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On the removability of a level set for solutions to fully nonlinear equations

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1 Introduction

In the early 20th century, Radó [20] proved the following theorem for complex analytic functions.

Theorem 1.1. *Let f be a continuous complex-valued function in a domain $\Omega \subset \mathbb{C}$. If f is analytic in $\Omega \setminus f^{-1}(0)$, then f is actually analytic in the whole domain Ω.*

This result says that a level set is always removable for continuous analytic functions. Later, an analogous result of Radó's result for harmonic functions has been obtained.

Theorem 1.2. *[1, 8, 17] Let u be a real-valued continuously differentiable function defined in a domain $\Omega \subset \mathbb{R}^n$. If u is harmonic in $\Omega \setminus u^{-1}(0)$, then it is harmonic in the whole domain Ω.*

Such removability problems have been intensively studied. The corresponding results for linear elliptic equations were proved by Šabat [21]. The case of p-Laplace equation has been treated in [13, 16]. Recently, Juutinen and Lindqvist [14] proved the removability of a level set for *viscosity* solutions to general quasilinear elliptic and parabolic equations. However, to the best of our knowledge, there are no results concerning such problems for fully nonlinear PDEs.

In this article, we study this type of removability results for *fully nonlinear* equations. The equations which we are concerned with are the following degenerate elliptic, fully nonlinear equation

$$F(x, u, Du, D^2u) = 0, \tag{1.1}$$

in $\Omega \subset \mathbb{R}^n$, or the parabolic one

$$u_t + F(t, x, u, Du, D^2u) = 0, \tag{1.2}$$

in $\mathcal{O} \subset \mathbb{R} \times \mathbb{R}^n$. In both equations, D means the derivation with respect to the space variables, that is,

$$Du := \left(\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n}\right)^T, \quad D^2u := \left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right)_{\substack{1 \le i \le n \\ 1 \le j \le n}}. \tag{1.3}$$

Here A^T denotes the transpose of a matrix A.

In the elliptic case, our problem is written as follows.

> **Problem:** Let $\Omega \subset \mathbb{R}^n$ be a domain. If a function u defined in Ω is a viscosity solution to (1.1) in $\Omega \setminus u^{-1}(0)$, then is it actually a viscosity solution to (1.1) in the *whole* domain Ω?

The problem for the parabolic case is similar. We shall obtain the removability results for (1.1) and (1.2). We also establish this type of removability result for *singular equations*, that is, equations where F is singular at $Du = 0$.

In the following section, we give some notations and state main results of this article. In section 3, we describe the definition and basic properties of viscosity solutions. Our main results are proved in section 4. We extend those removability results to the singular equations in section 5.

2 Notations and main results

We prepare some notations which are used in this article.

- $\mathbb{S}^{n \times n} := \{n \times n \text{ real symmetric matrix}\}$.
- For $X, Y \in \mathbb{S}^{n \times n}$, $\quad X \le Y \overset{\text{def}}{\iff} Y - X$ is non-negative definite.
  (i.e., $(Y - X)\xi \cdot \xi \ge 0$ for all $\xi \in \mathbb{R}^n$.)
- For $X \in \mathbb{S}^{n \times n}$,

$$\begin{aligned} \|X\| &:= \max\big\{|\lambda| \bigm| \lambda \text{ is an eigenvalue of } X.\big\} \\ &= \max\big\{|X\xi \cdot \xi| \bigm| |\xi| \le 1\big\}. \end{aligned} \tag{2.1}$$

- For $\xi, \eta \in \mathbb{R}^n$, $\xi \otimes \eta$ denotes the $n \times n$ matrix with the entries

$$(\xi \otimes \eta)_{ij} = \xi_i \eta_j \quad (i, j \in \{1, \dots, n\}). \tag{2.2}$$

- For $x \in \mathbb{R}^n$ and for $r > 0$,

$$B_r(x) := \{z \in \mathbb{R}^n \mid |z - x| < r\}. \tag{2.3}$$

- For $(t,x) \in \mathbb{R} \times \mathbb{R}^n$ and for $r > 0$,

$$B_r(t,x) := \{(s,z) \in \mathbb{R} \times \mathbb{R}^n \mid (s-t)^2 + |z-x|^2 < r^2\}. \tag{2.4}$$

- Let Ω be an open set in $\mathbb{R}^n$ or $\mathbb{R} \times \mathbb{R}^n$.

$$\mathrm{USC}(\Omega) := \{u : \Omega \to [-\infty, \infty),\ \text{upper semicontinuous}\}, \tag{2.5}$$
$$\mathrm{LSC}(\Omega) := \{u : \Omega \to (-\infty, \infty],\ \text{lower semicontinuous}\}. \tag{2.6}$$

- For $u : \Omega \to \mathbb{R}$, $q \in \mathbb{R}^n$, $X \in \mathbb{S}^{n\times n}$, $\hat{x} \in \Omega$,

$(q, X) \in J^{2,+}u(\hat{x}) \overset{\text{def}}{\Longleftrightarrow}$

$$u(x) \le u(\hat{x}) + q \cdot (x - \hat{x}) + \frac{1}{2} X(x-\hat{x}) \cdot (x-\hat{x}) + o(|x-\hat{x}|^2) \quad \text{as } x \to \hat{x}, \tag{2.7}$$

$(q, X) \in J^{2,-}u(\hat{x}) \overset{\text{def}}{\Longleftrightarrow}$

$$u(x) \ge u(\hat{x}) + q \cdot (x - \hat{x}) + \frac{1}{2} X(x-\hat{x}) \cdot (x-\hat{x}) + o(|x-\hat{x}|^2) \quad \text{as } x \to \hat{x}. \tag{2.8}$$

- For $u : \Omega \to \mathbb{R}$, $\hat{x} \in \Omega$,

$$\begin{aligned} \overline{J}^{2,+}u(x) := \{(q,X) \in \mathbb{R}^n \times \mathbb{S}^{n\times n} \mid &\text{ there exists a sequence} && (2.9)\\ &\{(x_n, q_n, X_n)\} \subset \Omega \times \mathbb{R}^n \times \mathbb{S}^{n\times n} \text{ such that } (q_n, X_n) \in J^{2,+}u(x_n) \\ &\text{and } x_n \to x, u(x_n) \to u(x), q_n \to q, X_n \to X.\}, \\ \overline{J}^{2,-}u(x) := \{(q,X) \in \mathbb{R}^n \times \mathbb{S}^{n\times n} \mid &\text{ there exists a sequence} && (2.10)\\ &\{(x_n, q_n, X_n)\} \subset \Omega \times \mathbb{R}^n \times \mathbb{S}^{n\times n} \text{ such that } (q_n, X_n) \in J^{2,-}u(x_n) \\ &\text{and } x_n \to x, u(x_n) \to u(x), q_n \to q, X_n \to X.\}. \end{aligned}$$

Here we state the result concerning the removability of a level set for solutions to (1.1).

Theorem 2.1. *Let Ω be a domain in $\mathbb{R}^n$. We suppose that $F = F(x, r, q, X)$ satisfies the following conditions.*

(A1) F is a continuous function defined in $\Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}^{n\times n}$.

(A2) F is degenerate elliptic, i.e.,

$$F(x, r, q, X) \ge F(x, r, q, Y) \tag{2.11}$$

for every $x \in \Omega$, $r \in \mathbb{R}$, $q \in \mathbb{R}^n$, $X, Y \in \mathbb{S}^{n\times n}$ with $X \le Y$.

(A3) $F(x,0,0,O) = 0$ *for every* $x \in \Omega$.

(A4) There exists a constant $\alpha > 2$ *such that for every compact subset* $K \Subset \Omega$ *we can find positive constants* ε, C *and a continuous, non-decreasing function* $\omega_K : [0,\infty) \to [0,\infty)$ *which satisfy* $\omega_K(0) = 0$ *and the following:*

$$F(y,s,j|x-y|^{\alpha-2}(x-y),Y) - F(x,r,j|x-y|^{\alpha-2}(x-y),X) \qquad (2.12)$$
$$\leq \omega_K(|r-s| + j|x-y|^{\alpha-1} + |x-y|)$$

whenever $x,y \in K$, $r,s \in (-\varepsilon,\varepsilon)$, $j \geq C$, $X,Y \in \mathbb{S}^{n\times n}$ *and*

$$-\left(j + j(\alpha-1)|x-y|^{\alpha-2}\right) I_{2n} \leq \begin{pmatrix} X & O \\ O & -Y \end{pmatrix} \qquad (2.13)$$
$$\leq \left(j(\alpha-1)|x-y|^{\alpha-2} + 2j(\alpha-1)^2|x-y|^{2\alpha-4}\right) \begin{pmatrix} I_n & -I_n \\ -I_n & I_n \end{pmatrix}$$

holds.

If $u \in C^1(\Omega)$ *is a viscosity solution to (1.1) in* $\Omega \setminus u^{-1}(0)$, *then* u *is a viscosity solution to (1.1) in the whole domain* Ω.

Remark 2.1. We remark about the regularity assumption on u. This theorem also holds if we only assume that u is continuously differentiable on some neighborhood of $\{u = 0\}$ instead of assuming that $u \in C^1(\Omega)$. However, one can not weaken the differentiability assumption. More precisely, if we replace $u \in C^1(\Omega)$ by $u \in C^{0,1}(\Omega)$, the conclusion fails to hold. Define the function u by

$$u(x) = |x_1|, \quad x = (x_1, \ldots, x_n) \in \Omega = B_1 = \{|x| < 1\}. \qquad (2.14)$$

It is easily checked that u satisfies $-\Delta u = 0$ in $\Omega \setminus u^{-1}(0) = B_1 \setminus \{x_1 = 0\}$ in the classical sense as well as in the viscosity sense. But u does not satisfy $-\Delta u = 0$ in B_1 in the viscosity sense.

In Theorem 2.1, the conditions (A1) and (A2) are quite natural, and it is necessary to assume (A3) since the function $u \equiv 0$ must be a solution to (1.1). However, the condition (A4) seems to be complicated and artificial. For the particular case that F can be expressed as $F(x,r,q,X) = \tilde{F}(q,X)$ or $\tilde{F}(q,X) + f(r)$, the hypotheses can be simplified as follows.

Corollary 2.2. *Let* Ω *be a domain in* $\mathbb{R}^n$. *We suppose that* $\tilde{F} = \tilde{F}(q,X)$ *and* $f = f(r)$ *satisfy the following conditions.*

(B1) $\tilde{F}$ *is a continuous function defined in* $\mathbb{R}^n \times \mathbb{S}^{n\times n}$ *and* f *is a continuous function defined in* $\mathbb{R}$.

(B2) $\tilde{F}$ *is degenerate elliptic.*

(B3) $\tilde{F}(0,O)+f(0)=0$.

If $u\in C^1(\Omega)$ *is a viscosity solution to*

$$\tilde{F}(Du,D^2u)+f(u)=0 \tag{2.15}$$

in $\Omega\setminus u^{-1}(0)$, *then* u *is a viscosity solution to (2.15) in the whole domain* Ω.

Next we state our removability result for parabolic equations (1.2).

Theorem 2.3. *Let* $\mathcal{O}$ *be a domain in* $\mathbb{R}\times\mathbb{R}^n$. *We suppose that the conditions given below are satisfied.*

(C1) F *is a continuous function defined in* $\mathcal{O}\times\mathbb{R}\times\mathbb{R}^n\times\mathbb{S}^{n\times n}$.

(C2) F *is degenerate elliptic.*

(C3) $F(t,x,0,0,O)=0$ *for every* $(t,x)\in\mathcal{O}$.

(C4) There exists a constant $\alpha>2$ *such that for every compact subset* $K\Subset\mathcal{O}$ *we can find positive constants* ε, C *and a continuous, non-decreasing function* $\omega_K:[0,\infty)\to[0,\infty)$ *which satisfy* $\omega_K(0)=0$ *and the following:*

$$\begin{aligned} F(t',y,s,j|x-y|^{\alpha-2}(x-y),Y)-F(t,x,r,j|x-y|^{\alpha-2}(x-y),X)& \\ \le\omega_K(|t-t'|+|r-s|+j|x-y|^{\alpha-1}+|x-y|)& \end{aligned} \tag{2.16}$$

whenever $(t,x),(t',y)\in K$, $r,s\in(-\varepsilon,\varepsilon)$, $j\ge C$, $X,Y\in\mathbb{S}^{n\times n}$ *and*

$$\begin{aligned} -\left(j+j(\alpha-1)|x-y|^{\alpha-2}\right)I_{2n}\le\begin{pmatrix}X&O\\O&-Y\end{pmatrix}& \\ \le\left(j(\alpha-1)|x-y|^{\alpha-2}+2j(\alpha-1)^2|x-y|^{2\alpha-4}\right)\begin{pmatrix}I_n&-I_n\\-I_n&I_n\end{pmatrix}& \end{aligned} \tag{2.17}$$

holds.

If $u\in C^1(\mathcal{O})$ *is a viscosity solution to (1.2) in* $\mathcal{O}\setminus u^{-1}(0)$, *then* u *is a viscosity solution to (1.2) in the whole domain* $\mathcal{O}$.

Remark 2.2. For F of the form $\tilde{F}(q,X)+f(r)$, a level set of a viscosity solution to (1.2) is always removable if we assume the continuity of $\tilde{F}$ and f, the degenerate ellipticity of $\tilde{F}$, and $\tilde{F}(0,O)+f(0)=0$ only, as in the elliptic case.

Example 2.1. Utilizing Theorem 2.1 or Corollary 2.2, and Theorem 2.3, one sees that our removability results can be applied to many well-known equations. Here are the examples.

(i) Laplace equation $-\Delta u = 0$, cf. [1, 8, 17].

(ii) The heat equation $u_t - \Delta u = 0$.

(iii) Poisson equation $-\Delta u = f(u)$, where $f(0) = 0$ and f is continuous, for example, $f(u) = |u|^{p-1}u$ $(p > 0)$.

(iv) Linear elliptic equations

$$-\sum_{i,j=1}^{n} a_{ij}(x)D_{ij}u(x) + \sum_{i=1}^{n} b_i(x)D_iu(x) + c(x)u(x) = 0, \tag{2.18}$$

cf. Šabat [21].

(v) Quasilinear elliptic equations

$$-\sum_{i,j=1}^{n} a_{ij}(x,u,Du)D_{ij}u(x) + b(x,u,Du) = 0, \tag{2.19}$$

such as the minimal surface equation $-\operatorname{div}(Du/\sqrt{1+|Du|^2}) = 0$, p-Laplace equation $-\Delta_p u := -\operatorname{div}(|Du|^{p-2}Du) = 0$ $(p \geq 2)$ and ∞-Laplace equation $\sum_{i,j=1}^{n} D_iuD_juD_{ij}u = 0$, cf. Juutinen and Lindqvist [14]. We note that our result does not contain theirs, but that is because they utilize the quasilinear nature of the equation.

(vi) Quasilinear parabolic equations, such as p-Laplace diffusion equation $u_t - \Delta_p u = 0$.

(vii) Pucci's equation, which is an important example of fully nonlinear uniformly elliptic equation,

$$-\mathcal{M}^+_{\lambda,\Lambda}(D^2u) = f(u), \qquad -\mathcal{M}^-_{\lambda,\Lambda}(D^2u) = f(u), \tag{2.20}$$

where $\mathcal{M}^+_{\lambda,\Lambda}$, $\mathcal{M}^-_{\lambda,\Lambda}$ are the so-called Pucci extremal operators with parameters $0 < \lambda \leq \Lambda$ defined by

$$\mathcal{M}^+_{\lambda,\Lambda}(X) = \Lambda\sum_{e_i>0} e_i + \lambda\sum_{e_i<0} e_i, \quad \mathcal{M}^-_{\lambda,\Lambda}(X) = \lambda\sum_{e_i>0} e_i + \Lambda\sum_{e_i<0} e_i, \tag{2.21}$$

for $X \in \mathbb{S}^{n\times n}$ (see [2, 19]). Here $e_1, \ldots, e_n$ are the eigenvalues of X.

(viii) Monge-Ampère equation

$$\det D^2u = f(u). \tag{2.22}$$

When we are concerned with (2.22), we look for solutions in the class of convex functions. It is known that the equation (2.22) is not elliptic on all C^2 functions; it is degenerate elliptic for only C^2 *convex* functions. In this case, the condition (A2) is not satisfied. However, modifying our argument below appropriately, one can also apply Theorem 2.1 to (2.22) and obtain the removability result.

(ix) The parabolic Monge-Ampère equation $u_t - (\det D^2u)^{1/n} = 0$.

(x) k-Hessian equation

$$F_k[u] = S_k(\lambda_1, \dots, \lambda_n) = f(u), \tag{2.23}$$

where $\lambda = (\lambda_1, \dots, \lambda_n)$ denotes the eigenvalues of D^2u and S_k $(k = 1, \dots, n)$ denotes the k-th elementary symmetric function, that is,

$$S_k(\lambda) = \sum \lambda_{i_1} \cdots \lambda_{i_k}, \tag{2.24}$$

where the sum is taken over increasing k-tuples, $1 \le i_1 < \cdots < i_k \le n$. Thus $F_1[u] = \Delta u$ and $F_n[u] = \det D^2u$, which we have seen before. This equation has been intensively studied, see for example [3, 24, 25, 26].

(xi) Gauss curvature equation

$$\det D^2u = f(u)\left(1 + |Du|^{(n+2)/2}\right). \tag{2.25}$$

(xii) Gauss curvature flow equation $u_t - \det D^2u/\left(1 + |Du|^2\right)^{(n+1)/2} = 0$.

(xiii) k-curvature equation

$$H_k[u] = S_k(\kappa_1, \dots, \kappa_n) = f(u), \tag{2.26}$$

where $\kappa_1, \dots, \kappa_n$ denote the principal curvatures of the graph of the function u, and S_k is the k-th elementary symmetric function. The mean, scalar and Gauss curvature equation correspond respectively to the special cases $k = 1, 2, n$ in (2.26). For the classical Dirichlet problem for k-curvature equations in the case that $2 \le k \le n-1$, see for instance [4, 11, 23].

In the last section, we also prove the removability of a level set for solutions to the singular equations such as p-Laplace diffusion equation where $1 < p < 2$. See Theorems 5.2 and 5.4, and subsequent remarks.

3 The notion of viscosity solutions

In this section we recall the notion of viscosity solutions to the fully nonlinear equations, (1.1) and (1.2). The theory of viscosity solutions to fully nonlinear equations was developed by Crandall, Evans, Ishii, Jensen, Lions and others. See, for example, [6, 7, 9, 12].

First we define a viscosity solution to (1.1).

Definition 3.1. Let Ω be a domain in $\mathbb{R}^n$. Assume that (A1) and (A2) in Theorem 2.1 are satisfied.

(i) A function $u \in \mathrm{USC}(\Omega)$ is said to be a *viscosity subsolution* to (1.1) in Ω if $u \not\equiv -\infty$ and for any function $\varphi \in C^2(\Omega)$ and any point $x_0 \in \Omega$ which is a maximum point of $u - \varphi$, we have

$$F(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)) \leq 0. \tag{3.1}$$

(ii) A function $u \in \mathrm{LSC}(\Omega)$ is said to be a *viscosity supersolution* to (1.1) in Ω if $u \not\equiv \infty$ and for any function $\varphi \in C^2(\Omega)$ and any point $x_0 \in \Omega$ which is a minimum point of $u - \varphi$, we have

$$F(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)) \geq 0. \tag{3.2}$$

(iii) A function $u \in C^0(\Omega)$ is said to be a *viscosity solution* to (1.1) in Ω if it is both a viscosity subsolution and supersolution to (1.1) in Ω.

We omit the proof of the following proposition.

Proposition 3.2. *Let Ω be a domain in $\mathbb{R}^n$ and assume (A1) and (A2) in Theorem 2.1 are satisfied. If $u \in USC(\Omega)$ (resp. $u \in LSC(\Omega)$) is a viscosity subsolution (resp. viscosity supersolution) to (1.1) in Ω, then $F(\hat{x}, u(\hat{x}), q, X) \leq 0$ (resp. $F(\hat{x}, u(\hat{x}), q, X) \geq 0$) for every $\hat{x} \in \Omega$ and every $(q, X) \in \overline{J}^{2,+}u(\hat{x})$ (resp. $(q, X) \in \overline{J}^{2,-}u(\hat{x})$).*

Next we introduce another notion of viscosity solutions to the elliptic equation (1.1), which we call relaxed viscosity solutions. The difference between the definition of viscosity solutions and the following one is that nothing is required if the test function φ satisfies $D\varphi(x_0) = 0$.

Definition 3.3. Let Ω be a domain in $\mathbb{R}^n$. Assume that (A1) and (A2) in Theorem 2.1 are satisfied.

(i) A function $u \in \mathrm{USC}(\Omega)$ is said to be a *relaxed viscosity subsolution* to (1.1) in Ω if $u \not\equiv -\infty$ and for any function $\varphi \in C^2(\Omega)$ and any point $x_0 \in \Omega$, which is a maximum point of $u - \varphi$ and satisfies $D\varphi(x_0) \neq 0$, we have

$$F(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)) \leq 0. \tag{3.3}$$

(ii) A function $u \in \mathrm{LSC}(\Omega)$ is said to be a *relaxed viscosity supersolution* to (1.1) in Ω if $u \not\equiv \infty$ and for any function $\varphi \in C^2(\Omega)$ and any point $x_0 \in \Omega$ which is a minimum point of $u - \varphi$ and satisfies $D\varphi(x_0) \neq 0$, we have

$$F(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)) \geq 0. \tag{3.4}$$

(iii) A function $u \in C^0(\Omega)$ is said to be a *relaxed viscosity solution* to (1.1) in Ω if it is both a relaxed viscosity subsolution and supersolution to (1.1) in Ω.

It is trivial that if u is a viscosity solution, then it is a relaxed viscosity solution. We shall show in the following section that under some assumptions, the notion of viscosity solutions and that of relaxed viscosity solutions are equivalent, which is proved for the case of quasilinear equations in [14]. Namely, we require no testing at all at the points where the gradient of φ vanishes in the definition of viscosity solutions. See Proposition 4.1.

Furthermore, utilizing this definition, we can define the notion of viscosity solutions to singular equations in the sense that $F(x, r, q, X)$ in (1.1) is defined and degenerate elliptic only on $\{q \neq 0\}$, for example, p-Laplace equation in the case $1 < p < 2$. In section 5, we state the Radó type removability result for singular equations.

In the last part of this section, we recall the definition of viscosity solutions to the parabolic equation (1.2).

Definition 3.4. Let $\mathcal{O}$ be a domain in $\mathbb{R} \times \mathbb{R}^n$. We assume (C1) and (C2) are satisfied.

(i) A function $u \in \mathrm{USC}(\mathcal{O})$ is said to be a *viscosity subsolution* to (1.2) in $\mathcal{O}$ if $u \not\equiv -\infty$ and for any function $\varphi \in C^2(\mathcal{O})$ and any point $(t_0, x_0) \in \mathcal{O}$ which is a maximum point of $u - \varphi$, we have

$$\varphi_t(t_0, x_0) + F(t_0, x_0, u(t_0, x_0), D\varphi(t_0, x_0), D^2\varphi(t_0, x_0)) \leq 0. \tag{3.5}$$

(ii) A function $u \in \mathrm{LSC}(\mathcal{O})$ is said to be a *viscosity supersolution* to (1.2) in $\mathcal{O}$ if $u \not\equiv \infty$ and for any function $\varphi \in C^2(\mathcal{O})$ and any point $(t_0, x_0) \in \mathcal{O}$ which is a minimum point of $u - \varphi$, we have

$$\varphi_t(t_0, x_0) + F(t_0, x_0, u(t_0, x_0), D\varphi(t_0, x_0), D^2\varphi(t_0, x_0)) \geq 0. \tag{3.6}$$

(iii) A function $u \in C^0(\mathcal{O})$ is said to be a *viscosity solution* to (1.2) in $\mathcal{O}$ if it is both a viscosity subsolution and supersolution to (1.2) in $\mathcal{O}$.

4 Proof of the main results

In this section we prove Theorem 2.1 and Corollary 2.2. The proof of Theorem 2.3 is similar to that of Theorem 2.1, so that we omit the proof of Theorem 2.3. See [22] for the detail.

First we show the removability of a level set for solutions to (1.1), Theorem 2.1. Our idea of the proof is adapted from that of Juutinen and Lindqvist [14].

We shall show that u is a viscosity subsolution to (1.1) in the whole domain Ω. To the contrary, we suppose that there exist a point $x_0 \in \Omega$ and a function $\varphi \in C^2(\Omega)$ such that

$$u(x_0) = \varphi(x_0), \tag{4.1}$$

$$u(x) < \varphi(x) \quad \text{for } x \in \Omega \setminus \{x_0\}, \tag{4.2}$$

and that

$$\mu := F(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)) > 0. \tag{4.3}$$

Here we note that $u(x_0)$ must be 0 since u is a viscosity subsolution to (1.1) in $\Omega \setminus u^{-1}(0)$.

$\boxed{\textbf{Case 1.}}$ We assume that $D\varphi(x_0) \neq 0$. Then it holds from (4.1) and (4.2) that $Du(x_0) = D\varphi(x_0) \neq 0$. Here we used the assumption that u is a differentiable function.

Therefore it follows from the implicit function theorem that $\{u = 0\}$ and $\{\varphi = 0\}$ are a C^1-hypersurface and a C^2-hypersurface in some neighborhood of x_0, respectively. This fact, together with (4.1) and (4.2), implies that there exist positive constants ρ and $\rho' \in (0, \rho/2)$ and a point $z \in \{\varphi < 0\}$ such that

$$B_{\rho'}(z) \subset \{\varphi < 0\} \cap B_\rho(x_0) \subset \{u < 0\} \cap B_\rho(x_0) \tag{4.4}$$

and $x_0 \in \partial B_{\rho'}(z)$ (see [14, Figure 3.1]). Without loss of generality, we may assume that $x_0 = 0$ and $z = (0, \ldots, 0, \rho')$.

For $\delta \in (0, \rho')$, we define ψ_δ by

$$\psi_\delta(x) = \varphi(x) - \left(\delta^2 x_n - \frac{\delta}{2}|x|^2\right). \tag{4.5}$$

Then $w_\delta := u - \psi_\delta$ satisfies the following:

(i) $D_n w_\delta(0) = D_n(u - \varphi)(0) + \delta^2 = \delta^2 > 0.$

(ii) $w_\delta(0) = u(0) - \varphi(0) = 0.$

(iii) if $\delta^2 x_n = \delta|x|^2/2$, i.e., $x \in \partial B_\delta(0,\dots,0,\delta)$, then

$$w_\delta(x) = u(x) - \varphi(x) \le 0. \tag{4.6}$$

Thus there exists a point $\tilde{x}_\delta \in B_\delta(0,\dots,0,\delta)$ such that

$$\sup\{w_\delta(x) \mid x \in \overline{B_\delta(0,\dots,0,\delta)}\} = w_\delta(\tilde{x}_\delta). \tag{4.7}$$

Since $\tilde{x}_\delta \in B_\delta(0,\dots,0,\delta) \subset B_{\rho'}(z) \subset \{u < 0\}$ and u is a viscosity subsolution to (1.1) in $\Omega \setminus u^{-1}(0)$, we have

$$F(\tilde{x}_\delta, u(\tilde{x}_\delta), D\psi_\delta(\tilde{x}_\delta), D^2\psi_\delta(\tilde{x}_\delta)) \le 0. \tag{4.8}$$

We see that $\tilde{x}_\delta \to 0$ as $\delta \to +0$. And furthermore,

$$u(\tilde{x}_\delta) \to u(0) = 0, \tag{4.9}$$

$$D\psi_\delta(\tilde{x}_\delta) = D\varphi(\tilde{x}_\delta) - \delta^2(0,\dots,0,1)^T + \delta\tilde{x}_\delta \to D\varphi(0), \tag{4.10}$$

$$D^2\psi_\delta(\tilde{x}_\delta) = D^2\varphi(\tilde{x}_\delta) + \delta I_n \to D^2\varphi(0) \tag{4.11}$$

as $\delta \to +0$. Taking $\delta \to +0$ in (4.8), we obtain by the condition (A1) that

$$F(0,0,D\varphi(0),D^2\varphi(0)) = \mu \le 0, \tag{4.12}$$

which is contradictory to (4.3).

Case 2. We assume that $D\varphi(x_0) = 0$. As is mentioned in the previous section, under some hypotheses we need no testing at all if $D\varphi = 0$ in the definition of viscosity solutions. Indeed we have the following proposition.

Proposition 4.1. *Suppose that (A1) and (A2) in Theorem 2.1 and the conditions given below are satisfied.*

(A3)' $F(x,r,0,O) = 0$ *for every* $x \in \Omega$ *and every* $r \in \mathbb{R}$.

(A4)' There exists a constant $\alpha > 2$ *such that for every compact subset* $K \Subset \Omega \times \mathbb{R}$ *we can find a constant* $C > 0$ *and a continuous, non-decreasing function* $\omega_K : [0,\infty) \to [0,\infty)$ *which satisfy* $\omega_K(0) = 0$ *and the following:*

$$\begin{aligned} F(y,s,j|x-y|^{\alpha-2}(x-y),Y) - F(x,r,j|x-y|^{\alpha-2}(x-y),X) \\ \le \omega_K(|r-s| + j|x-y|^{\alpha-1} + |x-y|) \end{aligned} \tag{4.13}$$

whenever $(x,r),(y,s) \in K$, $j \ge C$, $X,Y \in \mathbb{S}^{n\times n}$ *and*

$$\begin{aligned} -\left(j + j(\alpha-1)|x-y|^{\alpha-2}\right) I_{2n} \le \begin{pmatrix} X & O \\ O & -Y \end{pmatrix} \\ \le \left(j(\alpha-1)|x-y|^{\alpha-2} + 2j(\alpha-1)^2|x-y|^{2\alpha-4}\right) \begin{pmatrix} I_n & -I_n \\ -I_n & I_n \end{pmatrix} \end{aligned} \tag{4.14}$$

holds.

Then $u \in C(\Omega)$ is a relaxed viscosity subsolution (resp. supersolution, solution) to (1.1) if and only if it is a viscosity subsolution (resp. supersolution, solution) to (1.1).

Proof. We prove the subsolution case only. Other cases can be proved similarly. The "if" part is trivial.

To prove the "only if" part, we argue by contradiction. We suppose that there exist a point $x_0 \in \Omega$ and a function $\varphi \in C^2(\Omega)$ such that

$$D\varphi(x_0) = 0, \tag{4.15}$$

$$u(x_0) = \varphi(x_0), \tag{4.16}$$

$$u(x) < \varphi(x) \quad \text{for } x \in \Omega \setminus \{x_0\}, \tag{4.17}$$

and that

$$\mu := F(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)) > 0. \tag{4.18}$$

Fix a constant $R > 0$ such that $B_R := B_R(x_0) \Subset \Omega$.

We use the technique that we double the number of variables and penalize the doubling, as discussed in [7]. For $j \in \mathbb{N}$, we define $\psi_j = \psi_j(x, y)$ by

$$\psi_j(x, y) = \frac{j}{\alpha}|x - y|^\alpha \tag{4.19}$$

and set

$$w_j(x, y) = u(x) - \varphi(y) - \psi_j(x, y). \tag{4.20}$$

Then there exists $(x_j, y_j) \in \overline{B_R} \times \overline{B_R}$ which satisfies

$$w_j(x_j, y_j) = \max_{(x,y)\in\overline{B_R}\times\overline{B_R}} w_j(x, y). \tag{4.21}$$

One can show the following:

$$\lim_{j\to\infty} \frac{j}{\alpha}|x_j - y_j|^\alpha = 0, \quad \lim_{j\to\infty}(x_j, y_j) = (x_0, x_0), \tag{4.22}$$

see [7, Proposition 3.7]. Thus $(x_j, y_j) \in B_R \times B_R$ for sufficiently large j. From now on we assume j is sufficiently large. Since $w_j(x_j, y) \le w_j(x_j, y_j)$ for every point $y \in B_R$, we have

$$\varphi(y) \ge \varphi(y_j) + \psi_j(x_j, y_j) - \psi_j(x_j, y). \tag{4.23}$$

for all $y \in B_R$. We denote the right hand side of (4.23) by $\Psi_j(y)$.

It follows from (4.23) and the equality $\varphi(y_j) = \Psi_j(y_j)$ that

$$D\varphi(y_j) = D\Psi_j(y_j) = j|x_j - y_j|^{\alpha-2}(x_j - y_j), \tag{4.24}$$

$$\begin{aligned} D^2\varphi(y_j) &\geq D^2\Psi_j(y_j) \\ &= -j|x_j - y_j|^{\alpha-2}I_n \\ &\qquad - j(\alpha-2)|x_j - y_j|^{\alpha-4}(x_j - y_j) \otimes (x_j - y_j). \end{aligned} \tag{4.25}$$

We first deal with the case that $x_j = y_j$ for infinitely many j's. Passing to a subsequence if necessary, we may assume that $x_j = y_j$ for all $j \in \mathbb{N}$. By (3.24) and (3.25), we obtain that $D\varphi(y_j) = 0$ and $D^2\varphi(y_j) \geq O$. Therefore the conditions (A2) and (A3)' yield

$$F(y_j, \varphi(y_j), D\varphi(y_j), D^2\varphi(y_j)) \leq F(y_j, \varphi(y_j), 0, O) = 0 \tag{4.26}$$

for all $j \in \mathbb{N}$. As $j \to \infty$, it follows from (4.22) and (A1) that

$$\mu = F(x_0, \varphi(x_0), D\varphi(x_0), D^2\varphi(x_0)) \leq 0, \tag{4.27}$$

which contradicts (4.18).

Next we consider the case that there exists $j_0 \in \mathbb{N}$ such that $x_j \neq y_j$ for all $j \geq j_0$. By the maximum principle for semicontinuous functions (see [7]), we have that there exist $X_j, Y_j \in \mathbb{S}^{n\times n}$ such that

$$(D_x\psi_j(x_j, y_j), X_j) \in \overline{J}^{2,+}u(x_j), \tag{4.28}$$

$$(-D_y\psi_j(x_j, y_j), Y_j) \in \overline{J}^{2,-}\varphi(y_j), \tag{4.29}$$

$$-(j + \|A_j\|)I_{2n} \leq \begin{pmatrix} X_j & O \\ O & -Y_j \end{pmatrix} \leq A_j + \frac{1}{j}A_j{}^2, \tag{4.30}$$

where $A_j = D^2\psi_j(x_j, y_j) = \begin{pmatrix} D^2_{xx}\psi_j(x_j, y_j) & D^2_{xy}\psi_j(x_j, y_j) \\ D^2_{yx}\psi_j(x_j, y_j) & D^2_{yy}\psi_j(x_j, y_j) \end{pmatrix}$. In this case ψ_j is defined by (4.19), so that we can calculate the last inequality (4.30) as

$$\begin{aligned} &-\left(j + j(\alpha-1)|x_j - y_j|^{\alpha-2}\right)I_{2n} \leq \begin{pmatrix} X_j & O \\ O & -Y_j \end{pmatrix} \\ &\leq j\left(|x_j - y_j|^{\alpha-2} + 2|x_j - y_j|^{2\alpha-4}\right)\begin{pmatrix} I_n & -I_n \\ I_n & -I_n \end{pmatrix} \\ &\quad + j(\alpha-2)\left(|x_j - y_j|^{\alpha-4} + 2\alpha|x_j - y_j|^{2\alpha-6}\right) \\ &\qquad \times \begin{pmatrix} (x_j - y_j) \otimes (x_j - y_j) & -(x_j - y_j) \otimes (x_j - y_j) \\ -(x_j - y_j) \otimes (x_j - y_j) & (x_j - y_j) \otimes (x_j - y_j) \end{pmatrix} \\ &\leq \left(j(\alpha-1)|x_j - y_j|^{\alpha-2} + 2j(\alpha-1)^2|x_j - y_j|^{2\alpha-4}\right)\begin{pmatrix} I_n & -I_n \\ I_n & -I_n \end{pmatrix}. \end{aligned} \tag{4.31}$$

Next, since $x_j \neq y_j$ for $j \geq j_0$, it holds that

$$D_x\psi_j(x_j, y_j) = -D_y\psi_j(x_j, y_j) = j|x_j - y_j|^{\alpha-2}(x_j - y_j) \neq 0, \tag{4.32}$$

for $j \geq j_0$. From (4.18), (4.28), (4.29) and the fact that u is a relaxed viscosity subsolution to (1.1), it follows that

$$F(x_j, u(x_j), j|x_j - y_j|^{\alpha-2}(x_j - y_j), X_j) \leq 0, \tag{4.33}$$

$$F(y_j, \varphi(y_j), j|x_j - y_j|^{\alpha-2}(x_j - y_j), Y_j) \geq \mu \tag{4.34}$$

for $j \geq j_0$. Moreover, by (4.15), (4.22) and (4.24)

$$j|x_j - y_j|^{\alpha-2}(x_j - y_j) = D\varphi(y_j) \to D\varphi(x_0) = 0 \quad \text{as } j \to \infty, \tag{4.35}$$

and thus

$$j|x_j - y_j|^{\alpha-1} \to 0 \quad \text{as } j \to \infty. \tag{4.36}$$

Finally, by (4.16), (4.22), (4.33), (4.34), (4.36) and the condition (A4)', we obtain

$$\begin{aligned} \mu \leq\ & F(y_j, \varphi(y_j), j|x_j - y_j|^{\alpha-2}(x_j - y_j), Y_j) \\ & - F(x_j, u(x_j), j|x_j - y_j|^{\alpha-2}(x_j - y_j), X_j) \\ \leq\ & \omega_K(|u(x_j) - \varphi(y_j)| + j|x_j - y_j|^{\alpha-1} + |x_j - y_j|) \to 0 \end{aligned} \tag{4.37}$$

as $j \to \infty$. We reach a contradiction. □

Let us mention again that if u is assumed to be a viscosity subsolution to (1.1) in $\{u \neq 0\}$, then $u(x_0)$ and $\varphi(x_0)$ must be 0. Therefore, in our setting the inequalities (4.27) and (4.37) hold if we only assume (A3) and (A4) instead of (A3)' and (A4)'. Thus we conclude that u is a viscosity subsolution to (1.1) in the whole domain Ω and it can be proved by analogous arguments that u is a supersolution to (1.1) in Ω. This completes the proof of Theorem 2.1.

Next we prove Corollary 2.2. It is enough to check that (A1), (A2), (A3) and (A4) are satisfied when we set $F(x, r, q, X) = \tilde{F}(q, X) + f(r)$. It is trivial that our conditions (B1), (B2) and (B3) imply (A1), (A2) and (A3) respectively. (A4) follows from the conditions (B1) and (B2), and the fact that (2.13) implies $X \leq Y$.

5 Removability results for singular equations

In this section we focus on the fully nonlinear equations (1.1), (1.2) which are *singular* in the sense that F is not defined on $\{Du = 0\}$. Typical examples are

p-Laplace equation $-\Delta_p u = 0$ and p-Laplace diffusion equation $u_t - \Delta_p u = 0$ where $1 < p < 2$, and the mean curvature flow equation

$$u_t - |Du| \operatorname{div}\left(\frac{Du}{|Du|}\right) = 0 \tag{5.1}$$

which says that every level set $\Gamma_c := \{u(t,\cdot) = c\}$ moves by its mean curvature provided $|Du| \neq 0$ on Γ_c. It is important to study singular equations because such equations appear in physics and geometry.

Hereafter we deal with the particular case that F depends only on Du and D^2u variable. The equations we consider are

$$F(Du, D^2u) = 0, \tag{5.2}$$

$$u_t + F(Du, D^2u) = 0. \tag{5.3}$$

Let us remark that F is not necessarily geometric in the sense of [5]. The notion of viscosity solutions to singular equations, (5.2) and (5.3), is due to Ohnuma and Sato [18] (see also [10, 15]). Let us recall the definition. We introduce some notations and state the assumptions on F.

We define $\mathcal{F}(F)$ and Σ by

$$\mathcal{F}(F) = \Big\{ f \in C^2([0,\infty)) \,\Big|\, f(0) = f'(0) = f''(0) = 0, \tag{5.4}$$

$$f''(r) > 0 \text{ for all } r > 0, \text{ and } \lim_{x\to 0} F(Df(|x|), D^2 f(|x|)) = 0 \Big\},$$

$$\Sigma = \{\sigma \in C^1(\mathbb{R}) \mid \sigma(0) = \sigma'(0) = 0,\ \sigma(t) = \sigma(-t) > 0 \text{ for all } t > 0\}. \tag{5.5}$$

We suppose that $F = F(q, X)$ satisfies the following:

(D1) F is a continuous function defined in $(\mathbb{R}^n \setminus \{0\}) \times \mathbb{S}^{n\times n}$.
(D2) F is degenerate elliptic.
(D3) $\mathcal{F}(F) \neq \emptyset$, and if $f \in \mathcal{F}(F)$ and $a > 0$ then $af \in \mathcal{F}(F)$.

A function u is said to be a viscosity solution to the singular elliptic equation (5.2) if u is a relaxed viscosity solution, which is defined in Definition 3.3, to (5.2). More precisely, we give a definition as follows.

Definition 5.1. Let Ω be a domain in $\mathbb{R}^n$. Assume that (D1), (D2) and (D3) are satisfied.

(i) A function $u \in \mathrm{USC}(\Omega)$ is said to be a *viscosity subsolution* to (5.2) in Ω if $u \not\equiv -\infty$ and for any function $\varphi \in C^2(\Omega)$ and any point $x_0 \in \Omega$, which is a maximum point of $u - \varphi$ and satisfies $D\varphi(x_0) \neq 0$, we have

$$F(D\varphi(x_0), D^2\varphi(x_0)) \leq 0. \tag{5.6}$$

(ii) A function $u \in \mathrm{LSC}(\Omega)$ is said to be a *viscosity supersolution* to (5.2) in Ω if $u \not\equiv \infty$ and for any function $\varphi \in C^2(\Omega)$ and any point $x_0 \in \Omega$ which is a minimum point of $u - \varphi$ and satisfies $D\varphi(x_0) \neq 0$, we have

$$F(D\varphi(x_0), D^2\varphi(x_0)) \geq 0. \tag{5.7}$$

(iii) A function $u \in C^0(\Omega)$ is said to be a *viscosity solution* to (5.2) in Ω if it is both a viscosity subsolution and supersolution to (5.2) in Ω.

Here is our Radó type removability result for (5.2).

Theorem 5.2. *Let Ω be a domain in $\mathbb{R}^n$. We suppose that (D1), (D2) and (D3) are satisfied. If $u \in C^1(\Omega)$ is a viscosity solution to (5.2) in $\Omega \setminus u^{-1}(0)$, then u is a viscosity solution to (5.2) in the whole domain Ω.*

Since the proof of this theorem is the same as Case 1 in the proof of Theorem 2.1, we omit the proof. Theorem 5.2 can be applied, for example, to p-Laplace equation where $1 < p < 2$. We note that for $p \geq 2$, p-Laplace equation has no singularity at $Du = 0$ and has been already covered by Theorem 2.1.

Next we give the notion of viscosity solutions to the singular parabolic equation (5.3). Let $\mathcal{O}$ be a domain in $\mathbb{R} \times \mathbb{R}^n$. We say that a function $\varphi \in C^2(\mathcal{O})$ is *admissible* if for any $(\hat{t}, \hat{x}) \in \mathcal{O}$ with $D\varphi(\hat{t}, \hat{x}) = 0$, there exist $f \in \mathcal{F}(F)$, $\sigma \in \Sigma$ and a constant $\rho > 0$ such that $B_\rho(\hat{t}, \hat{x}) \subset \mathcal{O}$ and

$$|\varphi(t, x) - \varphi(\hat{t}, \hat{x}) - \varphi_t(\hat{t}, \hat{x})(t - \hat{t})| \leq f(|x - \hat{x}|) + \sigma(t - \hat{t}) \tag{5.8}$$

for all $(t, x) \in B_\rho(\hat{t}, \hat{x})$.

Definition 5.3. Let $\mathcal{O}$ be a domain in $\mathbb{R} \times \mathbb{R}^n$. We assume (D1), (D2) and (D3) are satisfied.

(i) A function $u \in \mathrm{USC}(\mathcal{O})$ is said to be a *viscosity subsolution* to (5.3) in $\mathcal{O}$ if $u \not\equiv -\infty$ and for any admissible function $\varphi \in C^2(\mathcal{O})$ and any point $(t_0, x_0) \in \mathcal{O}$ which is a maximum point of $u - \varphi$, we have

$$\begin{cases} \varphi_t(t_0, x_0) + F(D\varphi(t_0, x_0), D^2\varphi(t_0, x_0)) \leq 0 & \text{if } D\varphi(t_0, x_0) \neq 0, \\ \varphi_t(t_0, x_0) \leq 0 & \text{if } D\varphi(t_0, x_0) = 0. \end{cases} \tag{5.9}$$

(ii) A function $u \in \mathrm{LSC}(\mathcal{O})$ is said to be a *viscosity supersolution* to (5.3) in $\mathcal{O}$ if $u \not\equiv \infty$ and for any admissible function $\varphi \in C^2(\mathcal{O})$ and any point $(t_0, x_0) \in \mathcal{O}$ which is a minimum point of $u - \varphi$, we have

$$\begin{cases} \varphi_t(t_0, x_0) + F(D\varphi(t_0, x_0), D^2\varphi(t_0, x_0)) \geq 0 & \text{if } D\varphi(t_0, x_0) \neq 0, \\ \varphi_t(t_0, x_0) \geq 0 & \text{if } D\varphi(t_0, x_0) = 0. \end{cases} \tag{5.10}$$

(iii) A function $u \in C^0(\mathcal{O})$ is said to be a *viscosity solution* to (5.3) in $\mathcal{O}$ if it is both a viscosity subsolution and supersolution to (5.3) in $\mathcal{O}$.

We state the removability of a level set for (5.3). The proof of this theorem is given in [22].

Theorem 5.4. *Let $\mathcal{O}$ be a domain in $\mathbb{R} \times \mathbb{R}^n$. We suppose that (D1), (D2) and (D3) are satisfied. If $u \in C^1(\mathcal{O})$ is a viscosity solution to (5.3) in $\mathcal{O} \setminus u^{-1}(0)$, then u is a viscosity solution to (5.3) in the whole domain $\mathcal{O}$.*

Remark 5.1. This theorem is applicable to various equations such as p-Laplace diffusion equation where $1 < p < 2$ and the mean curvature flow equation (5.1).

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