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Nonlinear Diffusion with a Stationary Level Surface *

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Abstract

We consider nonlinear diffusion of some substance in a bounded C^2 container. Suppose that, initially, the container is empty and, at all times, its boundary is kept at density 1. We show that if the container contains a proper sub- C^2 domain having constant boundary density at each given time, then the container must be a ball.

Key words. Nonlinear diffusion equation, overdetermined problems, stationary level surfaces.

AMS subject classifications. Primary 35K60; Secondary 35B40.

1 Introduction

This is based on the author's recent work with R. Magnanini [MS5]. In the previous paper [MS3], we considered the solution $u = u(x,t)$ of the following initial-boundary value problem for the heat equation:

$$u_t = \Delta u \quad \text{in } \Omega \times (0, +\infty), \tag{1.1}$$

$$u = 1 \quad \text{on } \partial\Omega \times (0, +\infty), \tag{1.2}$$

$$u = 0 \quad \text{on } \Omega \times \{0\}, \tag{1.3}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with $N \geq 2$, and we obtained

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Theorem 1.1 ([MS3]) *Let Ω be a bounded domain in $\mathbb{R}^N$, $N \geq 2$, satisfying the exterior sphere condition and suppose that D is a domain, with boundary ∂D, satisfying the interior cone condition, and such that $\overline{D} \subset \Omega$.*

Assume that the solution u of problem (1.1)-(1.3) *satisfies the following condition:*

$$u(x,t) = a(t), \quad (x,t) \in \partial D \times (0,+\infty), \tag{1.4}$$

for some function $a : (0,+\infty) \to (0,+\infty)$. Then Ω must be a ball.

We recall that Ω satisfies the *exterior sphere condition* if for every $y \in \partial\Omega$ there exists a ball $B_r(z)$ such that $\overline{B_r(z)} \cap \overline{\Omega} = \{y\}$, where $B_r(z)$ denotes an open ball centered at $z \in \mathbb{R}^N$ and with radius $r > 0$. Also, D satisfies the *interior cone condition* if for every $x \in \partial D$ there exists a finite right spherical cone K_x with vertex x such that $K_x \subset \overline{D}$ and $\overline{K_x} \cap \partial D = \{x\}$.

Here we introduce an outline of the proof of Theorem 1.1 by using a result in [MS4]. The proof is essentially based on three ingredients.

One ingredient is a result of Varadhan [Va] which shows that, as $t \to 0^+$, the function $-4t \log u(x,t)$ converges uniformly on $\overline{\Omega}$ to the function $d(x)^2$, where

$$d(x) = \text{dist }(x, \partial\Omega), \quad x \in \Omega. \tag{1.5}$$

Here in order to apply the result of Varadhan we have used the assumption that Ω satisfies the exterior sphere condition. Hence, by (1.4) there exists $R > 0$ satisfying

$$d(x) = R \ \text{ for every } \ x \in \partial D. \tag{1.6}$$

The second ingredient is a couple of *balance laws* proved in [MS1] and [MS2] (see [MS3] for another proof). For $x_0 \in \Omega$, $\nabla u(x_0, t) = 0$ for every $t > 0$ if and only if

$$\int_{\partial B_r(x_0)} (x - x_0) u(x,t)\, dS_x = 0, \ \text{ for every } r \in [0, d(x_0)) \text{ and every } t > 0. \tag{1.7}$$

With the aid of the interior cone condition of D, by combining (1.7) and (1.6) with the initial behavior of u proved in Varadhan [Va], we see that for every point $x_0 \in \partial D$ there exists a time $t_0 > 0$ satisfying $\nabla u(x_0, t_0) \neq 0$, which implies that ∂D

is analytic. Thus, by using the exterior sphere condition of Ω again, we conclude that $\partial\Omega$ is analytic and parallel to ∂D. Another balance law is stated as follows: Let G be a domain in $\mathbb{R}^N$. For $x_0 \in G$, a solution $v = v(x,t)$ of the heat equation in $G \times (0,+\infty)$ is such that $v(x_0,t) = 0$ for every $t > 0$ if and only if

$$\int_{\partial B_r(x_0)} v(x,t)\, dS_x = 0, \text{ for every } r \in [0, \text{dist }(x_0, \partial G)) \text{ and every } t > 0. \tag{1.8}$$

Let $P, Q \in \partial\Omega$ be two distinct points, and let $p, q \in \partial D$ be the points such that

$$\overline{B_R(p)} \cap \partial\Omega = \{P\} \quad \text{and} \quad \overline{B_R(q)} \cap \partial\Omega = \{Q\}.$$

Consider the function $v = v(x,t)$ defined by

$$v(x,t) = u(x+p,t) - u(x+q,t) \quad \text{for} \quad (x,t) \in B_R(0) \times (0,+\infty).$$

Since v satisfies the heat equation and $v(0,t) = a(t) - a(t) = 0$ for every $t > 0$, it follows from (1.8) that

$$t^{-\frac{N+1}{4}} \int_{B_R(p)} u(x,t)\, dx = t^{-\frac{N+1}{4}} \int_{B_R(q)} u(x,t)\, dx \quad \text{for every} \quad t > 0.$$

Therefore, by using a result in [MS4], letting $t \to 0^+$ yields that

$$C(N) \left\{ \prod_{j=1}^{N-1} \left[\frac{1}{R} - \kappa_j(P) \right] \right\}^{-\frac{1}{2}} = C(N) \left\{ \prod_{j=1}^{N-1} \left[\frac{1}{R} - \kappa_j(Q) \right] \right\}^{-\frac{1}{2}}, \tag{1.9}$$

where $\kappa_j(x), j = 1, \ldots, N-1$, denotes the j-th principal curvature of the surface $\partial\Omega$ at the point $x \in \partial\Omega$, and where $C(N)$ is a positive constant depending only on N (see [MS4], Theorem 4.2).

The third ingredient is Aleksandrov's sphere theorem [Alek], p. 412. A special case of this theorem is the well-known *Soap-Bubble Theorem* (see also [R]). Finally, by applying Aleksandrov's sphere theorem to the fact that $\prod_{j=1}^{N-1} \left[\frac{1}{R} - \kappa_j(x)\right]$ is constant for $x \in \partial\Omega$, we conclude that $\partial\Omega$ must be a sphere. (See [MS3] and [MS4] for the details.)

We observe that Varadhan's result, a couple of balance laws, and Aleksandrov's sphere theorem play a key role in the above proof. Among these we can not expect a couple of balance laws for *nonlinear* diffusion equations.

In this article we consider the solution $u = u(x,t)$ of the following initial-boundary value problem for the nonlinear diffusion equation:

$$u_t = \Delta\phi(u) \quad \text{in} \quad \Omega \times (0, +\infty), \tag{1.10}$$

$$u = 1 \quad \text{on} \quad \partial\Omega \times (0, +\infty), \tag{1.11}$$

$$u = 0 \quad \text{on} \quad \Omega \times \{0\}, \tag{1.12}$$

where Ω is a bounded C^2 domain in $\mathbb{R}^N$ with $N \geq 2$, and where $\phi : \mathbb{R} \to \mathbb{R}$ is such that

$$\phi \in C^2(\mathbb{R}), \quad \phi(0) = 0, \text{ and} \tag{1.13}$$

$$0 < \delta_1 \leq \phi'(s) \leq \delta_2 \text{ for } s \in \mathbb{R}, \tag{1.14}$$

where δ_1, δ_2 are positive constants. By the maximum principle we get

$$0 < u < 1 \text{ in } \Omega \times (0, +\infty). \tag{1.15}$$

Let $\Phi = \Phi(s)$ be a function defined by

$$\Phi(s) = \int_1^s \frac{\phi'(\xi)}{\xi}\, d\xi \text{ for } s > 0. \tag{1.16}$$

Note that if $\phi(s) \equiv s$, then $\Phi(s) = \log s$.

Our result corresponding to Varadhan's one is

Theorem 1.2 ([MS5]) *Let u be the solution of problem* (1.10)-(1.12). *Then, as $t \to 0^+$, the function $-4t\Phi(u(x,t))$ converges to the function $d(x)^2$ uniformly on every compact set in Ω.*

The symmetry result corresponding to Theorem 1.1 is

Theorem 1.3 ([MS5]) *Let D be a bounded C^2 domain in $\mathbb{R}^N$ satisfying $\overline{D} \subset \Omega$.*

Assume that the solution u of problem (1.10)-(1.12) *satisfies the following condition:*

$$u(x,t) = a(t), \quad (x,t) \in \partial D \times (0, +\infty), \tag{1.17}$$

for some function $a : (0, +\infty) \to (0, +\infty)$. Then Ω must be a ball.

Remark. Let us give two remarks concerning Theorem 1.1 and Theorem 1.3. Since we can not expect the balance laws for nonlinear equations and we used the balance law (1.7) to obtain the regularity of ∂D, we assume that both ∂D and $\partial\Omega$ are C^2 smooth in Theorem 1.3. So, as far as problem (1.1)-(1.3) is concerned, Theorem 1.1 is stronger than Theorem 1.3. Furthermore, in problem (1.1)-(1.3), the same method of the proof as in Theorem 1.1 also yields

Theorem 1.4 *Let Ω be a bounded domain in $\mathbb{R}^N$, $N \geq 2$, satisfying the exterior sphere condition and suppose that D is a domain, with boundary ∂D, satisfying the interior cone condition, and such that $\overline{D} \subset \Omega$. Let Γ be a connected component of ∂D satisfying*

$$\text{dist}\ (\Gamma, \partial\Omega) = \text{dist}\ (\partial D, \partial\Omega).$$

Assume that the solution u of problem (1.1)-(1.3) *satisfies the following condition:*

$$u(x,t) = a(t), \quad (x,t) \in \Gamma \times (0, +\infty), \tag{1.18}$$

for some function $a : (0,+\infty) \to (0,+\infty)$. Then Ω must be either a ball or an annulus.

2 Outline of proofs of Theorems 1.3 and 1.2

In this section we give an outline of proofs. For the details, see [MS5].

Proof of Theorem 1.3. By using Theorem 1.2, we get (1.6). Furthermore, with the aid of the C^2 smoothness assumption of both ∂D and $\partial\Omega$, we see that $\partial\Omega$ is parallel to ∂D. Then, by applying the method of moving planes to problem (1.10)-(1.12) directly, we conclude that Ω must be a ball. See Serrin [Ser] for the method of moving planes.

Proof of Theorem 1.2. Let $g = g(s)$ be the inverse function of Φ. Then

$$s = \Phi(g(s)) = \int_1^{g(s)} \frac{\phi'(\xi)}{\xi}\, d\xi.$$

Differentiating in s yields

$$g(s) = \phi'(g(s))g'(s). \tag{2.1}$$

As in Freidlin and Wentzell [FW], for $0 < \varepsilon < 1$, define the function $u^\varepsilon = u^\varepsilon(x,t)$ by

$$u^\varepsilon(x,t) = u(x, \varepsilon^2 x) \text{ for } (x,t) \in \Omega \times (0,+\infty).$$

Then u^ε satisfies

$$\begin{aligned} u_t^\varepsilon &= \varepsilon^2 \Delta\phi(u^\varepsilon) && \text{in} && \Omega \times (0,+\infty), && (2.2)\\ u^\varepsilon &= 1 && \text{on} && \partial\Omega \times (0,+\infty), && (2.3)\\ u^\varepsilon &= 0 && \text{on} && \Omega \times \{0\}. && (2.4)\end{aligned}$$

Moreover, we define the function $v^\varepsilon = v^\varepsilon(x,t)$ by

$$v^\varepsilon(x,t) = -\varepsilon^2 \Phi(u^\varepsilon(x,t)) \text{ for } (x,t) \in \Omega \times (0,+\infty).$$

Then $u^\varepsilon = g\left(-\varepsilon^{-2} v^\varepsilon\right)$. With the aid of (2.1), we have

$$\begin{aligned} v_t^\varepsilon &= \varepsilon^2 \phi' \Delta v^\varepsilon - |\nabla v^\varepsilon|^2 && \text{in} && \Omega \times (0,\infty), && (2.5)\\ v^\varepsilon &= 0 && \text{on} && \partial\Omega \times (0,\infty), && (2.6)\\ v^\varepsilon &= +\infty && \text{on} && \Omega \times \{0\}, && (2.7)\end{aligned}$$

where $\phi' = \phi'\left(g\left(-\varepsilon^{-2} v^\varepsilon\right)\right)$.

Consider the limit problem as $\varepsilon \to 0^+$

$$\begin{aligned} v_t &= -|\nabla v|^2 && \text{in} && \Omega \times (0,\infty), && (2.8)\\ v &= 0 && \text{on} && \partial\Omega \times (0,\infty), && (2.9)\\ v &= +\infty && \text{on} && \Omega \times \{0\}. && (2.10)\end{aligned}$$

This problem has a unique viscosity solution

$$v(x,t) = \frac{1}{4t} d(x)^2. \tag{2.11}$$

The uniqueness is proved by Crandall, Lions, and Souganidis [CrLS]. With the help of Crandall, Ishii, and Lions [CrIL] we can prove that the function given by (2.11) is a viscosity solution of problem (2.8)-(2.10).

By applying the comparison principle to $u(x, t+h)$ and $u(x,t)$ for $h > 0$, we get

$$u_t > 0 \text{ and } \Delta\phi(u) > 0 \text{ in } \Omega \times (0,+\infty). \tag{2.12}$$

Set $w = \phi(u)$. Then $w_t = \phi'(u)\Delta w$ and by (1.14)

$$\delta_1 \Delta w \le w_t \le \delta_2 \Delta w \quad \text{in} \quad \Omega \times (0, +\infty). \tag{2.13}$$

Let w_j $(j = 1, 2)$ solve the problems:

$$(w_j)_t = \delta_j \Delta(w_j) \qquad \text{in} \quad \Omega \times (0, +\infty), \tag{2.14}$$

$$w_j = \phi(1) \qquad \text{on} \quad \partial\Omega \times (0, +\infty), \tag{2.15}$$

$$w_j = 0 \qquad \text{on} \quad \Omega \times \{0\}. \tag{2.16}$$

Hence, in view of (2.13), from the comparison principle we get

Lemma 2.1

$$w_1 \le w \le w_2 \quad in \quad \Omega \times (0, +\infty).$$

We observe that the following hold:

$$\delta_1 s \le \phi(s) \le \delta_2 s \qquad \text{for} \quad s \ge 0, \tag{2.17}$$

$$-\delta_1 \log s \le -\Phi(s) \le -\delta_2 \log s \qquad \text{for} \quad 0 < s \le 1, \tag{2.18}$$

$$e^{\frac{s}{\delta_1}} \le g(s) \le e^{\frac{s}{\delta_2}} \qquad \text{for} \quad -\infty < s \le 0. \tag{2.19}$$

Let $w_j^\varepsilon = w_j^\varepsilon(x, t)$, $(j = 1, 2)$ be the functions defined by

$$w_j^\varepsilon(x, t) = w_j(x, \varepsilon^2 t).$$

With the aid of (2.17) and (2.18), it follows from Lemma 2.1 that

$$-\varepsilon^2 \delta_1 \log\left(\frac{w_2^\varepsilon}{\delta_1}\right) \le v^\varepsilon \le -\varepsilon^2 \delta_2 \log\left(\frac{w_1^\varepsilon}{\delta_2}\right) \quad \text{in} \quad \Omega \times (0, +\infty). \tag{2.20}$$

By a result in Crandall, Lions, and Souganidis [CrLS], we obtain that, as $\varepsilon \to 0^+$, the functions $-\varepsilon^2 \delta_j \log w_j^\varepsilon$ converge to the function $\frac{1}{4t} d(x)^2$ uniformly on $\overline{\Omega} \times [\tau, T]$ for each $0 < \tau < T < +\infty$, since their results work for the equation $v_t = \varepsilon^2 \delta_j \Delta v - |\nabla v|^2$ with $v = -\varepsilon^2 \delta_j \log\left(\frac{w_j^\varepsilon}{\phi(1)}\right)$. Therefore we obtain

Lemma 2.2

$$\frac{\delta_1}{\delta_2} \cdot \frac{1}{4t} d(x)^2 \le \liminf_{\varepsilon \to 0^+} v^\varepsilon(x, t) \le \limsup_{\varepsilon \to 0^+} v^\varepsilon(x, t) \le \frac{\delta_2}{\delta_1} \cdot \frac{1}{4t} d(x)^2 \quad in \quad \Omega \times (0, +\infty).$$

Hence this lemma yields

Lemma 2.3 *For any compact set K in $\Omega \times (0, +\infty)$, there exist three constants $\varepsilon_0 = \varepsilon_0(K)$, $c_1 = c_1(K)$, and $c_2 = c_2(K)$ satisfying*

$$\varepsilon_0 > 0, \ 0 < c_1 \leq c_2 < +\infty,$$

and, if $O < \varepsilon \leq \varepsilon_0$,

$$0 < c_1 \leq v^\varepsilon \leq c_2 \quad in \ \ K.$$

The key point in the proof of Theorem 1.2 is to obtain the following gradient estimate:

Lemma 2.4 *For any compact set K in $\Omega \times (0, +\infty)$, there exist two constants $\varepsilon_1 = \varepsilon_1(K)$ and $c_3 = c_3(K)$ satisfying*

$$0 < \varepsilon_1 \leq \varepsilon_0, \ c_3 > 0,$$

and, if $O < \varepsilon \leq \varepsilon_1$,

$$|\nabla v^\varepsilon| \leq c_3 \quad in \ \ K.$$

Then, by combining Lemmas 2.3 and 2.4 with Gilding's result [Gild] we have

Lemma 2.5 *For any compact set K in $\Omega \times (0, +\infty)$, there exist two constants $\varepsilon_2 = \varepsilon_2(K)$ and $c_4 = c_4(K)$ satisfying*

$$0 < \varepsilon_2 \leq \varepsilon_1, \ c_4 > 0,$$

and, if $O < \varepsilon \leq \varepsilon_2$,

$$|v^\varepsilon(x,t) - v^\varepsilon(x,s)| \leq c_4|t-s|^{\frac{1}{2}} \quad for \ \ (x,t), (x,s) \in K.$$

Thus, Lemmas 2.3, 2.4, and 2.5 imply

Theorem 2.6 *As $\varepsilon \to 0^+$, $v^\varepsilon(x,t)$ converges to $\frac{1}{4t}d(x)^2$ uniformly on every compact set in $\Omega \times (0, +\infty)$.*

In conclusion, setting $t = 1$ and $\varepsilon^2 = t$ in Theorem 2.6 yields Theorem 1.2.

It remains to prove Lemma 2.4. We use Bernstein's technique. (See Evans and Ishii [EI], Koike [Koi], Evans and Souganidis [ES], and Lions, Souganidis, and Vazquez [LSV] for the technique.) Let $K \subset B_R(0) \times [2\tau, T]$ for some $R > 0$, $0 < \tau < 2\tau < T$. Take $\zeta \in C^\infty(B_{2R}(0) \times (\tau, T])$ satisfying

$$0 \le \zeta \le 1 \text{ and } \zeta_t \ge 0 \text{ in } B_{2R}(0) \times (\tau, T],$$
$$\zeta = 1 \text{ on } B_R(0) \times [2\tau, T], \text{ and } \operatorname{supp} \zeta \subset B_{2R}(0) \times (\tau, T].$$

Consider the function $z = z(x,t)$ defined by

$$z = \zeta^2 |\nabla v^\varepsilon|^2 - \lambda v^\varepsilon, \tag{2.21}$$

where $\lambda > 0$ is a constant determined later, and $0 < \varepsilon \le \varepsilon_0$. Here, $\varepsilon_0 = \varepsilon_0(\overline{B_{2R}(0)} \times [\tau, T])$ is the constant in Lemma 2.3. Suppose that (x_0, t_0) is a point in $B_{2R}(0) \times (\tau, T]$ satisfying

$$\zeta(x_0, t_0) > 0 \text{ and } \max_{\overline{B_{2R}(0)} \times [\tau, T]} z = z(x_0, t_0).$$

At (x_0, t_0) we then have

$$z_t \ge 0, \ z_{x_i} = 0, \text{ and } \Delta z \le 0,$$

and hence

$$0 \le z_t - \varepsilon^2 \phi' \left(g(-\varepsilon^{-2} v^\varepsilon) \right) \Delta z.$$

By using (2.5) and by some calculation, we can conclude that there exist two positive constants A_1 and A_2 independent of (x_0, t_0) and ε such that at (x_0, t_0)

$$\lambda |\nabla v^\varepsilon|^2 \le A_1 |\nabla v^\varepsilon|^2 + A_2 \zeta |\nabla v^\varepsilon|^3 - 2\zeta^2 |\nabla v^\varepsilon|^2 \phi'' g' \Delta v^\varepsilon - \varepsilon^2 \phi' \zeta^2 |\nabla^2 v^\varepsilon|^2. \tag{2.22}$$

Here, we use the following key inequality:

$$0 < g'\left(-\varepsilon^{-2} v^\varepsilon\right) = \frac{g\left(-\varepsilon^{-2} v^\varepsilon\right)}{\phi'} \le \frac{1}{\delta_1} e^{-\frac{v^\varepsilon}{\varepsilon^2 \delta_2}} \le \frac{1}{\delta_1} e^{-\frac{c_1}{\varepsilon^2 \delta_2}}, \tag{2.23}$$

where $c_1 = c_1(\overline{B_{2R}(0)} \times [\tau, T])$ is the constant in Lemma 2.3. With the aid of (2.23), we observe that there exists a positive constant A_3 independent of (x_0, t_0) and ε such that at (x_0, t_0)

$$-2\zeta^2 |\nabla v^\varepsilon|^2 \phi'' g' \Delta v^\varepsilon \le \zeta^2 (A_3 |\nabla v^\varepsilon|^4 + |\nabla^2 v^\varepsilon|^2) \cdot \frac{1}{\delta_1} e^{-\frac{c_1}{\varepsilon^2 \delta_2}},$$
$$\text{and } \ -\varepsilon^2 \phi' \zeta^2 |\nabla^2 v^\varepsilon|^2 \le -\varepsilon^2 \delta_1 \zeta^2 |\nabla^2 v^\varepsilon|^2.$$

Set

$$M = \max_{\overline{B_{2R}(0)} \times [\tau, T]} \zeta |\nabla v^\varepsilon| \quad \text{and} \quad \lambda = \frac{M^2+1}{2(c_2+1)},$$

where $c_2 = c_2(\overline{B_{2R}(0)} \times [\tau, T])$ is the constant in Lemma 2.3. Choose ε_* in $(0, \varepsilon_0]$ small to get

$$\frac{A_3}{\delta_1} e^{-\frac{c_1}{\varepsilon^2 \delta_2}} \leq \frac{1}{4(c_2+1)} \quad \text{and} \quad \frac{1}{\delta_1} e^{-\frac{c_1}{\varepsilon^2 \delta_2}} \leq \varepsilon^2 \delta_1$$

for all $\varepsilon \in (0, \varepsilon_*]$. Then, at (x_0, t_0), for any $\varepsilon \in (0, \varepsilon_*]$ we have from (2.22)

$$\frac{M^2+1}{4(c_2+1)} |\nabla v^\varepsilon|^2 \leq A_1 |\nabla v^\varepsilon|^2 + A_2 M |\nabla v^\varepsilon|^2. \tag{2.24}$$

We distinguish cases:

(i) $\nabla v^\varepsilon(x_0, t_0) \neq 0$,

(ii) $\nabla v^\varepsilon(x_0, t_0) = 0$.

In case (i), we get from (2.24)

$$\frac{M^2+1}{4(c_2+1)} \leq A_1 + A_2 M,$$

which yields the gradient estimate desired. In case (ii), since $\nabla v^\varepsilon(x_0, t_0) = 0$, we get

$$M^2 \leq \max z + \lambda \max v^\varepsilon \leq \lambda \max v^\varepsilon \leq \frac{M^2+1}{2(c_2+1)} \cdot c_2 \leq \frac{M^2}{2} + \frac{1}{2},$$

and hence $M \leq 1$. This completes the proof of Lemma 2.4.

Remark. Lions, Souganidis, and Vazquez [LSV] consider the pressure equation for the porous medium equation:

$$(v_m)_t = (m-1) v_m \Delta v_m + |\nabla v_m|^2 \quad \text{for} \quad m > 1,$$

and consider the asymptotic behavior as $m \to 1^+$. They get the interior gradient estimate for v_m independent of m by the technique similar to ours. We follow them, but we use inequality(2.23) in order to overcome the difficulty caused by $\phi' = \phi'\left(g\left(-\varepsilon^{-2} v^\varepsilon\right)\right)$ in equation (2.5).

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