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Min-max representation in ergodic type Bellman equation of first order under general stability conditions

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1 Introduction

Let us consider the following nonlinear partial differential equation:

$$\frac{1}{2}a\nabla W\cdot\nabla W+b\cdot\nabla W+V(x)=\Lambda \text{ in } \mathbb{R}^{N}, \tag{1.1}$$

where $a=a(x)=[a^{ij}(x)]$ is a $N\times N$ positive-definite symmetric matrix, $b:\mathbb{R}^{N}\rightarrow \mathbb{R}^{N}$ and $V:\mathbb{R}^{N}\rightarrow \mathbb{R}$. A pair of a function $W:\mathbb{R}^{N}\rightarrow \mathbb{R}$ and a constant Λ is considered as a solution of (1.1). Since (1.1) is equivalent to

$$\sup_{v\in \mathbb{R}^{N}}[(b(x)+v)\cdot\nabla W+V(x)-\frac{1}{2}a^{-1}v\cdot v]=\Lambda,$$

(1.1) can be seen as the ergodic type Bellman equation arising in the optimization problem of the long time average cost (cf. [1], [2]). Indeed, some solution $(W(x), \Lambda)$ is considered to characterize the optimal long time average:

$$\Lambda=\sup_{v.}\varlimsup_{T\rightarrow\infty}\frac{1}{T}\int_{0}^{T}(V(x_{t})-\frac{1}{2}a^{-1}(x_{t})v_{t}\cdot v_{t})dt, \tag{1.2}$$

where x_{t} is the solution of the controlled ordinary differential equation

$$\frac{dx_{t}}{dt}=b(x_{t})+v_{t},\ t\geq 0,\ x_{0}=x \tag{1.3}$$

and $v.$ is a control taking its value in the whole Euclidean space $\mathbb{R}^{N}$. In this report, we are concerned with (1.1) from an analytical point of view. We shall study the structure of viscosity solutions of (1.1) and a critical solution $(W^{*}(x), \Lambda^{*})$ which might be related to (1.2). As we shall see later, there is the smallest Λ^{*} among all the Λ associated with viscosity solutions. For the smallest value Λ^{*}, we shall obtain the min-max representation for Λ^{*}:

$$\Lambda^{*}=\sup_{\mu\in \mathcal{M}_{1}(\mathbb{R}^{N})}\inf_{W}\int_{\mathbb{R}^{N}}(G(W)+V)d\mu, \tag{1.4}$$

where $G(W)=(1/2)a\nabla W\cdot\nabla W+b\cdot\nabla W$, $\mathcal{M}_{1}(\mathbb{R}^{N})$ is the set of probability measures on $\mathbb{R}^{N}$ and the infimum in (1.4) is taken over some class of smooth functions on $\mathbb{R}^{N}$.

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In order to explain our motivation to obtain (1.4), we shall deduce (1.4) in a formal way from the second order equation via a kind of a singular limit. Let us consider the generalized eigenvalue problem:

$$\mathcal{L}_\epsilon^V \phi_\epsilon = \kappa_\epsilon \phi_\epsilon \text{ in } \mathbb{R}^N, \tag{1.5}$$

where $\mathcal{L}_\epsilon^V \phi = (\epsilon/2)\operatorname{tr}(aD^2\phi) + b\cdot\nabla\phi + (1/\epsilon)V\phi$. We are concerned with a positive eigenfunction $\phi_\epsilon : \mathbb{R}^N \to \mathbb{R}$ and an eigenvalue κ_ϵ satisfying (1.5). It is known that under certain assumptions, there exists the smallest κ_ϵ^* (principal eigenvalue) corresponding to positive eigenfunctions. In [7] and [8], they showed that κ_ϵ^* has the min-max representation:

$$\kappa_\epsilon^* = \sup_{\mu\in\mathcal{M}_1(\mathbb{R}^N)} \inf_{\phi>0} \int_{\mathbb{R}^N} \frac{\mathcal{L}_\epsilon^V \phi}{\phi} d\mu, \tag{1.6}$$

where the infimum is taken over some class of positive functions.

On the other hand, if we take the transformations $W_\epsilon = \epsilon \log\phi_\epsilon$, $\Lambda_\epsilon = \epsilon\kappa_\epsilon$, we can see that (1.5) is equivalent to

$$\frac{\epsilon}{2}\operatorname{tr}(aD^2W_\epsilon) + \frac{1}{2}a\nabla W_\epsilon\cdot\nabla W_\epsilon + b\cdot\nabla W_\epsilon + V = \Lambda_\epsilon. \tag{1.7}$$

We note that (1.1) can be obtained from (1.7) if $W_\epsilon(x)$ and Λ_ϵ converge to $W(x)$ and Λ as $\epsilon \to 0$, respectively, in certain sense. If we set $\Lambda_\epsilon^* = \epsilon\kappa_\epsilon^*$, the min-max formula (1.6) can be written by

$$\Lambda_\epsilon^* = \sup_{\mu\in\mathcal{M}_1(\mathbb{R}^N)} \inf_{W} \int_{\mathbb{R}^N} (\frac{\epsilon}{2}\operatorname{tr}(aD^2W) + \frac{1}{2}a\nabla W\cdot\nabla W + b\cdot\nabla W + V)d\mu. \tag{1.8}$$

By noting that Λ_ϵ^* is the smallest Λ_ϵ corresponding to solutions, it is natural to expect that Λ_ϵ^* converges to Λ^* as $\epsilon \to 0$. If this is true, we can deduce (1.4) from (1.8) by sending $\epsilon \to 0$.

Although one might be able to obtain (1.4) from the min-max formula of the second order equation, we shall try to show (1.4) directly within the framework of the first order equation. Our key idea is to regard the critical value Λ^* as a function on V, i.e., for a given V, we associate V to the critical value $\Lambda^* = \Lambda^*(V)$ of (1.1). The idea comes from the argument on the identification of a rate function in large deviation theory (cf. [9]. Also [4]). Indeed, we can have the same properties on $\Lambda^*(V)$ as in large deviation theory, such as the convexity of $\Lambda^*(V)$ with respect to V. Then, we can expect the min-max formula (1.1) by using the duality argument of convex analysis. In [6], we have the min-max formula (1.4) under some "stability" assumptions (the precise assumptions for the stability will be given later). In the present report, we shall give the similar result on the min-max representation without the stability assumptions.

2 Min-max representation

2.1 Structure of viscosity solutions

Let us recall the basic result on viscosity solutions of (1.1). We use the opposite inequalities for the definitions of viscosity solutions, i.e. $W \in C(\mathbb{R}^N)$ is a *viscosity subsolution* (*resp. viscosity supersolution*) of (1.1) if for any smooth function $\varphi : \mathbb{R}^N \to \mathbb{R}$,

$$\frac{1}{2}a(\hat{x})\nabla\varphi(\hat{x}) \cdot \nabla\varphi(\hat{x}) + b(\hat{x}) \cdot \nabla W(\hat{x}) + V(\hat{x}) \geq 0 \quad (\textit{resp.} \leq 0)$$

at any local maximum point (*resp.* local minimum point) $\hat{x}$ of $W - \varphi$. $W \in C(\mathbb{R}^N)$ is a viscosity solution if W is a viscosity subsolution and supersolution. We prefer the above definition to the usual one because of the relationships to the Dynamic Programming Principle.

We denote by $\mathcal{A}$ the set of Λ corresponding to viscosity solutions:

$$\mathcal{A} \equiv \{\Lambda \in \mathbb{R};\ \text{there exists } W \in C(\mathbb{R}^N) \text{ s.t. } W \text{ is a viscosity solution of (1.1) for } \Lambda\}.$$

We state the basic assumptions:

(A1) $a(x)$, $b(x)$ and $V(x)$ are locally Lipschitz continuous.

(A2) There exist $0 < \mu_1 < \mu_2$ such that

$$\mu_1|\xi|^2 \leq a(x)\xi \cdot \xi \leq \mu_2|\xi|^2, \ \forall x, \xi \in \mathbb{R}^N.$$

(A3) There exist $W_0 \in C(\mathbb{R}^N)$ and a constant Λ_0 such that

$$\frac{1}{2}a\nabla W_0 \cdot \nabla W_0 + b \cdot \nabla W_0 + V \leq \Lambda_0 \text{ in } \mathbb{R}^N \text{ (viscosity sense).}$$

Theorem 2.1 ([3], [5]). *Under* (A1) – (A3), *there exists a constant* Λ^* *such that* $\mathcal{A} = [\Lambda^*, \infty)$.

We call Λ^* the *critical value* of (1.1). For the later argument, we specify the dependence on V as $\Lambda^* = \Lambda^*(V)$.

Remark 2.2. (i) If a viscosity supersolution of (1.1) for Λ is given, we can construct a viscosity solution of (1.1) for the same Λ. Thus, we can replace 'viscosity solution' with 'viscosity supersolution' in the definition of $\mathcal{A}$ (See [5]).

(ii) Let $C_b(\mathbb{R}^N)$ be the set of bounded continuous functions on $\mathbb{R}^N$. If $V \in C_b(\mathbb{R}^N)$, then (A3) holds for $W_0(x) \equiv 0$.

2.2 Min-max representation under stability condition

In order to obtain the min-max formula, the following lemma is crucial. The same properties can be found in the discussion on the identification of a rate function in large deviation theory (cf. [9]). Once these properties are established, we can resort to the duality argument in convex analysis.

Lemma 2.3 ([6]. cf. [4]). *Suppose* (A1) *and* (A2). *For* V, V_1, $V_2 \in C_b(\mathbb{R}^N)$, *(i) – (iv) hold:*

(i) The mapping $V \mapsto \Lambda^*(V)$ *is convex on* $C_b(\mathbb{R}^N)$.
(ii) If $V_1 \leq V_2$, *then* $\Lambda^*(V_1) \leq \Lambda^*(V_2)$.
(iii) $\Lambda^*(V + \alpha) = \Lambda^*(V) + \alpha$ *for any* $\alpha \in \mathbb{R}$.
(iv) $|\Lambda^*(V_1) - \Lambda^*(V_2)| \leq \|V_1 - V_2\|_\infty$, *where* $\|V\|_\infty = \sup_x |V(x)|$.

From Lemma 2.3 and the duality argument, we can see that $\Lambda^*(V)$ has the following representation:

$$\Lambda^*(V) = \sup_{\mu \in C_b(\mathbb{R}^N)^*} \{\langle V, \mu \rangle - J(\mu)\}, \tag{2.1}$$

where $C_b(\mathbb{R}^N)^*$ is the dual space of $C_b(\mathbb{R}^N)$ and $J(\mu)$ is the convex dual of $\Lambda^*(V)$, i.e.,

$$J(\mu) = \sup_{V \in C_b(\mathbb{R}^N)} \{\langle V, \mu \rangle - \Lambda^*(V)\}.$$

We note that $C_b(\mathbb{R}^N)^*$ can be identified with the set of regular bounded finitely additive set functions on $\mathbb{R}^N$. So, $\mathcal{M}_1(\mathbb{R}^N) \subsetneq C_b(\mathbb{R}^N)^*$.

We can show that the supremum in (2.1) is attained at some $\bar{\mu} \in C_b(\mathbb{R}^N)^*$. Moreover, it can be proved that $\bar{\mu}$ is nonnegative and $\bar{\mu}(\mathbb{R}^N) = 1$ by using Lemma 2.3 (ii) and (iii). However it is not easy to see that $\bar{\mu}$ is actually a probability measure. We need the following "stability condition" to show $\bar{\mu}$ is a measure:

(ST) There exists a smooth function $W_0(x)$ such that

$$\frac{1}{2} a \nabla W_0 \cdot \nabla W_0(x) + b \cdot \nabla W_0(x) \to -\infty \ (|x| \to \infty)$$

Then, we can obtain the min-max representation of the critical value after the identification of $J(\mu)$.

Theorem 2.4 ([6]). *Suppose* (A1), (A2) *and* (ST) *hold. For any* $V \in C_b(\mathbb{R}^N)$, *the critical value* $\Lambda^*(V)$ *has the min-max representation*

$$\Lambda^*(V) = \sup_{\mu \in \mathcal{M}_1(\mathbb{R}^N)} \inf_{W \in \mathcal{G}} \int_{\mathbb{R}^N} (G(W) + V) d\mu = \sup_{\mu \in \mathcal{M}_1(\mathbb{R}^N)} \{\langle V, \mu \rangle - I(\mu)\}, \tag{2.2}$$

where $G(W) = (1/2) a \nabla W \cdot \nabla W + b \cdot \nabla W$ *and*

$$\mathcal{G} = \{W \in C^\infty(\mathbb{R}^N);\ G(W) \textit{ is bounded above in } \mathbb{R}^N\} \tag{2.3}$$

$$I(\mu) = -\inf_{W \in \mathcal{G}} \int G(W) d\mu.$$

The next example shows that (ST) requires some stability property for the underlying system.

Example 2.5. Suppose (A1) and (A2) hold. If there exists $\kappa > 0$ such that

$$b(x) \cdot x \le -\kappa |x|^2 \text{ for sufficiently large } |x|, \tag{2.4}$$

then (ST) holds. Indeed, if we set $W_0(x) = (\alpha/2)|x|^2$, $W_0(x)$ satisfies (ST) for $0 < \alpha < (2\kappa)/\mu_2$. In terms of the system dynamics (1.3), (2.4) means that the vector field has a strong force pushing back the trajectory to the origin. This implies a "nice stability condition".

2.3 Min-max representation without stability condition

In this section, we shall extend (2.2) without condition (ST).

Theorem 2.6. *Suppose* (A1) *and* (A2). *For any* $V \in C_b(\mathbb{R}^N)$, *the min-max representation* (2.2) *holds for the same class of the infimum as* (2.3).

Sketch of the proof: For $V \in C(\mathbb{R}^N)$, which is not necessarily bounded, Theorem 2.4 can be extended under the following stability condition (cf. [6]):

(ST)$'$ There exists a smooth function $W_0(x)$ such that

$$\frac{1}{2} a\nabla W_0 \cdot \nabla W_0(x) + b(x) \cdot \nabla W_0(x) + V(x) \to -\infty \ (|x| \to \infty).$$

More precisely, under (A1), (A2) and (ST)$'$, we have

$$\Lambda^*(V) = \sup_{\mathcal{M}_1(\mathbb{R}^N)} \inf_{W \in \mathcal{G}_V} \int_{\mathbb{R}^N} (G(W) + V) d\mu, \tag{2.5}$$

where

$$\mathcal{G}_V = \{W \in C^\infty(\mathbb{R}^N);\ G(W) + V \text{ is bounded above in } \mathbb{R}^N\}.$$

Note that $\mathcal{G}_V$ depends on V. Also, we note that the min-max formula (2.5) holds for $V \in C(\mathbb{R}^N)$ such that $V(x) \to -\infty$ as $|x| \to \infty$ because we can take $W_0(x) \equiv 0$ in (ST)$'$.

In order to prove (2.2) for $V \in C_b(\mathbb{R}^N)$ without the stability condition (ST)$'$, we approximate V by some sequence $\{V_n\} \subset C(\mathbb{R}^N)$ satisfying (i) $V_n(x) \to -\infty$ ($|x| \to \infty$), $n = 1, 2, \cdots$ (ii) $V_n(x) \uparrow V(x)$ ($n \uparrow \infty$) (pointwise). On the other hand, we can prove that a stability of the critical values for monotone sequence, i.e., $\Lambda^*(V_n) \uparrow \Lambda^*(V)$ as $n \uparrow \infty$. Note that this is not expected by Lemma 2.3 (iv) because V_n does not converge to V uniformly. Then, by using the approximation of V and the stability of the critical value in the above, we can prove that "$\leq$" in (2.2).

For the proof "$\geq$", we use a smooth supersolution near $\Lambda^*(V)$. Precisely, we can show for any $\epsilon > 0$, there exists a smooth function $W_\epsilon(x)$ such that

$$\frac{1}{2} a\nabla W_\epsilon \cdot \nabla W_\epsilon + b \cdot \nabla W_\epsilon + V \leq \Lambda^*(V) + \epsilon \text{ in } \mathbb{R}^N.$$

It is not difficult to prove that the right hand side of (2.2) is smaller than $\Lambda^*(V) + \epsilon$ by noting "$\inf\sup \geq \sup\inf$" in general. By sending ϵ to 0, we obtain "$\geq$" in (2.2) □

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