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SOME FIXED POINT THEOREMS FOR CONTRACTIVE TYPE MULTI-VALUED MAPPINGS

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ABSTRACT. In this paper, using more general mapping than Hausdorff metric we obtain fixed points for a multi-valued mapping.

1. INTRODUCTION

Fixed point theory has important applications in diverse disciplines of mathematics, statistics, engineering and economics in dealing with problems arising in approximation theory, potential theory, game theory, mathematical economics, etc. Many authors [1-13] have proved some fixed point theorems for various generalizations of contraction mapping in metric spaces. Extensions of the Banach contraction mapping principle to multi-valued mapping were initiated independently by Markin [9] and Nadler [10]. Further results on fixed points of contraction type multi-valued mappings were given by Ćirić [3], Dube and Singh [5], Kubiaczyk [7], Kubiak [8], Ray [11] and others.

In 1995, Chang et al. [2] proved the following theorem: Let (X, d) be a complete metric space and let $T : X \to CB(X)$ be a multi-valued mapping. If there exists $k \in (0,1)$ such that for all $x, y \in X$

$$H(Tx, Ty) \leq kd(x,y) + k \mid d(x,Tx) - d(y,Ty) \mid,$$

then T has a fixed point in X, where $CB(X)$ is the collection of all nonempty bounded closed subsets of X.

In 1996, Kada-Suzuki-Takahashi [6] introduced the concept of w-distance and, by using this concept, proved a nonconvex minimization theorem and some fixed point theorems in complete metric spaces.

In this paper, using more general mapping than Hausdorff metric we obtain fixed points for a multi-valued mapping.

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2. Preliminaries

Definition 2.1 [6]. Let (X,d) be a metric space. Then a function $p : X \times X \to [0,\infty)$ is called a w-distance on X if the following are satisfied:

(1) $p(x,z) \le p(x,y) + p(y,z)$ for all $x,y,z \in X$;
(2) for any $x \in X$, $p(x,\cdot) : X \to [0,\infty)$ is lower semicontinuous;
(3) for any $\epsilon > 0$, there exists $\delta > 0$ such that $p(z,x) \le \delta$ and $p(z,y) \le \delta$ imply $d(x,y) \le \epsilon$.

Example 2.2 [6]. Let (X,d) be a metric space. Then clearly d is a w-distance on X

Example 2.3. Let (X,d) be a metric space with a continuous w-distance p and a continuous w-distance r. Then $q : X \times X \to [0,\infty)$ defined by $q(x,y) = \max[p(x,y), r(y,x)]$ for all $x,y \in X$ is a w-distance on X.

The following is an easy consequence from the definition of w-distance p.

Lemma 2.4. *Let (X,d) be a metric space with a w-distance p. Let $\{x_n\}$ and $\{y_n\}$ be sequences in X, let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[0,\infty)$ converging to 0, and let $z \in X$. If $p(x_n, y_n) \le \alpha_n$ and $p(x_n, z) \le \beta_n$ for any $n \in N$, then $\{y_n\}$ converges to z.*

Now we recall the following generalization of Caristi's fixed point theorem in [6], which will be used in the proof of our results.

Lemma 2.5 [6]. *Let (X,d) be a complete metric space with a w-distance p, let $F : X \to X$ be a mapping and let $\varphi : X \to [0,\infty)$ be a proper lower semicontinuous function, bounded from below such that*

$$p(x,Fx) \le \varphi(x) - \varphi(Fx) \quad \textit{for all } x \in X.?? \tag{2.1}$$

Then there exists $x_0 \in X$ such that $Fx_0 = x_0$ and $p(x_0,x_0) = 0$.

Definition 2.6. Let (X,d) be a metric space with a w-distance p.

(i) For any $x \in X$ and $A \subseteq X$, $p(x,A) := \inf\{p(x,y) : y \in A\}$ and $p(A,x) := \inf\{p(y,x) : y \in A\}$.
(ii) $CB_p(X) = \{A \mid A :$ nonempty closed subset of X and $\sup_{x,y \in A} p(x,y) < \infty\}$.
(iii) For $A, B \in CB_p(X)$,

$$G(A,B) := \max\{\sup_{u\in A} p(u,B), \sup_{u\in A} p(B,u), \sup_{v\in B} p(v,A), \sup_{v\in B} p(A,v)\}.$$

Definition 2.7 [2]. Let $(X, \|\cdot\|)$ be a normed vector space, D a nonempty subset of X. For any given $x \in D$, the set

$$I_D(x) = \{x + a(y - x) : y \in D, a \geq 0\}$$

is called the inwardness set of D at x.

For given $x \in D$ and $a \geq 0$ we denote

$$I_{D,a}(x) = \{x + a(y - x) : y \in D\}.$$

3. Main results

Theorem 3.1. *Let (X, d) be a complete metric space with a continuous w-distance p and $T : X \to CB_p(X)$ be a multi-valued mapping such that*

$$\begin{aligned} G(Tx, Ty) \leq & k \max[p(x, y), p(y, x)] \\ & + k \mid \max[p(x, Tx), p(Tx, x)] \\ & - \max[p(y, Ty), p(Ty, y)] \mid \end{aligned} \tag{3.1}$$

for all $x, y \in X$ and some $k \in (0, 1)$,

$$\inf_{u \in A}\{\max[p(x, u), p(u, x)]\} \leq \max[p(x, A), p(A, x)] \tag{3.2}$$

for all $A \in CB_p(X)$ and each $x \in X$, and

$$x \mapsto \max[p(x, Tx), p(Tx, x)] \quad \text{for all } x \in X, \tag{3.3}$$

is lower semicontinuous.

Then T has a fixed point in X.

Proof. If $\max[p(x, Tx), p(Tx, x)] = 0$ for some $x \in X$, then, by Lemma 2.4, T has a fixed point in X. Next we may assume that $\max[p(x, Tx), p(Tx, x)] > 0$ for all $x \in X$. Take $\alpha > 1$ such that $\alpha \cdot k < 1$. By (3.2), for each $x \in X$ there exists $z_x \in Tx$ such that

$$0 < \max[p(x, z_x), p(z_x, x)] < \alpha \cdot \max[p(x, Tx), p(Tx, x)].$$

Since

$$\begin{aligned} \max[p(z_x, Tz_x), p(Tz_x, z_x)] \leq & G(Tx, Tz_x) \\ \leq & k \cdot \max[p(x, z_x), p(z_x, x)] \\ & + k \mid \max[p(x, Tx), p(Tx, x)] \\ & - \max[p(z_x, Tz_x), p(Tz_x, z_x)] \mid, \end{aligned}$$

we have

$$\begin{aligned}\max[p(z_x, Tz_x), p(Tz_x, z_x)] <& \alpha \cdot k \cdot \max[p(x, Tx), p(Tx, x)] \\ &+ k \mid \max[p(x, Tx), p(Tx, x)] \\ &\quad - \max[p(z_x, Tz_x), p(Tz_x, z_x)] \mid . \end{aligned} \tag{3.4}$$

Suppose that

$$\max[p(x, Tx), p(Tx, x)] < \max[p(z_x, Tz_x), p(Tz_x, z_x)].$$

Then, by (3.4), we obtain $1 < \alpha \cdot k$, which is a contradiction. Thus we have

$$\max[p(z_x, Tz_x), p(Tz_x, z_x)] \leq \max[p(x, Tx), p(Tx, x)]$$

and

$$\begin{aligned}\max[p(x, z_x), p(z_x, x)] < (1+k)(\frac{1}{\alpha} - k)^{-1}\{ &\max[p(x, Tx), p(Tx, x)] \\ &- \max[p(z_x, Tz_x), p(Tz_x, z_x)]\}.\end{aligned}$$

Define a function $\varphi : X \to [0, \infty)$ by

$$\varphi(x) = (1+k)(\frac{1}{\alpha} - k)^{-1}\{\max[p(x, Tx), p(Tx, x)]\}.$$

for all $x \in X$. Define a mapping $F : X \to X$ by $Fx = z_x$ for all $x \in X$. Then, we have

$$\max[p(x, Fx), p(Fx, x)] \leq \varphi(x) - \varphi(Fx)$$

for all $x \in X$. Since all conditions of Lemma 2.5 are satisfied, there exists $u \in X$ such that $u = Fu = z_u \in Tu$. Therefore T has a fixed point in X. □

Corollary 3.2 [2]. *Let (X, d) be a complete metric space and let $T : X \to CB(X)$ be a multi-valued mapping. If there exists $k \in (0, 1)$ such that for all $x, y \in X$*

$$H(Tx, Ty) \leq kd(x, y) + k \mid d(x, Tx) - d(y, Ty) \mid,$$

then T has a fixed point in X.

Proof. Since the metric d is a w-distance and all conditions of Theorem 3.1 are satisfied, Corollary 3.2 follows from Theorem 3.1. □

The following simple example shows that Theorem 3.1 is more general than Theorem 2 of Chang et al. [2].

Example. Let $X = [0,1]$ be the closed bounded interval with the usual metric and let $p : X \times X \to [0,\infty)$ be a mapping defined by $p(x,y) = y$ for all $x, y \in X$. Suppose that $T : X \to CB_p(X)$ is a multi-valued mapping such that $Tx = \{\frac{k}{2}x\}$ for all $x \in X$, where k is a fixed element of $(0,1)$. Then all conditions of Theorem 3.1 are satisfied but not satisfied all conditions of Corollary 3.2, since p is a w-distance but not a metric.

Theorem 3.3. *Let $(X, \|\cdot\|)$ be a Banach space, d be a metric on X induced by the norm $\|\cdot\|$ as $d(x,y) = \|x-y\|$ with a continuous w-distance p, and D be a nonempty closed subset of X. Assume that the w-distance p satisfies*

(i) *for any $x \in X$, $p(x,y) = p(x-y,0) = p(0,y-x)$,*

(ii) *for any $x \in X$ and for any $\alpha > 0$, $p(\alpha x, \alpha y) = \alpha p(x,y)$.*

(iii) *for each $s \in X$,*

$$\inf_{v\in D}\{\max[p(s,v),p(v,s)]\} \le \max[p(s,D),p(D,s)].$$

Let $T : D \to CB_p(X)$ be a multi-valued mapping satisfying (3.1), (3.2), (3.3) and the following condition: there exists a constant $\delta \in [0, \frac{1-k}{1+k})$ such that

$$\inf_{h\in(0,1]}\{\sup_{z\in Tx}\frac{1}{h}\max[p((1-h)x+hz,D),p(D,(1-h)x+hz)]\} \le \delta\cdot\max[p(x,Tx),p(Tx,x)] \quad (3.5)$$

for each x in D.

Then T has a fixed point in X.

Proof. Assume that $\max[p(x,Tx),p(Tx,x)] > 0$ for all $x \in D$. Let $q, \eta, \alpha \in (0,1)$ be such that $q < \frac{1-k}{1+k} - \delta$, $\eta = q + \delta$ and $k < \alpha < \frac{1-\eta}{1+\eta}$. Then we obtain $k < \frac{1-\eta}{1+\eta}$. By (3.5), for any $\epsilon \in (0,q)$ and $x \in D$, there exists $h \in (0,1]$ such that

$$\begin{aligned}&\sup_{z\in Tx}\{\max[p((1-h)x+hz,D),p(D,(1-h)x+hz)]\}\\ &< h\{\delta\cdot\max[p(x,Tx),p(Tx,x)]\\ &\qquad + (q-\epsilon)\cdot\max[p(x,Tx),p(Tx,x)]\}\\ &= h(\eta-\epsilon)\cdot\max[p(x,Tx),p(Tx,x)]. \end{aligned} \quad (3.6)$$

By (3.2), choosing $z \in Tx$ such that

$$\max[p(x,z),p(z,x)] < (1+h\epsilon)\max[p(x,Tx),p(Tx,x)] \quad (3.7)$$

and for this z, taking $y \in D$ in (3.6) from (iii), we have

$$\max[p((1-h)x+hz-y,0),p(0,(1-h)x+hz-y)] < h(\eta-\epsilon)\max[p(x,Tx),p(Tx,x)]. \quad (3.8)$$

From (3.8) and $z \in Tx$, we get $y \neq x$. Letting $u = (1-h)x + hz$, we obtain

$$\begin{aligned}\max[p(u,y),p(y,u)] <& h \cdot \eta \cdot \max[p(x,Tx),p(Tx,x)] \\ &- h \cdot \epsilon \cdot \max[p(x,Tx),p(Tx,x)] \\ \leq& h \cdot \eta \cdot \max[p(x,z),p(z,x)] \\ &- h \cdot \epsilon \cdot \max[p(x,Tx),p(Tx,x)] \\ =& \eta \cdot \max[p(u,x),p(x,u)] \\ &- h \cdot \epsilon \cdot \max[p(x,Tx),p(Tx,x)]. \end{aligned} \tag{3.9}$$

Thus we have

$$\begin{aligned}\max[p(x,y),p(y,x)] \leq& \max[p(x,u),p(u,x)] \\ &+ \max[p(u,y),p(y,u)] \\ \leq& (1+\eta)\max[p(x,u),p(u,x)]. \end{aligned} \tag{3.10}$$

From (3.1) and $k < \alpha$, we get

$$\begin{aligned} l =& \alpha \cdot \max[p(x,y),p(y,x)] + k \mid \max[p(x,Tx),p(Tx,x)] \\ &- \max[p(y,Ty),p(Ty,y)] \mid - G(Tx,Ty) > 0. \end{aligned}$$

By (3.2), there exists $b \in Ty$ such that

$$\max[p(z,b),p(b,z)] < G(Tx,Ty) + l. \tag{3.11}$$

Thus we have

$$\begin{aligned}\max[p(y,Ty),p(Ty,y)] \leq& \max[p(y,b),p(b,y)] \\ \leq& \max[p(y,u),p(u,y)] \\ &+ \max[p(u,z),p(z,u)] \\ &+ \max[p(z,b),p(b,z)]. \end{aligned} \tag{3.12}$$

Using (3.7), (3.9), (3.11) and (3.12), we obtain

$$\begin{aligned}\max[p(y,Ty),p(Ty,y)] <& (\eta - 1) \cdot \max[p(x,u),p(u,x)] \\ &+ \max[p(x,Tx),p(Tx,x)] \\ &+ \alpha \cdot \max[p(x,y),p(y,x)] \\ &+ k \mid \max[p(x,Tx),p(Tx,x)] \\ &\quad - \max[p(y,Ty),p(Ty,y)] \mid . \end{aligned}$$

From (3.10) and $\eta < 1$, we get

$$\begin{aligned}\max[p(y,Ty),p(Ty,y)] <& \Big(\alpha + \frac{\eta-1}{\eta+1}\Big)\max[p(x,y),p(y,x)] \\ &+ \max[p(x,Tx),p(Tx,x)] \\ &+ k \mid \max[p(x,Tx),p(Tx,x)] \\ &\quad - \max[p(y,Ty),p(Ty,y)] \mid .\end{aligned}$$

Suppose that

$$\max[p(x,Tx),p(Tx,x)] \leq \max[p(y,Ty),p(Ty,y)].$$

Then we have

$$\begin{aligned}\max[p(y,Ty),p(Ty,y)] <& \frac{1}{1+k}\Big(\alpha + \frac{\eta-1}{\eta+1}\Big)\max[p(x,y),p(y,x)] \\ &+ \max[p(x,Tx),p(Tx,x)] \\ <& \max[p(x,Tx),p(Tx,x)].\end{aligned}$$

This is a contradiction. Thus we get

$$\max[p(y,Ty),p(Ty,y)] < \max[p(x,Tx),p(Tx,x)]$$

and

$$\begin{aligned}&\frac{1}{1+k}\Big(\frac{1-\eta}{\eta+1} - \alpha\Big) \cdot \max[p(x,y),p(y,x)] \\ &\leq \max[p(x,Tx),p(Tx,x)] - \max[p(y,Ty),p(Ty,y)]. \end{aligned} \tag{3.13}$$

On the other hand, from (3.7), (3.8) and (3.13), there exists a function

$$F : D \to D \tag{3.14}$$

such that for any $x \in D$, $Fx := y$, $y \neq x$ and

$$\max[p((x,Fx),p(Fx,x)] < \varphi(x) - \varphi(Fx),$$

where

$$\varphi(x) = (1+k)\Big(\frac{1-\eta}{1+\eta} - \alpha\Big)^{-1}\max[p(x,Tx),p(Tx,x)].$$

Thus by Lemma 2.5, there exists $v \in D$ such that $Fv = v$. This is a contradiction. This completes the proof. □

Using Theorem 3.3, we have the following corollary

Corollary 3.4 [2]. *Let $(X, \|\cdot\|)$ be a Banach space, d be a metric induced by the norm as $d(x,y) = \|x-y\|$, D be a nonempty closed subset of X and $T : D \to CB(X)$ be a multi-valued mapping satisfying the following conditions :*
(i) *there exists a constant $k \in (0,1)$ such that for any $x, y \in D$*

$$H(Tx, Ty) \leq k\|x-y\| + k \mid d(x, Tx) - d(y, Ty) \mid; \tag{3.15}$$

(ii) *there exists a constant $\delta \in [0, \frac{1-k}{1+k})$ such that*

$$\inf_{k\in(0,1]} \sup_{z\in Tx} \frac{1}{k} d((1-k)x + kz), D) \leq \delta \cdot d(x, Tx).$$

Then T has a fixed point in X.

Theorem 3.5. *Let $(X, \|\cdot\|)$ be a normed space with a continuous w-distance p connecting with a metric d induced by the norm $\|\cdot\|$ as $d(x,y) = \|x-y\|$, D be a convex subset of X, $x \in D$ and $A \in CB_p(X)$. Then*

$$\begin{aligned} &\inf_{h\in(0,1]} \sup_{z\in A} \frac{1}{h}\{\max[p((1-h)x + hz, D), p(D, (1-h)x + hz)]\} \\ &= \inf_{a\geq 0} \sup_{z\in A}\{\max[p(z, I_{D,a}(x)), p(I_{D,a}(x), z)]\}, \end{aligned} \tag{i}$$

$$\begin{aligned} &\inf_{a\geq 0} \sup_{z\in A}\{\max[p(z, I_{D,a}(x)), p(I_{D,a}(x), z)]\} \\ &\geq \sup_{z\in A}\{\max[p(z, I_D(x)), p(I_D(x), z)]\}. \end{aligned} \tag{ii}$$

Proof. Since

$$\begin{aligned} &\inf_{h\in(0,1]} \sup_{z\in A} \frac{1}{h}\{\max[p((1-h)x + hz, D), p(D, (1-h)x + hz)]\} \\ &= \inf_{a\geq 1} \sup_{z\in A}\{\max[p(z, I_{D,a}(x)), p(I_{D,a}(x), z)]\} \end{aligned}$$

and

$$\max[p(z, I_{D,1}(x)), p(I_{D,1}(x), z)] \leq \max[p(z, I_{D,a}(x)), p(I_{D,a}(x), z)],$$

for all $a \in R$ with $0 \leq a < 1$ and for all $x \in D$, $z \in A$, we obtain (i). By elementary calculus, we obtain (ii). □

Using Theorem 3.3 and Theorem 3.5 we have the following Theorem.

Theorem 3.6. *Let $(X, \|\cdot\|)$ be a Banach space with a continuous w-distance p connecting with a metric d induced by the norm $\|\cdot\|$ as $d(x,y) = \|x-y\|$, D be a nonempty closed convex subset of X satisfying (iii) in Theorem 3.3 and $T : D \to CB_p(X)$ be a multi-valued mapping satisfying (3.1), (3.2), (3.3) and the following condition:*

$$\text{there exists a constant } \delta \in [0, \frac{1-k}{1+k}) \text{ such that}$$
$$\inf_{a\geq 0} \sup_{z\in Tx} \{\max[p(z, I_{D,a}(x)), p(I_{D,a}(x), z)]\}$$
$$\leq \delta \cdot \max[p(x,Tx), p(Tx,x)]\} \quad \text{for all } x \in D.$$

Then T has a fixed point in D.

From Theorem 3.6 we have the following corollary.

Corollary 3.7 [2]. *Let $(X, \|\cdot\|)$ be a Banach space, D be a nonempty closed convex subset of X and $T : D \to CB(X)$ be a mapping satisfying (3.15) and the following condition:*

$$\text{there exists a constant } \delta \in [0, \frac{1-k}{1+k}) \text{ such that}$$
$$\inf_{a\geq 0} \sup_{z\in Tx} d(z, I_{D,a}(x)) \leq \delta d(x, Tx) \quad \text{for all } x \in D.$$

Then T has a fixed point in D.

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