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Asymptotic profile for solutions of Keller-Segel model

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1 Introduction

We consider the following reaction-diffusion equation:

$$\text{(KS)} \quad \begin{cases} u_t = \nabla \cdot \left(\nabla u^m - u^{q-1}\nabla v\right), & x \in \mathbb{R}^N,\ 0 < t < \infty, \\ 0 = \Delta v - v + u, & x \in \mathbb{R}^N,\ 0 < t < \infty, \\ u(x,0) = u_0(x), & x \in \mathbb{R}^N, \end{cases}$$

where $N \geq 1$, $m \geq 1$ and $q > \{m + \frac{2}{N}, \frac{3}{2}\}$. The initial data u_0 is a non-negative function in $L^1 \cap L^\infty(\mathbb{R}^N)$ with $u_0^m \in H^1(\mathbb{R}^N)$. This equation was proposed by Keller-Segel [9] to describe the motion of the chemotaxis molds, and nowadays it is called Keller-Segel model.

The first equation of (KS) without the perturbation term is written as follows:

$$\text{(PM)} \qquad \psi_t(x,t) = \Delta \psi^m(x,t).$$

It is known that (PM) has the exact solution $V(x,t;M)$ with self-similarity, called *Barenblatt solution.*

For (KS), for $q > 2$ in [14], for $q = 2$ in [15], and for $\frac{3}{2} < q < 2$ in [16], it was shown that the exponent $q = m + \frac{2}{N}$ represents so called Fujita's one which divides the situation between the global existence and finite time blow-up to a solution of (KS). Specifically, it was proved in [14]–[16] that under the assumption $q > \frac{3}{2}$;
(i) when $q < m + \frac{2}{N}$, (KS) is globally solvable without the any restriction on the size of the initial data, and
(ii) when $m \geq 1$ and $q \geq m + \frac{2}{N}$, (KS) is globally solvable for small $L^{\frac{N(q-m)}{2}}$- initial data. Furthermore, the decay of solution in $L^p(\mathbb{R}^N)(1 < p < \infty)$ was shown.

In the present article, we shall consider the above case (ii) and obtain the asymptotic profile of the solution $u(t)$ with a definite convergence rate in $L^p(\mathbb{R}^N)$. More precisely, we shall show that

(I) for (KS) with $m > 1$, we obtain the optimal convergence rate such as

$$\lim_{t\to\infty} t^{\sigma_m(1-\frac{1}{p})} \left\| u(\cdot,t) - V(\cdot,t; \|u_0\|_{L^1(\mathbb{R}^N)}) \right\|_{L^p(B_{t,R})} = 0 \qquad \text{for } 1 < p < \infty$$

with $B_{t,R} := \{x \in \mathbb{R}^N;\ |x| < Rt^{\frac{1}{N(m-1)+2}}\}$, where $V(x,t;M)$ is the well-known *Barenblatt solution* of (PM) such that $\displaystyle\int_{\mathbb{R}^N} V(x,t;M)\, dx = M$. For detail, see (2.4) in the next section.

We also discuss the semi-linear case: $m = 1$ of (KS) and

(II) for (KS) with $m = 1$, we prove that

$$\lim_{t\to\infty} t^{\frac{N}{2}(1-\frac{1}{p})}\|u(\cdot,t) - MG_t(\cdot)\|_{L^p(B_{t,R})} = 0 \qquad \text{for } 1 < p < \infty,$$

where $G_t(x)$ is the heat kernel and $M = \|u_0\|_{L^1(\mathbf{R}^N)}$.

We thus propose the method to prove the asymptotic profile with the optimal convergence rate without "comparison principles and the representation formula of solutions." In many systems, it is difficult to show that a comparison principle holds. Our method could be applied to other nonlinear systems which do not make comparison principles ensure.

2 Results

Throughout this article, we deal with the weak solution of (KS). Our definition of the weak solution now reads:

Definition 1 *Let $m \geq 1$, $q > 1$ and let $u_0 \in L^1 \cap L^\infty(\mathbf{R}^N)$ with $u_0^m \in H^1(\mathbf{R}^N)$ and $u_0 \geq 0$. A pair (u, v) of non-negative functions defined in $\mathbf{R}^N \times [0, T)$ is called a weak solution of* (KS) *on $[0, T)$ if*

i) $u \in L^\infty(0, T; L^1 \cap L^\infty(\mathbf{R}^N))$, $\nabla u^m \in L^2(0, T; L^2(\mathbf{R}^N))$,

ii) $v \in L^\infty(0, T; H^1(\mathbf{R}^N))$,

iii) (u, v) *satisfies the equations in the sense of distribution, i.e., that*

$$\int_0^\infty \int_{\mathbf{R}^N} \left(\nabla u^m \cdot \nabla\varphi - u^{q-1}\nabla v \cdot \nabla\varphi - u\varphi_t\right)\, dxdt = \int_{\mathbf{R}^N} u_0(x)\varphi(x,0)\, dx,$$

$$\int_{\mathbf{R}^N} \left(\nabla v \cdot \nabla\psi + v\psi - u\psi\right)(t)\, dx = 0 \qquad \textit{for a.a. } t \in (0,T)$$

for all functions $\varphi \in C_0^\infty(\mathbf{R}^N \times [0,T))$ and $\psi \in C_0^\infty(\mathbf{R}^N)$.

We introduce the existence and decay property of a weak solution (u, v). The following proposition is a direct consequence of [10],[14]–[16].

Proposition 2.1 ([10],[14]–[16]) *Let $1 \leq p < \infty$, $N \geq 1$, $m \geq 1$, $q > \frac{3}{2}$ and $q \geq m + \frac{2}{N}$, $\ell \geq \frac{N(q-m)}{2}$ (≥ 1). Suppose that the initial data u_0 is non-negative everywhere. Then, there exist an absolute constant M and a positive number ε depending only on M, p, N, m, ℓ such that if $u_0 \in L^1 \cap L^\ell(\mathbf{R}^N)$ satisfies that*

$$\|u_0\|_{L^1(\mathbf{R}^N)} = M, \quad \|u_0\|_{L^\ell(\mathbf{R}^N)} \leq \varepsilon, \tag{2.1}$$

then (KS) *has a weak solution (u, v) on $[0, \infty)$ with the following decay property: there exists a constant C_p depending only on p, $\|u_0\|_{L^p(\mathbf{R}^N)}$ together with N, m, q, M, $\|u_0\|_{L^{(N+2)q}(\mathbf{R}^N)}$*

such that

$$\|u(t)\|_{L^p(\mathbb{R}^N)} + \|v(t)\|_{L^p(\mathbb{R}^N)} \le C_p(1+t)^{-d} \qquad \textit{for all } 0 < t < \infty, \tag{2.2}$$

where

$$d = \sigma_m\left(1 - \frac{1}{p}\right), \quad \sigma_m = \frac{N}{N(m-1)+2}. \tag{2.3}$$

Remark 1 (i) The decay rate d depends on m, N but not on q.
(ii) The above convergence rate d seems to be optimal. In fact, for $m = 1$, we find that $\sigma_m = \frac{N}{2}$ whose decay rate d coincides with the L^1-L^p estimate for the linear heat equation.

We introduce the self-similar solution $V(x,t;M)$ by Barenblatt [1]:

$$V(x,t;M) := \frac{1}{t^{\sigma_m}}\left(\beta^2 M^{\frac{2\sigma_m(m-1)}{N}} - \frac{\sigma_m(m-1)}{2mN}\cdot\frac{|x|^2}{t^{\frac{2\sigma_m}{N}}}\right)_+^{\frac{1}{m-1}}, \tag{2.4}$$

where β is the parameter. In this article, we take β in such a way that $V(x,t;M)$ satisfies $\int_{\mathbb{R}^N} V(x,t;M)\,dx = M$ for all $t > 0$. We call the above function $V(x,t;M)$ *the Barenblatt solution.* Moreover, it is known that $V(x,t;M)$ is the weak solution for the Cauchy problem of (PM) corresponding to the initial data δM, where δ is the Dirac mass at the origin.

We denote the heat kernel $G_t(x)$ by $G_t(x) := \frac{1}{(4\pi t)^{\frac{N}{2}}}\exp\left(-\frac{|x|^2}{4t}\right)$.

We now give two main theorems. The first one is for the quasilinear case of $m > 1$.

Theorem 2.2 (asymptotic profile: Barenblatt solution) *Let the same assumption as that in Proposition 2.1 hold. In addition, let $m > 1$ and $q > m + \frac{2}{N}$. Then, the weak solution u obtained in Proposition 2.1 satisfies that*

$$\lim_{t\to\infty} t^{\sigma_m(1-\frac{1}{p})}\left\|u(\cdot,t) - V(\cdot,t;\|u_0\|_{L^1(\mathbb{R}^N)})\right\|_{L^p(B_{t,R})} = 0, \qquad 1 < p < \infty \tag{2.5}$$

for all $R > 0$, where σ_m is the exponent defined in (2.3) *and $B_{t,R}$ is the ball defined by*

$$B_{t,R} := \{x \in \mathbb{R}^N;\ |x| < Rt^{\frac{1}{N(m-1)+2}}\}. \tag{2.6}$$

Remark 2 (i) The solution of (PM) has the similar property as Theorem 2.2. Indeed, for the solution ψ of (PM), it holds

$$\lim_{t\to\infty} t^{\sigma_m(1-\frac{1}{p})}\|\psi(\cdot,t) - V(\cdot,t;\|\psi(0)\|_{L^1(\mathbb{R}^N)})\|_{L^p(\mathbb{R}^N)} = 0 \tag{2.7}$$

for any $1 \le p \le \infty$. (we refer to Bénilan [2], Friedman-Kamin [5], Kamin [7], Kamin-Vazquez [8], Véron [17].) Hence, Theorem 2.2 implies that Δu^m is dominant to $\nabla(u^{q-1}\nabla v)$

in the case of "$q > m + \frac{2}{N}$ and small initial data",

(ii) Proposition 2.1 includes the case of $q = m + \frac{2}{N}$. On the other hand, Theorem 2.2 excludes the case of $q = m + \frac{2}{N}$.

The next theorem is for the semi-linear case of $m = 1$.

Theorem 2.3 (asymptotic profile: heat kernel) *Let the same assumption as that in Proposition 2.1 hold. In addition, let $m = 1$ and $q > 1 + \frac{2}{N}$. Then, the weak solution u obtained in Proposition 2.1 satisfies that*

$$(2.8) \qquad \lim_{t\to\infty} t^{\frac{N}{2}(1-\frac{1}{p})} \|u(\cdot,t) - \|u_0\|_{L^1(\mathbb{R}^N)} G_t(\cdot)\|_{L^p(B_{t,R})} = 0, \qquad 1 < p < \infty$$

for all $R > 0$, where $B_{t,R}$ is the ball defined in (2.6).

Remark 3 The asymptotic profile as (2.8) (in the whole domain) was firstly obtained by Nagai-Syukuinn-Umesako [12] for Keller-Segel model of parabolic-parabolic type. Their argument is based on the representation formula of solutions. On the other hand, we study the Keller-Segel model of parabolic-elliptic type and give another proof without using any representation formula of solutions.

To prove our main theorems, we make fully use of the scaling argument. Let us introduce rescaled functions w_k and z_k defined by

$$w_k(x,t) = k^N u(kx, k^{N(m-1)+2}t) \quad \text{and} \quad z_k(x,t) = k^N v(kx, k^{N(m-1)+2}t) \quad \text{for } k \geq 1.$$

Then we see that (KS) can be rewritten as

$$^w(\mathrm{KS}) \begin{cases} w_{kt} = \nabla \cdot \Big(\nabla (w_k)^m - k^{-N(q-m)} (w_k)^{q-1} \nabla z_k\Big), & (x,t) \in \mathbb{R}^N \times (0,\infty), \quad \cdots (1)_w \\ 0 = k^{-2}\Delta z_k - z_k + w_k, & (x,t) \in \mathbb{R}^N \times (0,\infty), \quad \cdots (2)_w \\ w_k(x,0) = k^N u_0(kx), & x \in \mathbb{R}^N \end{cases}$$

where $N \geq 1,\ m > 1,\ q > \frac{3}{2},\ q \geq m + \frac{2}{N}$.

It should be noted that $^w(\mathrm{KS})$ does not have any invariance under change of scaling more. However, under the hypothesis $q > m + \frac{2}{N}$, it has an advantage since we can gain the negative power $-N(q-m)$ to k of the coefficient $w_k^{q-1}\nabla z_k$ which may be regarded as the small perturbation term. Hence, for $q > m + \frac{2}{N}$ we [10] proved that the sequence $\{w_k\}_{k=1}^\infty$ is bounded in $L^\infty(\mathbb{R}^N \times (\delta, T))$ together with the fact that $\{w_k^m\}_{k=1}^\infty$ is also bounded in $H^1(\delta, T; L^2(\mathbb{R}^N)) \cap L^\infty(\delta, T; H^1(\mathbb{R}^N))$ for all $\delta > 0$. These bounds and the standard compactness argument yield a subsequence of $\{w_k\}_{k=1}^\infty$, which we denote by $\{w_k\}_{k=1}^\infty$ itself for simplicity, and a function $U(x,t)$ such that

$$(2.9) \qquad \|w_k(\cdot,t) - U(\cdot,t)\|_{L^p(B_R)} \to 0 \qquad \text{for all } 1 < p < \infty \quad \text{as } k \to \infty$$

with the ball $B_R := \{x \in \mathbb{R}^N; |x| < R\}$. Here, we may take arbitrary $R > 0$. On account of the negative power $-N(q-m)$ to k in $^w(\mathrm{KS})$ as is described above, we see that U is, in fact, a weak solution of (PM) with the property that $\|U(\cdot,t)\|_{L^1(\mathbf{R}^N)} = M =: \|u_0\|_{L^1(\mathbf{R}^N)}$. Furthermore, it turns out that both $U(\cdot,t)$ and $V(\cdot,t;M)$ converge to $M\delta$ in the sense of distributions as $t \downarrow 0$, which yields with the aid of uniqueness result due to Pierre [13] that $U(x,t) \equiv V(x,t;M)$. Now, taking $k = t^{\sigma_m/N}$ in (2.9) and then returning to our original solution u from the rescaled sequence $\{w_k\}_{k=1}^{\infty}$, we obtain the desired asymptotic profile such as (2.5).

We will use the simplified notations:

1) $\partial_t = \frac{\partial}{\partial t}$, $\partial_i = \frac{\partial}{\partial x_i}$, $\partial_{ij}^2 = \partial_i\partial_j$, $\nabla u = \left(\partial_1, \partial_2, \cdots\right)$, $\nabla^2 u = \left(\partial_{11}^2, \partial_{12}^2, \cdots\right)$,

2) $\|\cdot\|_{L^r} = \|\cdot\|_{L^r(\mathbf{R}^N)}, (1 \le r \le \infty)$, $\int \cdot\, dx := \int_{\mathbf{R}^N} \cdot\, dx$.

3) $Q_T := \mathbf{R}^N \times (0,T)$, $B_R := \{x \in \mathbf{R}^N; |x| < R\}$.

4) When the weak derivatives $\nabla u, \nabla^2 u$ and $\partial_t u$ are in $L^p(Q_T)$ for some $p \ge 1$, we say that $u \in W_p^{2,1}(Q_T)$, *i.e.*,

$$\begin{aligned} W_p^{2,1}(Q_T) \; := \; \Big\{ & u \in L^p(0,T;W^{2,p}(\mathbf{R}^N)) \cap W^{1,p}(0,T;L^p(\mathbf{R}^N)); \\ & \|u\|_{W_p^{2,1}(Q_T)} := \|u\|_{L^p(Q_T)} + \|\nabla u\|_{L^p(Q_T)} + \|\nabla^2 u\|_{L^p(Q_T)} + \|\partial_t u\|_{L^p(Q_T)} < \infty \Big\}. \end{aligned}$$

3 Outline of proof

Let us recall ^{w}(KS) introduced in Section 2. The problem ^{w}(KS) does not have any invariance under change of scaling. However, we can show that the sequence $\{w_k\}_{k=1}^{\infty}$ is uniformly bounded in $\mathbf{R}^N \times (\delta, T)$ together with the fact that

$$\text{(3.1)} \qquad \{(w_k)^m\}_{k=1}^{\infty} \text{ is also bounded in } H^1(\delta,T;L^2(\mathbf{R}^N)) \cap L^\infty(\delta,T;H^1(\mathbf{R}^N))$$

for all $0 < \delta < T < \infty$. By (3.1) and the standard compactness theorem, we find that there exist a subsequence, still denoted by $\{w_k\}$, and a function U on $\mathbf{R}^N \times (0,\infty)$ such that

$$\text{(3.2)} \qquad \|w_k(t) - U(t)\|_{L^p(B_R)} \to 0 \quad \text{with } 1 < p < \infty, \text{ as } k \to \infty$$

for all $0 < t < \infty$ and all $R > 0$, where $B_R := \{x \in \mathbf{R}^N;\ |x| < R\}$.

On account of the negative power $-N(q-m)$ to k of the coefficient $w_k^{q-1}\nabla z_k$, we may treat $k^{-N(q-m)}\nabla(w_k^{q-1}\nabla z_k)$ as the small perturbation term. As a result, we find that this function U satisfies (PM) in the following weak sense:

$$\text{(3.3)} \qquad \int_0^\tau \int_{\mathbf{R}^N} (U\varphi_t + U^m \Delta\varphi)\, dxdt \;=\; \int_{\mathbf{R}^N} U(x,\tau)\varphi(\cdot,\tau)\, dx - \|u_0\|_{L^1(\mathbf{R}^N)}\varphi(0,0)$$

for all C^∞ functions $\varphi(x,t)$ with compact support in $\mathbf{R}^N \times (0,T]$, and all $0 < \tau < T$. It should be noted that the Barenblatt solution $V(x,t;M)$ also satisfies (3.3).

Furthermore, it turns out that

$$\text{(H1)} \qquad U(t) \in L^1(\mathbf{R}^N) \quad \text{and} \quad \|U(t)\|_{L^1(\mathbf{R}^N)} = \|u_0\|_{L^1(\mathbf{R}^N)} \qquad \text{for all } 0 < t < \infty$$

with the property that

$$\text{(H2)} \qquad \lim_{t\downarrow 0}\int_{\mathbf{R}^N} U(x,t)\psi(x)\,dx \;=\; \|u_0\|_{L^1(\mathbf{R}^N)}\psi(0).$$

On the other hand, it is easy to show that

$$\text{(H3)} \qquad \lim_{t\downarrow 0}\int_{\mathbf{R}^N} V(x,t;\|u_0\|_{L^1(\mathbf{R}^N)})\psi(x)\,dx \;=\; \|u_0\|_{L^1(\mathbf{R}^N)}\psi(0)$$

for all $\varphi \in C_0^\infty(\mathbb{R}^N)$. Then, by the uniqueness theorem given by Dahlberg-Kenig [4], we conclude that

$$\text{(3.4)} \qquad U(x,t) = V(x,t;\|u_0\|_{L^1(\mathbf{R}^N)}) \qquad \text{for all } (x,t) \in \mathbb{R}^N \times (0,T].$$

Combining (3.2) with (3.4), we have

$$\text{(3.5)} \qquad \|w_k(\cdot,1) - V(\cdot,1;\|u_0\|_{L^1})\|_{L^p(B_R)} \to 0, \qquad 1 < p < \infty$$

as $k \to \infty$ for all $R > 0$, where $B_R := \{x \in \mathbb{R}^N;\ |x| < R\}$. Now taking k as $k = t^{\frac{\sigma_m}{N}}$ in (3.5), we conclude that

$$t^{\sigma_m(1-\frac{1}{p})}\left\|u(\cdot,t) - V(\cdot,t;\|u_0\|_{L^1(\mathbf{R}^N)})\right\|_{L^p(B_{t,R})} \;\to\; 0 \quad \text{with } 1 < p < \infty, \text{ as } t \to \infty$$

for all $R > 0$, where $B_{t,R} := \{x \in \mathbb{R}^N;\ |x| < Rt^{\frac{1}{N(m-1)+2}}\}$. Thus, we obtain the optimal convergence rate.

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