数理解析研究所講究録
第 1544 巻 2007 年 206-211

OBSERVATION ON VARIOUS CONJUGATES OF QUASICONVEX FUNCTIONS

島根大学大学院総合理工学研究科 [1] 鈴木 聡 (Satoshi Suzuki)
島根大学大学院総合理工学研究科 [1] 黒川真史 (Masafumi Kurokawa)
島根大学総合理工学部 [2] 黒岩大史 (Daishi Kuroiwa)

[1] Interdisciplinary Graduate School of Science and Engineering, Shimane University
[2] Interdisciplinary Faculty of Science and Engineering, Shimane University

Abstract

We observe various conjugates and their biconjugacies of quasiconvex functions. Especially, we give a sufficient condition which assures biconjugacy is satisfied for 0-quasiconjugate.

1 Introduction

Throughout this paper, let f be a function from $\mathbb{R}^n$ to $(-\infty, \infty]$, and assume f is proper, that is, its domain $\mathrm{dom} f = \{x \in \mathbb{R}^n \mid f(x) < \infty\}$ is not empty. Remember that f is said to be convex if for all x_1, $x_2 \in \mathrm{dom} f$ and $\alpha \in (0,1)$,

$$f((1-\alpha)x_1 + \alpha x_2) \leq (1-\alpha)f(x_1) + \alpha f(x_2),$$

and its Fenchel conjugate function f^* is defined as follows: for any $\xi \in \mathbb{R}^n$,

$$f^*(\xi) = \sup\{\langle \xi, x\rangle - f(x) \mid x \in \mathrm{dom} f\}.$$

We know that $f^* : \mathbb{R}^n \to (-\infty, \infty]$ is proper convex lower semicontinuous, and also if f is lower semicontinuous, then we have

$$f = f^{**},$$

that is, lower semicontinuity and the biconjugacy are equivalent for any proper convex function. It is well-known that this property plays very important roles to consider dual problems of convex minimization problem.

Similar researches of conjugates of quasiconvex functions, have been observed, see [1, 2, 3, 4, 5]. Various types of conjugates are introduced, and biconjugacies of functions are investigated. In this paper, we give a sufficient condition which assures biconjugacy is satisfied for a notion of conjugate, called 0-quasiconjugate.

2 Conjugates of quasiconvex functions

Remember that f is said to be quasiconvex if, for all $x_1, x_2 \in \mathrm{dom}f$ and $\alpha \in (0,1)$,

$$f((1-\alpha)x_1 + \alpha x_2) \leq \max\{f(x_1), f(x_2)\},$$

or equivalently, for any $\alpha \in \mathbb{R}$, its level set

$$L_\alpha(f) = \{x \in \mathbb{R}^n \mid f(x) \leq \alpha\}$$

is a convex set. Clearly, the notion of quasiconvexity is a generalization of convexity. For quasiconvex functions, various conjugates have been defined. In this paper, we treat λ-quasiconjugate, quasiconjugate, and R-quasiconjugate. At first, we mention about λ-quasiconjugate.

Definition 1. *For any $\lambda \in \mathbb{R}$, the λ-quasiconjugate of f is the functional f^ν_λ : $\mathbb{R}^n \to (\infty, \infty]$ defined as follows: for any $\xi \in \mathbb{R}^n$,*

$$f^\nu_\lambda(x^*) = \lambda - \inf\{f(x) \mid \langle x^*, x\rangle \geq \lambda\}.$$

By Greenberg and Pierskalle, the normalized second quasiconjugate is introduced. Note that this notion is not given by two-times iteration of the same operation.

Definition 2. *The normalized second quasiconjugate of f is the functional $f^{\nu\nu}$: $\mathbb{R}^n \to (-\infty, \infty]$ defined as follows: for any $x \in \mathbb{R}^n$,*

$$f^{\nu\nu}(x) = \sup_{\lambda \in \mathbb{R}} (f^\nu_\lambda)^\nu_\lambda(x).$$

Evenly quasiconvexity, defined as follows, assures biconjugacy.

Definition 3. *A subset A of $\mathbb{R}^n$ is evenly convex if there exists a family of open half space such that A is equal to the intersection of the family of open half space.*

Definition 4. *A function f is evenly quasiconvex if for all $\alpha \in (-\infty, \infty]$, $L_\alpha(f)$ is evenly convex.*

Theorem 1. *If a function f is evenly quasiconvex, then $f^{\nu\nu} = f$.*

Theorem 2. *The following formula holds:*

$$f^{\nu\nu} = \max\{(f^\nu_{-1})^\nu_{-1}, (f^\nu_0)^\nu_0, (f^\nu_1)^\nu_1\}.$$

Next, we define notions of quasiconjugate and R-quasiconjugate, which are closely concerned with $(f^\nu_{-1})^\nu_{-1}, (f^\nu_1)^\nu_1$, and we state sufficient conditions to obtain that each biconjugates are equal to f.

Definition 5 ([4]). *Quasiconjugate of f is the functional $f^H : \mathbb{R}^n \to (-\infty, \infty]$ defined by*

$$f^H(\xi) = \begin{cases} -\inf\{f(\xi) \mid \langle x, \xi\rangle \geq 1\} & \text{if} \quad \xi \neq 0 \\ -\sup\{f(x) \mid x \in \mathbb{R}^n\} & \text{if} \quad \xi = 0. \end{cases}$$

The quasiconjugate of the function f^H is called the biquasiconjugate of f and denoted by f^{HH}.

Note that $f_1^\nu = 1 - f^H$ on $\mathbb{R}^n \setminus \{0\}$.

Definition 6. *We say that f achieves the maximum value at the infinite if $f(x_n) \to \sup\{f(x) \mid x \in \mathbb{R}^n\}$ for any sequence $\{x_n\}$ such that $\|x_k\| \to \infty$.*

Theorem 3. *Let f be a lower semicontinuous quasiconvex function satisfying*

$$f(0) = \inf\{f(x) | x \in \mathbb{R}^n \backslash \{0\}\}.$$

If f achieves the maximum value at the infinite, then $f^{HH} = f$.

Definition 7. *R-quasiconjugate of f is the functional $f^R : \mathbb{R}^n \to (-\infty, \infty]$ defined by*

$$f^R(\xi) = -\inf\{f(x) \mid \langle \xi, x\rangle \geq -1\}.$$

The R-quasiconjugate of the function f^R is called the R-biquasiconjugate of f and denoted by f^{RR}.

Note that $f_{-1}^\nu = -1 - f^R$ on $\mathbb{R}^n$.

Definition 8. *A subset A of $\mathbb{R}^n$ is R-evenly convex if the intersection of a family of open half spaces which closure do not contain 0.*

Definition 9. *A function f is R-evenly quasiconvex if, $L_\alpha(f)$ is R-evenly convex for all $\alpha \in (-\infty, \infty]$.*

Theorem 4. *If a function f is R-evenly quasiconvex, then $f^{RR} = f$.*

3 Main theorem

Motivated by Theorems 1, 2, 3, and 4 in the previous section, we consider biconjugacy for 0-quasiconjugate.

Example 1. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = (x - 1)^2 + y^2$. Then we can calculate conjugate*

$$f_0^\nu(a, b) = \begin{cases} -\dfrac{a^2}{a^2 + b^2} & \text{if} \quad a < 0 \\ 0 & \text{if} \quad a \geq 0. \end{cases}$$

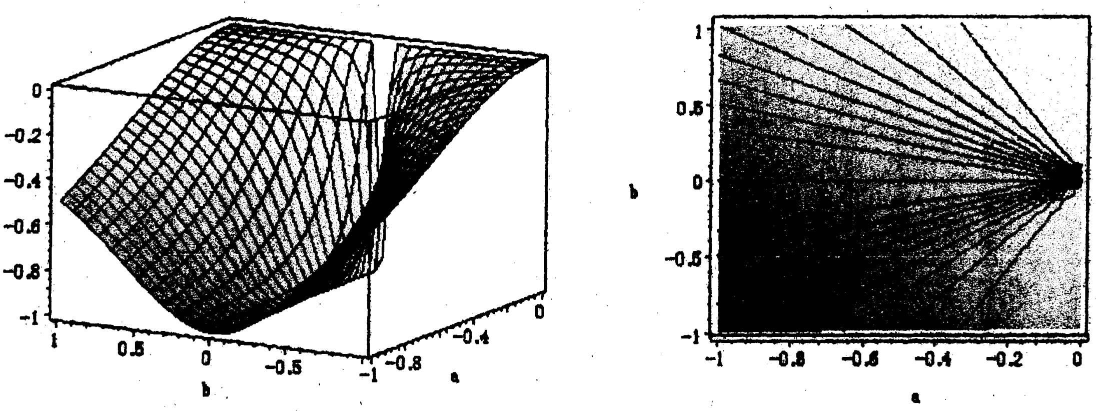

Figure: graph and contour graph of function f_0^ν on $(-\infty, 0) \times \mathbb{R}$.

Let $g = f_0^\nu$, then we have the conjugate g_0^ν as follows:

$$g_0^\nu(x,y) = \begin{cases} \dfrac{y^2}{x^2+y^2} & \text{if } x > 0 \\ 1 & \text{if } x \leq 0. \end{cases}$$

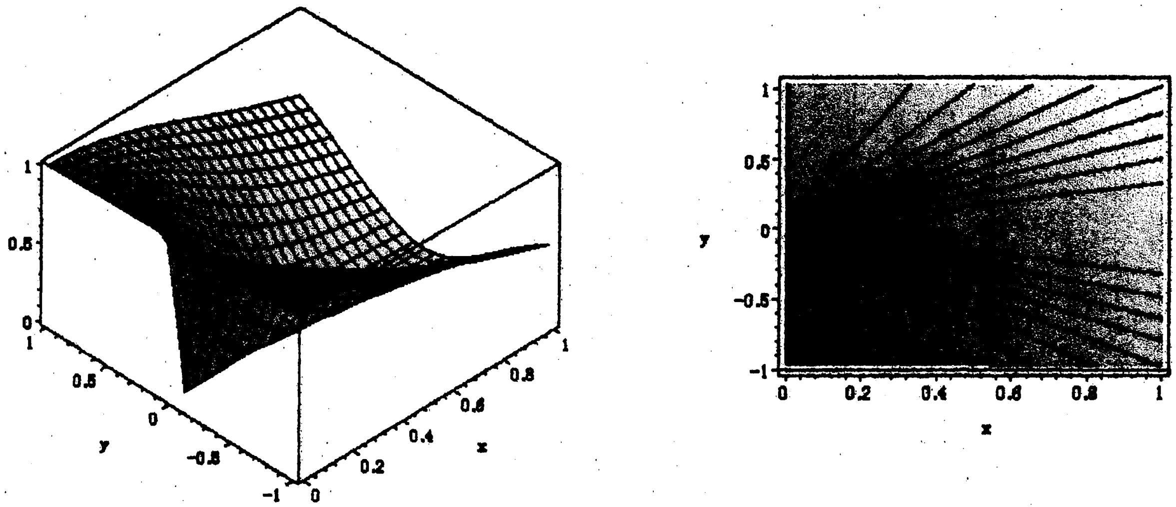

Figure: graph and contour graph of function g_0^ν on $(0, \infty) \times \mathbb{R}$.

From this, we have $(f_0^\nu)_0^\nu \neq f$. However, we can show $(g_0^\nu)_0^\nu = g$.

Inspired the example, we give a sufficient condition for biconjugacy. To the purpose, we show the following properties concerned with *convex cone.*

Lemma 1. *If K be a nonempty closed convex pointed cone in $\mathbb{R}^n$, then* intK^* *is not empty, where $K^* = \{x^* \in \mathbb{R}^n \mid \langle x^*, x\rangle \leq 0, \forall x \in K\}$.*

This proof is omitted. The following lemma is important to show our main result.

Lemma 2. *Let K be a nonempty closed convex pointed cone in $\mathbb{R}^n$. If $x_0 \notin K\backslash\{0\}$, then there exists $a \in \mathbb{R}^n$ such that $\langle a, x_0\rangle \geq 0 > \langle a, x\rangle$ for all $x \in K\backslash\{0\}$.*

Proof. From the assumption, we have $\text{int}K^* \neq \emptyset$ from Lemma 1 and $K = K^{**}$. Since $x_0 \notin K^{**}$, there exists $x^* \in K^*$ such that $\langle x_0, x^*\rangle > 0$. By continuity of the inner product, we can choose $r > 0$ such that $y^* \in B(x^*, r)$ implies $\langle x_0, y^*\rangle > 0$. Choose $z^* \in \text{int}K^*$ such that $z^* \neq x^*$, and let

$$a = \frac{\|x^* - z^*\|}{\|x^* - z^*\| + r}x^* + \frac{r}{\|x^* - z^*\| + r}z^*,$$

then we can check $a \in B(x^*, r)$ and $a \in \text{int}K^*$. Hence we have $\langle a, x_0\rangle > 0$ and $\langle a, x\rangle < 0$ for all $x \in K\backslash\{0\}$. □

Lemma 3. *The following formula holds*

$$f_0^\nu(0) = \max\{f_0^\nu(x^*) \mid x^* \in \mathbb{R}^n\}.$$

Proof. For any $x^* \in \mathbb{R}^n$,

$$f_0^\nu(x^*) = -\inf\{f(x) \mid \langle x^*, x\rangle \geq 0\},$$

and also

$$f_0^\nu(0) = -\inf\{f(x) \mid \langle 0, x\rangle \geq 0\} = -\inf\{f(x) \mid\in x \in \mathbb{R}\},$$

then we have $f_0^\nu(x^*) \leq f_0^\nu(0)$. □

Theorem 5. *Assume that $L_\alpha(f) \cup \{0\}$ is a closed convex pointed cone, or $\mathbb{R}^n$, for all $\alpha \in \mathbb{R}$. If $f(0) = \sup\{f(x) \mid x \in \mathbb{R}^n\backslash\{0\}\}$, then $f = (f_0^\nu)_0^\nu$.*

Proof. It is clear that $f \geq (f_0^\nu)_0^\nu$. Assume that there exists $x_0 \in \mathbb{R}^n$ such that $f(x_0) > (f_0^\nu)_0^\nu(x_0)$. By using Lemma 3 and $f(0) = \sup\{f(x) \mid x \in \mathbb{R}^n\backslash\{0\}\}$, we may assume $x_0 \neq 0$. Choose $\alpha \in \mathbb{R}^n$ satisfying

$$f(x_0) > \alpha > (f_0^\nu)_0^\nu(x_0)$$

Since $x_0 \notin L_\alpha(f) \cup \{0\}$, and $L_\alpha(f) \cup \{0\}$is a closed convex pointed cone, then

$$\exists a \in \mathbb{R}^n \text{ s.t. } \langle a, x_0\rangle \geq 0 > \langle a, x\rangle, \forall x \in L_\alpha(f)$$

by using Lemma 2. This shows

$$x \in L_\alpha(f) \implies \langle a, x\rangle < 0,$$

or equivalently,

$$\langle a, x\rangle \geq 0 \implies f(x) > \alpha.$$

Hence

$$(f_0^\nu)_0^\nu(x_0) = -\inf\{f_0^\nu(x^*) \mid \langle x^*, x_0\rangle \geq 0\} \geq -f_0^\nu(a) = \inf\{f(x) \mid \langle x, a\rangle \geq 0\} \geq \alpha,$$

this is contraction. □

References

[1] H. J. Greenberg and W. P. Pierskalle, Quasi-conjugate functions and surrogate duality, *Cah. Cent. Étud. Rech. Opér* **15** (1973) 437–448.

[2] J.-E. Martinez-Legaz, A generalized concept of conjugation, *Optimization: theory and algorithms (Confolant, 1981), 45–59, Lecture Notes in Pure and Appl. Math., 86, Dekker, New York, 1983.*

[3] I. Singer, The lower semi-continuous quasi-convex hull as a normalized second conjugate, *Nonlinear Analysis, Theory, Methods & Applications* **7** (1983) 1115–1121.

[4] P. T. Thach, Quasiconjugates of functions, duality relationship between quasiconvex minimization under a reverse convex constraint and quasiconvex maximization under a convex constraint, and applications, *JMAA* **159** (1991) 299–322.

[5] P. T. Thach, Diewert-Crouzeix conjugation for general quasiconvex duality and applications, *JOTA* **86** (1995) 719–743.