n\}$ is bounded above from one and $\lim_{n\to\infty} \beta_n = 1$.

Kamimura and Takahashi [12] considered the problem of finding an element u of a Banach space X satisfying $0 \in Au$, where $A \subset X \times X^*$ is a maximal monotone operator and X^* is the dual space of X. They studied the following algorithm:

$$\begin{cases} x_0 \in X \text{ chosen arbitrarily}, \\ 0 = v_n + \frac{1}{r_n}(Jy_n - Jx_n), \; v_n \in Ay_n, \\ H_n = \{z \in X : \langle y_n - z, v_n\rangle \ge 0\}, \\ W_n = \{z \in C : \langle x_n - z, Jx_0 - Jx_n\rangle \ge 0\}, \\ x_{n+1} = Q_{H_n \cap W_n} x_0, \end{cases} \tag{4.7}$$

where J is the duality mapping on X, $\{r_n\}$ is a sequence of positive real numbers and Q_K denotes the generalized projection from X onto a closed convex subset K of X; see the section 2 for more details. They proved that if $A^{-1}0 \neq \emptyset$ and $\liminf_{n\to\infty} r_n > 0$, then the sequence $\{x_n\}$ generated by (4.7) converges strongly to an element of $A^{-1}0$. This generalizes the result due to Solodov and Svaiter [27] in a Hilbert space.

Question 1. *Can we carry Theorem NT in Hilbert spaces over more general Banach spaces?*

The *crucial* key to solve this question is to show the *convexity* of C_n in (4.2) in general, which is not easy to prove it in Banach spaces. Professor H. K. Xu raised the following question to me:

Question 2. *Let C be a nonempty closed convex subset of a normed linear space X. For any choice of $a, b \in C$,*

$$C_{a,b} = \{z \in C : \|a - z\| \leq \|b - z\|\}$$

is a convex subset of C if and only if X is a Hilbert space.

Note that if X is a Hilbert space, then

$$z \in C_{a,b} \;\Leftrightarrow\; \langle b - a, z\rangle \leq \frac{1}{2}\left(\|b\|^2 - \|a\|^2\right).$$

So, $C_{a,b}$ is convex in C. However, the proof of the converse still remains open. Owing to these troubles, we need another hypotheses for mappings T. In view of this point, for relatively nonexpansive mappings, Matsushita and Takahashi [19] recently extended Nakajo and Takahashi's iteration process (4.2) to general Banach spaces as follows.

Theorem MT. ([19]) *Let X be a uniformly convex and uniformly smooth Banach space, let C be a nonempty closed convex subset of X, let $T : C \to C$ be a relatively nonexpansive mapping with $F(T) \neq \emptyset$, and let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \leq \alpha_n < 1$ and $\limsup_{n\to\infty} \alpha_n < 1$. Suppose that $\{x_n\}$ is given by*

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JTx_n), \\ H_n = \{z \in C : \phi(z, y_n) \leq \phi(z, x_n)\}, \\ W_n = \{z \in C : \langle x_n - z, Jx_0 - Jx_n\rangle \geq 0\}, \\ x_{n+1} = Q_{H_n \cap W_n} x_0, \end{cases} \tag{4.8}$$

where J is the normalized duality mapping. Then $\{x_n\}$ generated by (4.8) is an approximating fixed point sequence for T and strongly convergent to $Q_{F(T)}x_0$, where Q_K denotes the generalized projection from X onto a closed convex subset K of X.

As a special case, taking $\alpha_n = 0$ for all n in (4.8), the iteration scheme reduces to the following:

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ H_n = \{z \in C : \phi(z, Tx_n) \leq \phi(z, x_n)\}, \\ W_n = \{z \in C : \langle x_n - z, Jx_0 - Jx_n\rangle \geq 0\}, \\ x_{n+1} = Q_{H_n \cap W_n} x_0, \end{cases} \tag{4.9}$$

which generalizes the iteration scheme (4.3) in a Hilbert spaces. Also, they established that even though the condition of uniformly smooth of X is only weakened by the smooth condition of X, the sequence $\{x_n\}$ generated by (4.9) still converges strongly to $Q_{F(T)}x_0$.

Strong Convergence

Recently, Kim and Takahashi [15] generalized Matsushita and Takahashi's iteration process (4.8) to the following iteration process for a uniformly k-Lipschitzian mapping T which is relatively asymptotically nonexpansive.

Theorem KT. ([15]) *Let X be a uniformly convex and uniformly smooth Banach space, let C be a nonempty closed convex subset of X and let $T : C \to C$ be a uniformly k-Lipschitzian mapping which is relatively asymptotically nonexpansive. Assume that $F(T)$ is a nonempty bounded subset of C and $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$ such that $\limsup_{n\to\infty} \alpha_n < 1$ and $\beta_n \to 1$. Define a sequence $\{x_n\}$ in C by the algorithm:*

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JT^n z_n), \\ z_n = \beta_n x_n + (1-\beta_n)T^n x_n, \\ H_n = \{v \in C : \phi(v, y_n) \le \alpha_n \phi(v, x_n) + (1-\alpha_n)\phi(v, z_n) + \eta_n\}, \\ W_n = \{v \in C : \langle x_n - v, Jx_n - Jx_0\rangle \le 0\}, \\ x_{n+1} = Q_{H_n \cap W_n} x_0, \end{cases} \tag{4.10}$$

where J is the normalized duality mapping and

$$\eta_n = (1-\alpha_n)(k_n^2 - 1)\cdot \sup\{\phi(p, z_n) : p \in F(T)\}.$$

Then $\{x_n\}$ generated by (4.10) is an approximating fixed point sequence for T and strongly convergent to $Q_{F(T)}x_0$, where $Q_{F(T)}$ is the generalized projection from X onto $F(T)$.

Let C be a closed convex subset of a Hilbert space H and let $T : C \to C$ be an asymptotically nonexpansive mapping with $F(T) \neq \emptyset$. Then, after noticing that $\phi(x,y) = \|x-y\|^2$ for all $x, y \in H$, we see that $\|T^n x - T^n y\| \le k_n\|x-y\|$ is equivalent to $\phi(T^n x, T^n y) \le k_n^2 \phi(x,y)$. It is therefore easy to show that every asymptotically nonexpansive mapping is both uniformly k-Lipschitzian and relatively asymptotically nonexpansive. In fact, it suffices to show that $\hat{F}(T) \subset F(T)$. The inclusion follows easily from the well-known demiclosedness at zero of $I-T$ (c.f., [31]), where I denotes the identity operator.

Can we remove the hypothesis of *boundedness* of C in Theorem KX in Hilbert spaces? The question still remains open. However, if $F(T)$ is a nonempty bounded subset of C, we now give a partial answer with the following η_n instead of θ_n in (4.5), that is, a Hilbert space's version in a case when $\beta_n = 1$ for all n in Theorem KT.

Corollary KT. ([15]) *Let C be a nonempty closed convex subset of a Hilbert space H and let $T : C \to C$ be an asymptotically nonexpansive mapping. Assume that $F(T)$ is a nonempty bounded subset of C. Assume also that $\{\alpha_n\}$ is a sequence in $[0,1]$ such that $\limsup_{n\to\infty} \alpha_n < 1$. Define a sequence $\{x_n\}$ in C by the following algorithm:*

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ y_n = \alpha_n x_n + (1-\alpha_n)T^n x_n, \\ C_n = \{z \in C : \|y_n - z\|^2 \le \|x_n - z\|^2 + \eta_n\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{cases} \tag{4.11}$$

where

$$\eta_n = (1-\alpha_n)(k_n^2-1)\cdot\sup\{\|x_n-p\|^2 : p\in F(T)\},$$

then $\{x_n\}$ *in* C *generated by (4.11) is an approximating fixed point sequence for* T *and strongly convergent to* $P_{F(T)}x_0$.

Very recently, Xu [33] also constructed the following iteration to guarantee strong convergence for a single nonexpansive mapping $T : C \to C$ with $F(T) \neq \emptyset$ in Banach spaces.

Theorem X. ([33]) *Let* X *be a real smooth and uniformly convex Banach space,* C *a nonempty closed convex subset of* X*, and* $T : C \to C$ *a nonexpansive mapping such that* $F(T) \neq \emptyset$*. Define a sequence* $\{x_n\}$ *in* C *by the algorithm:*

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ H_n = \overline{co}\{v \in C : \|v - Tv\| \le t_n\|x_n - Tx_n\|\}, \\ W_n = \{v \in C : \langle x_n - v, Jx_0 - Jx_n\rangle \ge 0\}, \\ x_{n+1} = Q_{H_n\cap W_n}x_0, \end{cases} \tag{4.12}$$

where $\{t_n\}$ *is a sequence in* $(0,1)$ *so that* $t_n \to 0$*. Then* $\{x_n\}$ *is an approximating fixed point sequence for* T *and strongly convergent to* $Q_{F(T)}x_0$*, where* $Q_{F(T)}$ *is the generalized projection from* X *onto* $F(T)$.

The following question is naturally invoked.

Question 3. *Does Theorem X still remain true for asymtotically nonexpansive mappings?.*

5 Proof of Question 3

In this section, we give a positive answer for Question 3 which is reformulated as follows.

Theorem 5.1. *Let* X *be a uniformly convex and smooth Banach space, let* C *be a nonempty closed convex subset of* X *and let* $T : C \to C$ *be an asymptotically nonexpansive mapping. Assume that* $F(T)$ *is nonempty. Define a sequence* $\{x_n\}$ *in* C *by the algorithm:*

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ H_n = \overline{co}\{v \in C : \|v - T^nv\| \le t_n\|x_n - T^nx_n\|\}, \\ W_n = \{v \in C : \langle x_n - v, Jx_0 - Jx_n\rangle \ge 0\}, \\ x_{n+1} = Q_{H_n\cap W_n}x_0, \end{cases}$$

where $\{t_n\}$ *is a sequence in* $(0,1)$ *so that* $t_n \to 0$*. Then* $\{x_n\}$ *is an approximating fixed point sequence for* T *and strongly convergent to* $Q_{F(T)}x_0$*, where* $Q_{F(T)}$ *is the generalized projection from* X *onto* $F(T)$.

Proof. First we show that $F(T) \subset H_n \cap W_n$ and x_{n+1} is well defined. As a matter of fact, it is clear that $F(T) \subset H_n$ for all n. Also, clearly, $F(T) \subset W_0 = C$ and $x_1 = P_{H_0 \cap W_0} x_0$ is well defined. Assume now that $F(T) \subset W_n$ and x_{n+1} is well defined. We inductively need to prove that $F(T) \subset W_{n+1}$ and x_{n+2} is well defined. In fact, since $x_{n+1} = Q_{H_n \cap W_n} x_0$, by Proposition 3.1 (a), we get

$$\langle x_{n+1} - z, Jx_0 - Jx_{n+1} \rangle \geq 0 \tag{5.1}$$

for all $z \in H_n \cap W_n$. As $F(T) \subset H_n \cap W_n$, (5.1) holds for all $z \in F(T)$. Thus, $F(T) \subset W_{n+1}$ and $x_{n+2} = Q_{H_{n+1} \cap W_{n+1}} x_0$ is well defined.

Now we claim that $\{x_n\}$ is bounded. As a matter of fact, by the definition of W_n, we have $x_n = Q_{W_n} x_0$ and so

$$\phi(x_n, x_0) \leq \phi(y, x_0)$$

for all $y \in W_n$. In particular, since $F(T) \subset W_n$, we get

$$\phi(x_n, x_0) \leq \phi(p, x_0) \quad (p \in F(T)). \tag{5.2}$$

This implies the boundedness of $\{x_n\}$ and so is $\{T^m x_n : n, m \geq 1\}$. Next we show that $\|x_{n+1} - x_n\| \to 0$. For this end, noticing that $x_n = Q_{W_n} x_0$ and $x_{n+1} \in H_n \cap W_n \subset W_n$, we get

$$\phi(x_n, x_0) = \inf_{y \in W_n} \phi(y, x_0) \leq \phi(x_{n+1}, x_0)$$

which shows that the sequence $\{\phi(x_n, x_0)\}$ is increasing (and also bounded) and so $\lim_{n\to\infty} \phi(x_n, x_0)$ exists. Applying (b) of Proposition 3.1, we have

$$\begin{aligned} \phi(x_{n+1}, x_n) &= \phi(x_{n+1}, Q_{W_n} x_0) \leq \phi(x_{n+1}, x_0) - \phi(Q_{W_n} x_0, x_0) \\ &= \phi(x_{n+1}, x_0) - \phi(x_n, x_0) \to 0. \end{aligned}$$

By Proposition 3.4, we have

$$\|x_{n+1} - x_n\| \to 0. \tag{5.3}$$

We now claim that $\{x_n\}$ is an approximating fixed point sequence of T. Let $\tilde{C}$ be a bounded closed convex subset of C which contains all the points x_n and $T^m x_n$ for all n, m and let $d = \operatorname{diam}(\tilde{C})$. Since $x_{n+1} \in H_n$ and by definition of H_n, we have

$$\|x_{n+1} - \sum_{i=1}^{\ell} \lambda_i z_i\| < t_n \tag{5.4}$$

where $\lambda_i > 0$ satisfying $\sum_{i=1}^{\ell} \lambda_i = 1$ and each $z_i \in C$ satisfies

$$\|z_i - T^m z_i\| < t_n \|x_n - T^m x_n\| \leq d\, t_n. \tag{5.5}$$

Then it follows from Lemma 2.4 of [29] that there exists a continuous strictly increasing function γ (depending only on d) with $\gamma(0) = 0$ and such that for any fixed $n \geq 1$,

$$\begin{aligned} &\left\| T^n \left(\sum_{i=1}^{m} \mu_i v_i \right) - \sum_{i=1}^{m} \mu_i T^n v_i \right\| \\ \leq\ & k_n \gamma^{-1} \left(\max_{1 \leq i,j \leq m} [\|v_i - v_j\| - \|T^n v_i - T^n v_j\|] + (1 - k_n^{-1}) d \right) \end{aligned} \tag{5.6}$$

for all integers $m > 1$, all points $\{v_i\}$ in $\tilde{C}$, and all nonnegative numbers $\{\mu_i\}$ such that $\sum_{i=1}^{m} \mu_i = 1$. Then, since $t_n \to 0$ and $k_n \to 1$, it follows easily from (5.4)-(5.6)that

$$\begin{aligned}
\|x_{n+1} - T^{n+1}x_{n+1}\| \le & \left\| x_{n+1} - \sum_{i=1}^{\ell} \lambda_i z_i \right\| + \left\| \sum_{i=1}^{\ell} \lambda_i \left(z_i - T^{n+1} z_i\right) \right\| \\
& + \left\| \sum_{i=1}^{\ell} \lambda_i T^{n+1} z_i - T^{n+1} \left(\sum_{i=1}^{\ell} \lambda_i z_i \right) \right\| + \left\| T^{n+1} \left(\sum_{i=1}^{\ell} \lambda_i z_i \right) - T^{n+1} x_{n+1} \right\| \\
\le & \ (t_n + k_{n+1} t_n) + d\, t_{n+1} + \\
& k_{n+1}\gamma^{-1} \left(\max_{1 \le i,j \le \ell} [\|z_i - z_j\| - \|T^{n+1}z_i - T^{n+1}z_j\|] + (1 - k_{n+1}^{-1})d \right) \\
\le & \ (1 + k_{n+1})t_n + dt_{n+1} + \\
& k_{n+1}\gamma^{-1} \left(\max_{1 \le i,j \le \ell} [\|z_i - T^{n+1}z_i\| - \|z_j - T^{n+1}z_j\|] + (1 - k_{n+1}^{-1})d \right) \\
\le & \ (1 + k_{n+1})t_n + dt_{n+1} + k_{n+1} \cdot \gamma^{-1} \left[d(2t_{n+1} + 1 - k_{n+1}^{-1}) \right] \to 0.
\end{aligned}$$

This combined with (5.3) yields

$$\begin{aligned}
\|x_n - Tx_n\| \ \le & \ \|x_n - x_{n+1}\| + \|x_{n+1} - T^{n+1}x_{n+1}\| \\
& + \|T^{n+1}x_{n+1} - T^{n+1}x_n\| + \|T^{n+1}x_n - Tx_n\| \\
\le & \ (1 + k)\|x_n - x_{n+1}\| + \|x_{n+1} - T^{n+1}x_{n+1}\| \\
& + k\|T^n x_n - x_n\| \to 0,
\end{aligned} \tag{5.7}$$

recalling that T is k-uniformly Lipschitzian for some $k > 0$. Therefore, $\{x_n\}$ is an approximating fixed point sequence for T.

Finally let us prove that $x_n \to q = Q_{F(T)}x_0$. As a similar proof of Theorem 2 in [31], we have $\omega_w(x_n) \subset F(T)$. Indeed, let $p \in \omega_w(x_n)$, i.e., there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup p$. Set $z_k := x_{n_k}$ for all k. We shall prove that $T^n x \to x$. Since $z_k \rightharpoonup x$, for each integer $k \ge 1$, there exists a convex combination $y_k = \sum_{i=1}^{m(k)} \lambda_i^{(k)} z_{i+k}$, $\lambda_i^{(k)} \ge 0$ and $\sum \lambda_i^{(k)} = 1$, such that

$$\|y_k - x\| < 1/k. \tag{5.8}$$

By (5.7), since $\|x_n - Tx_n\| \to 0$, it easily follows that

$$\|z_k - T^n z_k\| \to 0 \tag{5.9}$$

as $k \to \infty$ for any fixed $n \ge 1$. Note that, by (5.9), for arbitrary given $\epsilon > 0$, there exists $N = N(\epsilon, n)$ such that $\|z_k - T^n z_k\| < \epsilon$ for all $k \ge N$. Applying (5.6) again, together with this fact, yields

$$
\begin{aligned}
\|y_k - T^n y_k\| &\le \left\| \sum_{i=1}^{m(k)} \lambda_i^{(k)} (z_{i+k} - T^n z_{i+k}) \right\| + \left\| \sum_{i=1}^{m(k)} \lambda_i^{(k)} T^n z_{i+k} - T^n y_k \right\| \\
&\le \|z_{i+k} - T^n z_{i+k}\| + k_n \gamma^{-1} \Big(\max_{1\le i,j \le m(k)} [\|z_{i+k} - z_{j+k}\| - \|T^n z_{i+k} \\
&\quad - T^n z_{j+k}\|] + (1 - k_n^{-1})d \Big) \\
&\le \|z_{i+k} - T^n z_{i+k}\| + k_n \gamma^{-1} \Big(\max_{1\le i,j \le m(k)} [\|z_{i+k} - T^n z_{i+k}\| \\
&\quad + \|z_{j+k} - T^n z_{j+k}\|] + (1 - k_n^{-1})d \Big) \\
&\le \epsilon + k_n \gamma^{-1} \left(2\epsilon + (1 - k_n^{-1})d \right) \qquad (k \ge N). \qquad (5.10)
\end{aligned}
$$

Taking the limit in (5.10) as $k \to \infty$, we obtain for each $n \ge 1$

$$
\limsup_{k\to\infty} \|y_k - T^n y_k\| \le k_n \gamma^{-1} \left((1 - k_n^{-1})d \right). \qquad (5.11)
$$

Noticing that

$$
\begin{aligned}
\|x - T^n x\| &\le \|x - y_k\| + \|y_k - T^n y_k\| + \|T^n y_k - T^n x\| \\
&\le (1 + k_n)\|x - y_k\| + \|y_k - T^n y_k\| \\
&\le (1 + k_n)/k + \|y_k - T^n y_k\| \qquad \text{(by using (5.8))}
\end{aligned}
$$

and (5.11), it follows that

$$
\limsup_{n\to\infty} \|x - T^n x\| \le \gamma^{-1}(0) = 0.
$$

This shows that $T^n x \to x$ and so $x \in F(T)$. Let $q = Q_{F(T)} x_0$. By (5.2), we see that $\phi(x_n, x_0) \le \phi(q, x_0)$ for all n. Applying Lemma 3.3 (with $K = F(T)$), we conclude that $x_n \to q = Q_{F(T)} x_0$. □

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