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Asymptotic solutions of a class of Hamilton-Jacobi equations*

大阪大学大学院基礎工学研究科 市原直幸 (Naoyuki Ichihara) †
Graduate School of Engineering Science,
Osaka University

概 要

We study the long time behavior of viscosity solutions to some Cauchy problem for Hamilton-Jacobi equations. The generalized dynamical approach due to Davini and Siconolfi is adopted. Contrary to the periodic situation they dealt with, we consider Hamilton-Jacobi equations having some non-periodic perturbations in both Hamiltonian and initial data. We also discuss the representation of corresponding asymptotic solutions.

1 Introduction.

This paper is concerned with Hamilton-Jacobi equations of the form

$$\text{(1)} \qquad \begin{cases} u_t + H(x, Du) - f(x) = 0 & \text{in } \mathbb{R}^n \times (0, +\infty), \\ u(\cdot, 0) = u_0(\cdot) & \text{on } \mathbb{R}^n, \end{cases}$$

where the Hamiltonian $H = H(x,p)$ is assumed to be $\mathbb{Z}^n$-periodic in x and convex and coercive in p. The function f, regarded as a perturbation of the original Hamiltonian H, is allowed to be non-periodic. The initial datum u_0 is assumed to behave like a $\mathbb{Z}^n$-periodic function as $|x| \to +\infty$. More precise conditions on these functions will be stated in the next section.

The objective of this paper is to investigate the large time behavior of continuous viscosity solutions of (1), namely we seek for a constant $c \in \mathbb{R}$ and a function $v(\cdot)$ on $\mathbb{R}^n$ such that as $t \to +\infty$,

$$\text{(2)} \qquad u(x,t) + ct - v(x) \longrightarrow 0 \qquad \text{uniformly on compact subsets of } \mathbb{R}^n.$$

The function $v(x) - ct$ is called the asymptotic solution of the Cauchy problem (1). While the constant c does not depend on initial data, v may change according to the choice of u_0.

*Joint work with Hitoshi Ishii.

†JSPS Post-Doctoral Research Fellow. E-mail: naoyuki@sigmath.es.osaka-u.ac.jp. Supported in part by the JSPS Research Fellowship for Young Scientists.

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Researches on the large time behavior of viscosity solutions to Hamilton-Jacobi equations have been growing in recent years. The first attempt to attack such problem was made by Fathi [7, 8] in the framework of his weak KAM theory. Recently, Davini-Siconolfi [6] improved his results; they proved the convergence (2) for Hamilton-Jacobi equations in the unit torus $\mathbb{T}^n$ with convex and coercive Hamiltonian (i.e., the case where $f = 0$ and u_0 is $\mathbb{Z}^n$-periodic). Their idea is based on the study of PDE aspects of the Aubry-Mather theory developed by Fathi-Siconolfi [10]. Concerning asymptotic problems in non-compact regions, Fujita-Ishii-Loreti [12] and Ishii [14] treat Hamilton-Jacobi equations on Euclidean n space $\mathbb{R}^n$. See also [11] for viscous version of this problem.

On the other hand, by another approach based mainly on PDE techniques, Namah-Roquejoffre [16], Barles-Souganidis [3, 4] and Barles-Roquejoffre [2] investigate same kinds of asymptotic problems under a different sort of assumptions on Hamiltonians admitting, in some cases, non-convex ones.

Motivated by the paper of Davini-Siconolfi [6], we deal with a perturbed version of their asymptotic problem by using the former approach of dynamical systems. In order to clarify the motivation as well as the novelty of this paper, we start with the case where $f = 0$ in (1):

$$(3) \qquad \begin{cases} u_t + H(x, Du) = 0 & \text{in } \mathbb{R}^n \times (0, +\infty), \\ u(\,\cdot\,, 0) = u_0(\,\cdot\,) & \text{on } \mathbb{R}^n. \end{cases}$$

Suppose that u_0 is continuous and $\mathbb{Z}^n$-periodic. Then, the problem is reduced to that of [6], which can be rewritten in our context as follows:

Theorem 1.1 (c.f. Theorem 5.7 of [6]). *Assume that u_0 is continuous and $\mathbb{Z}^n$-periodic, and let $\hat{u}$ be the unique $\mathbb{Z}^n$-periodic continuous viscosity solution of (3). We define c by*

$$(4) \qquad c := \inf\{a \in \mathbb{R}\,;\, H(x, Dv) = a \ \text{ in } \mathbb{R}^n \text{ has a } \mathbb{Z}^n\text{-periodic subsolution}\}.$$

Then, there exists a $\mathbb{Z}^n$-periodic viscosity solution $\hat{v}$ of the Hamilton-Jacobi equation

$$(5) \qquad -c + H(x, Dv) = 0 \qquad \text{in } \mathbb{R}^n$$

such that as $t \to +\infty$,

$$(6) \qquad \hat{u}(x, t) + ct - \hat{v}(x) \longrightarrow 0 \qquad \text{uniformly in } \mathbb{R}^n.$$

So, Cauchy problem (1) is a perturbed version of (3). However, we emphasize that this is not a simple generalization of [6]. Indeed, it is known that the convergence of the form (2) easily fails in non-periodic situations.

One of the fundamental differences between [6] and the present paper can be explained as follows. Due to the lack of uniqueness of solutions to the Hamilton-Jacobi equation in the limit as $t \to +\infty$:

$$(7) \qquad -c + H(x, Dv) - f(x) = 0 \qquad \text{in } \mathbb{R}^n,$$

it is important to find an appropriate uniqueness set in $\mathbb{R}^n$ called the (projected) Aubry set. That is to say, this set, say U, plays a significant role in establishing the comparison theorem of the form

$$v_1 \le v_2 \quad \text{on } \mathcal{U} \quad \Longrightarrow \quad v_1 \le v_2 \quad \text{on } \mathbb{R}^n$$

for solutions v_1, v_2 to (7). Remark that $\mathcal{U}$ is a closed set and is characterized as

$$\mathcal{U} = \{y \in \mathbb{R}^n \,;\text{ there is no subsolution strict at } y\}.$$

Note also that, in the periodic setting, $\mathcal{U}$ becomes the totality of points at which there is no $\mathbb{Z}^n$-periodic strict subsolution (see Section 5 for details).

We may classify $\mathcal{U}$ as the following three possible situations:
Case (A): $\mathcal{U}$ is non-empty and compact.
Case (B): $\mathcal{U}$ is empty.
Case (C): $\mathcal{U}$ is non-empty and non-compact.
Davini-Siconolfi [6] stays in the case (A) by virtue of the compactness of the state space $\mathbb{T}^n$. But, once the periodicity has been broken by a perturbation f, (B) or (C) occurs and the situation changes completely. That is the main difference between [6] and ours. We also point out that the papers [12, 14] are still in the case (A) although they treat equations in the whole space $\mathbb{R}^n$.

In this paper, we restrict ourselves to the case (B) by adding an additional assumption. The study of asymptotic problems when (C) takes place will be left in future investigation. Note that our work is closely related to the literature [2] which also treats the case (B) by another approach in a slightly different setting.

Before closing this introductory section, we make a brief comment on the representation of asymptotic solutions of (1). Since $\mathcal{U}$ is empty in our case, we have no representation formula for asymptotic solutions in the classical sense. So, getting such a formula in some sense is much of interest. It turns out in Section 5 that our uniqueness set is hidden at the "infinity". By taking account of this fact, we can establish a comparison theorem (Proposition 5.4) which makes us possible to specify solutions of (7) in terms of their behavior as $|x| \to +\infty$, and to get the representation formula (Proposition 6.3) of asymptotic solutions as well.

This paper is organized as follows. The next section is devoted to preliminaries. The main theorem is stated precisely at the end of the section. We discuss, in Section 3, the additive eigenvalue problem (7). Section 4 is concerned with some properties of curves in $\mathbb{R}^n$ that will be useful in the sequel. In Section 5, we determine the uniqueness set for the equation (7). The proof of the main theorem and the representation formula for v are given in the last section. We also collect some fundamental facts in Appendix.

2 Preliminaries.

Let $C(\mathbb{R}^n)$ be the totality of continuous functions on $\mathbb{R}^n$ equipped with the topology of locally uniform convergence, that is, we say a family of functions $\{u_j\}_{j\in\mathbb{N}} \subset C(\mathbb{R}^n)$ converges to a function u in $C(\mathbb{R}^n)$ if and only if $u_j(x) \to u(x)$ as $j \to +\infty$ uniformly on any compact subsets of $\mathbb{R}^n$. We often use the following subclasses of $C(\mathbb{R}^n)$:

$$\begin{aligned} BC(\mathbb{R}^n) &:= \{u \in C(\mathbb{R}^n)\,;\, |u|_\infty := \sup_{x\in\mathbb{R}^n} |u(x)| < +\infty\}, \\ BUC(\mathbb{R}^n) &:= \{u \in BC(\mathbb{R}^n)\,;\, u \text{ is uniformly continuous}\}, \\ C_c(\mathbb{R}^n) &:= \{u \in BUC(\mathbb{R}^n)\,;\, \mathrm{supp}(u) \text{ is compact}\}. \end{aligned}$$

Throughout this paper, we identify functions on the unit torus $\mathbb{T}^n$ with their $\mathbb{Z}^n$-periodic extension to the whole space $\mathbb{R}^n$.

For a closed interval J in the real line, the set of all absolutely continuous functions on J with values in $\mathbb{R}^n$ is denoted by $AC(J,\mathbb{R}^n)$. For given $-\infty \le S < T \le +\infty$ and $x, y \in \mathbb{R}^n$, we set

$$\begin{aligned} \mathcal{C}([S,T];x) &:= \{\gamma \in AC([S,T],\mathbb{R}^n)\,;\, \gamma(T) = x\}, \\ \mathcal{C}([S,T];y,x) &:= \{\gamma \in AC([S,T],\mathbb{R}^n)\,;\, \gamma(S) = y \text{ and } \gamma(T) = x\}. \end{aligned}$$

Let us consider the Cauchy problem (1). In this paper, the notion of solution, subsolution and supersolution will be interpreted in the viscosity sense. The standing assumptions on the Hamiltonian $H_f(x,p) := H(x,p) - f(x)$ and initial data are the following.

Assumption 1.
(H1) $H \in C(\mathbb{R}^n \times \mathbb{R}^n)$.
(H2) H is coercive, i.e. $\lim_{|p|\to+\infty} \inf_{x\in\mathbb{R}^n} H(x,p) = +\infty$.
(H3) $H(x,\cdot)$ is strictly convex in p for every $x \in \mathbb{R}^n$.
(H4) $H(\cdot,p)$ is $\mathbb{Z}^n$-periodic in x for every $p \in \mathbb{R}^n$.
(f1) $f \in C_c(\mathbb{R}^n)$ and $f \ge 0$.
(u1) $\lim_{R\to+\infty} \sup_{|x|\ge R} |u_0(x) - \hat{u}_0(x)| = 0$ for some $\mathbb{Z}^n$-periodic function $\hat{u}_0 \in BUC(\mathbb{R}^n)$ not exceeding u_0 on $\mathbb{R}^n$.

Remark 2.1. Assumption (u1) can be weakened if we impose a slightly stronger assumption on the Hamiltonian. See Section 6 for details.

The existence, uniqueness and the dynamic programming principle of solutions to (1) are standard in the theory of viscosity solutions.

Theorem 2.2. *Suppose that (H1)-(H4) and (f1) hold. Then, for every $u_0 \in BUC(\mathbb{R}^n)$, the function $u : \mathbb{R}^n \times [0,+\infty) \longrightarrow \mathbb{R}$ defined by*

$$u(x,t) := \inf\left\{ \int_{-t}^{0} L_f(\gamma(s),\dot\gamma(s))\,ds + u_0(\gamma(-t)) \;\middle|\; \gamma \in \mathcal{C}([-t,0];x)\right\} \tag{8}$$

is the unique solution of (1) in the class $C(\mathbb{R}^n)$, where L_f stands for the Lagrangian associated with H_f, i.e., $L_f(x,\xi) := L(x,\xi) + f(x) = \sup_{p\in\mathbb{R}^n}(\xi\cdot p - H(x,p)) + f(x)$.

Moreover, for all $t, s > 0$ and $x \in \mathbb{R}^n$, u satisfies

$$u(x,s+t) := \inf\left\{ \int_{-t}^{0} L_f(\gamma(r),\dot\gamma(r))\,dr + u(\gamma(-t),s) \;\middle|\; \gamma \in \mathcal{C}([-t,0];x)\right\}. \tag{9}$$

Let c be the constant defined by (4) and consider the Hamilton-Jacobi equation (7). We define $\mathcal{U}_f$ by

$$\mathcal{U}_f := \{y \in \mathbb{R}^n\,;\ \text{there is no subsolution of (7) strict at } y\}. \tag{10}$$

Here, we say a subsolution ϕ of (7) is strict in a subset $D \subset \mathbb{R}^n$ if there exists $\delta > 0$ such that $-c + H_f(y, D\phi(y)) \le -\delta$ for all $y \in D$ in the viscosity sense.

We also make an additional assumption in order to exclude the case (C) from our consideration.

Assumption 2. $\mathcal{U}_0 = \emptyset$, where $\mathcal{U}_0$ is defined by (10) with $f = 0$.

Remark. It is not difficult to check that Assumption 2 is equivalent to assume that $\mathcal{U}_f = \emptyset$ and $\mathrm{supp}(f) \cap \mathcal{U}_0 = \emptyset$. A natural interpretation of Assumption 2 will be given in Section 4 (see Remark 4.4).

The next example is one of the most typical and simplest ones satisfying Assumptions 1 and 2.

Example. Let $n = 1$ and $H(x,p) := |p-1|^2 - V(x)$, where $V \in C(\mathbb{R})$ is non-negative, $\mathbb{Z}$-periodic and $\min_{x\in\mathbb{R}} V(x) = 0$. Suppose that $\int_0^1 \sqrt{V(x)}\,dx < 1$. Then, we can check that $c > 0$ (see for example [15]). It is easily seen that the function $v(x) := x + \int_0^x \sqrt{V(z)}\,dz$ is a subsolution of (5) strict in $\mathbb{R}$. In particular, $\mathcal{U}_0 = \emptyset$, where $\mathcal{U}_0$ is defined by (10) with $f = 0$. Since $\mathcal{U}_f \subset \mathcal{U}_0$, we have $\mathcal{U}_f = \emptyset$.

Suppose now that $\int_0^1 \sqrt{V(x)}\,dx \ge 1$. Then, we have $c = 0$ and $\mathcal{U}_0 = V^{-1}(0) := \{y \in \mathbb{R}^n\,;\ V(y) = 0\} \neq \emptyset$. Thus, $\mathcal{U}_f = V^{-1}(0) \setminus \mathrm{supp}(f) \neq \emptyset$ and this gives an example of the case (C).

We are now in position to formulate our main result (Theorem 2.4).

Proposition 2.3. *Suppose that Assumptions 1 and 2 hold and let u be the unique solution of (1). Then, $u(x,t) + ct$ is bounded and uniformly continuous on $\mathbb{R}^n \times [0, +\infty)$.*

Proof. The proof will be postponed until Section 6. □

Theorem 2.4. *Under Assumptions 1 and 2, there exists a solution v of (7) such that the convergence (2) holds.*

Notice. In order to prove Theorem 2.4, we can assume $c = 0$ without loss of generality. Indeed, it suffices to consider the Hamiltonian $H_f - c$ and the solution $u(x,t) + ct$ in place of H_f and $u(x,t)$, respectively. Thus, we henceforth assume that $c = 0$ for the simplicity of description.

3 Additive eigenvalue problems.

In this section, we study the solvability of Hamilton-Jacobi equation (7). Since v in (2) is expected to be bounded in view of Proposition 2.3, we seek for solutions in the class $BC(\mathbb{R}^n)$.

For this purpose, we start with the following equation called the additive eigenvalue problem:

$$H_f(x, Dv(x)) = a \qquad \text{in } \mathbb{R}^n, \tag{11}$$

where unknowns are $a \in \mathbb{R}$ and $v \in C(\mathbb{R}^n)$. The solvability of (11) in the class $C(\mathbb{R}^n)$ is known (see [9] or Theorem 2.1 of [2]).

Theorem 3.1. *For $g \in BUC(\mathbb{R}^n)$, we define the critical eigenvalue $a_g \in \mathbb{R}$ by*

$$a_g := \inf\{a \in \mathbb{R}\,;\, H(x, Dv) - g(x) = a \ \textit{ in } \mathbb{R}^n \textit{ has a subsolution}\}.$$

Then, for every $a \geq a_g$, the equation $H(x, Dv) - g(x) = a$ in $\mathbb{R}^n$ has continuous solutions.

Remark here that by virtue of the coercivity of $H_f(x,p)$ in p, every solution of (11) is uniformly Lipschitz continuous with a universal constant $M > 0$ depending only on H_f and a. However, (11) may not have bounded solutions even in the case where $f = 0$. Actually, the solvability of (11) in the class $BC(\mathbb{R}^n)$ is closely related to the structure of the non-perturbed additive eigenvalue problem

$$H(x, Dv(x)) = a \qquad \text{in } \mathbb{R}^n. \tag{12}$$

It is known (e.g. [10, 13]) that (12) has bounded solutions if and only if $a = 0$ (recall that $c = 0$ by normalization).

We now claim that our perturbed problem (11) has a bounded solution only if $a = 0$.

Lemma 3.2. *Suppose that (11) has a bounded solution. Then, $a = 0$.*

Proof. Let v be a bounded solution of (11). Let $e \in \mathbb{Z}^n \setminus \{0\}$ and define v_k, $f_k \in BC(\mathbb{R}^n)$, $k \in \mathbb{N}$, by $v_k(x) := v(x + ke)$ and $f_k(x) := f(x + ke)$, respectively. Then, v_k is a solution of

$$H(x, Dv_k(x)) - f_k(x) = a \qquad \text{in } \mathbb{R}^n,$$

and $\{v_k(\cdot) - v_k(0)\}_{k \in \mathbb{N}}$ is uniformly bounded and equi-continuous on $\mathbb{R}^n$. Hence, there is an increasing sequence $k_j \to +\infty$ such that $v_{k_j}(\cdot) - v_{k_j}(0) \to w$ in $C(\mathbb{R}^n)$ for some $w \in BC(\mathbb{R}^n)$ as $j \to +\infty$. In the limit as $j \to +\infty$, we see that w is a bounded solution of (12), which implies that $a = 0$. □

Thus, in the rest of this section, we concentrate on the equation

$$H_f(x, Dv(x)) = 0 \qquad \text{in } \mathbb{R}^n. \tag{13}$$

Proposition 3.3. *Let a_f be the critical eigenvalue of (11). Then, (13) has bounded subsolutions if and only if $a_f \leq 0$.*

Proof. It is clear by the definition of a_f that the existence of bounded subsolutions of (13) implies $a_f \leq 0$. So, it remains to prove that $a_f \leq 0$ implies the existence of bounded subsolutions. We shall construct one by a cut off argument.

Fix a (possibly unbounded) subsolution $v \in C(\mathbb{R}^n)$ of (13) and let $\overline{v} \in BC(\mathbb{R}^n)$ be any $\mathbb{Z}^n$-periodic solution of

$$H(x, Dv(x)) = 0 \qquad \text{in } \mathbb{R}^n. \tag{14}$$

By adding a constant in advance, we may assume that $v \leq \overline{v}$ on supp(f).

Choose next $A > 0$ so that $\overline{v} - A \leq v$ on supp(f) and define $\underline{v} \in BC(\mathbb{R}^n)$ by

$$\underline{v}(x) := \min\left\{ \max\{v(x), \overline{v}(x) - A\}, \overline{v}(x)\right\}, \quad x \in \mathbb{R}^n.$$

It is standard to show that $w(x) := \max\{v(x), \overline{v}(x) - A\}$ is a subsolution of (13) since $\overline{v} - A$ is also a subsolution of (13). Moreover, from the study of semicontinuous viscosity solutions for Hamilton-Jacobi equations with convex Hamiltonians due to Barron and Jensen [5], we can prove that $\underline{v}(x) := \min\{w(x), \overline{v}(x)\}$ is also a subsolution of (13). Hence, $\underline{v}$ is a bounded subsolution of (13). □

Corollary 3.4. *Under Assumption 1, (13) has bounded subsolutions.*

Proof. Let a_0 be the critical eigenvalue of (12). Then, we can see that $a_0 = a_f$ by the same argument as in the proof of Lemma 3.2. Since $a_0 \leq 0$, the claim is obvious from the previous proposition. □

Once the existence of a bounded subsolution of (13) has been guaranteed, it is not hard to construct bounded solutions of (13). We will discuss this point in Section 5.

The following lemma will be used in the next section.

Lemma 3.5. *For any compact subset $K \subset \mathbb{R}^n$, there exists a bounded subsolution ϕ of (13) strict in K.*

Proof. For $y \in K$ and a subsolution ϕ_y of (13) strict and C^1 at y, there exist $r_y > 0$ and $\delta_y > 0$ such that

$$H_f(x, D\phi_y(x)) \leq -\delta_y \qquad \text{for all } x \in B(y, r_y),$$

where $B(y, r_y)$ stands for the closed ball in $\mathbb{R}^n$ centered at y with radius r_y. Choose a finite covering $\{B(y_i, r_{y_i})\}_{i=1}^m$ of K and define $\phi \in C(\mathbb{R}^n)$ by

$$\phi(x) := \sum_{i=1}^m \lambda_i \phi_{y_i}(x) \qquad x \in \mathbb{R}^n,$$

where $\sum_{i=1}^m \lambda_i = 1$ and $\lambda_i > 0$ for all $i = 1, \ldots, m$. By the convexity of H, we can check that ϕ is a subsolution of (13). Moreover, for any $x \in K$, there exists a number j such that $x \in B(y_j, r_{y_j})$ and

$$\begin{aligned} H_f(x, D\phi(x)) &\leq \sum_{i \neq j} \lambda_i H_f(x, D\phi_{y_i}(x)) + \lambda_j H_f(x, D\phi_{y_j}(x)) \\ &\leq -\lambda_j \delta_{y_j} \leq -\min_i \lambda_i \delta_{y_i} < 0. \end{aligned}$$

Similarly as in the proof of Proposition 3.3, we can construct a bounded subsolution of (13) equating ϕ on K. Hence, we have completed the proof. □

4 Curves in $\mathbb{R}^n$.

This section is devoted to some properties of curves in $\mathbb{R}^n$. It turns out in Proposition 4.3 that Assumption 2 is concerned with their long time behavior.

Lemma 4.1. *Let S and T be such that $-\infty \le S < S+1 \le T \le +\infty$, and suppose that a curve $\eta \in AC([S,T],\mathbb{R}^n)$ satisfies*

$$\int_a^b L_f(\eta(s),\dot\eta(s))\,ds \le C_f \qquad S<\forall a<\forall b<T \tag{15}$$

for some constant $C_f>0$. Then, for every $\varepsilon>0$, there exists $M_\varepsilon>0$ depending only on C_f, H_f and ε such that

$$\int_a^b |\dot\eta(s)|\,ds \le \varepsilon + M_\varepsilon(b-a) \qquad S<\forall a<\forall b<T.$$

Proof. This lemma is a direct consequence of Proposition 5.9 in [14]. □

Lemma 4.2. *Let $\eta \in AC([S,T],\mathbb{R}^n)$ be any curve such that*

$$\int_a^b L(\eta(s),\dot\eta(s))\,ds \le C_0 \qquad S<\forall a<\forall b<T \tag{a}$$

for some constant $C_0>0$. Then, η satisfies (15) for some constant $C_f>0$.

Proof. Since $\mathrm{supp}(f)\cap\mathcal{U}_0=\emptyset$, we can show similarly as in the proof of Lemma 3.5 that

$$H(x,D\phi(x)) \le -\delta \qquad \text{on} \quad \mathrm{supp}(f),$$

for some $\delta>0$ and a bounded subsolution ϕ of (14).

We set $I := \{s\in[S,T]\,;\,\eta(s)\in\mathrm{supp}(f)\}$. Then,

$$\begin{aligned}\phi(\eta(T))-\phi(\eta(S)) &\le \int_S^T \{L(\eta(s),\dot\eta(s))+H(\eta(s),D\phi(\eta(s))\}\,ds\\ &\le C_0-\delta\, m(I),\end{aligned}$$

where $m(I)$ denotes the Lebesgue measure of I. Thus, we have $m(I)\le\delta^{-1}(C_0+2|\phi|_\infty)<+\infty$, and for all $S<a<b<T$,

$$\int_a^b L_f(\eta(s),\dot\eta(s))\,ds \le C_0+\int_a^b f(\eta(s))\,ds \le C_0+\delta^{-1}|f|_\infty(C+2|\phi|_\infty),$$

which implies (15) since the right-hans side is independent of $a<b$. □

Proposition 4.3. *Let $\eta\in AC((-\infty,0],\mathbb{R}^n)$ be any curve satisfying (15) with $S=-\infty$ and $T=0$. Then, for every compact set $K\subset\mathbb{R}^n$, we have*

$$\tau := \sup\{t>0\,;\,\eta(-t)\in K\} < +\infty.$$

In particular, $|\eta(-t)| \longrightarrow +\infty$ as $t\to+\infty$.

Proof. Suppose that $\tau = +\infty$. Then, there exists a positive diverging sequence $\{t_k\}_{k\in\mathbb{N}}$ such that $\eta(-t_k) \in K$ for all $k \in \mathbb{N}$. In particular, by taking a subsequence if necessary, we may assume that $\eta(-t_k) \longrightarrow z$ for some $z \in K$ as $k \to +\infty$.

In view of Lemma 3.5, we can take a bounded subsolution ϕ of (13) such that

$$H_f(x, D\phi(x)) \leq -\delta \qquad \text{in } B(z, 4r)$$

for some $\delta > 0$ and $r > 0$. By renumbering $\{t_k\}_{k\in\mathbb{N}\cup\{0\}}$ if necessary, we may assume that $\eta(-t_0) \notin B(z, 3r)$ and $\eta(-t_k) \in B(z, r)$ for all $k \in \mathbb{N}$. Let us now set $\sigma_0 := t_0$ and define inductively σ_k and τ_k by

$$\begin{aligned}
\sigma_k &:= \min\{t > t_k\,;\, \eta(-t) \notin B(z, 3r)\},\\
\tau_k &:= \max\{\sigma_{k-1} < t < t_k\,;\, \eta(-t) \notin B(z, 3r)\}.
\end{aligned}$$

We set $\sigma_k := +\infty$ if $\{\cdots\} = \emptyset$. Since $\eta(-t_k) \in B(z, r)$, we see by Lemma 4.1 that

$$4r \leq \int_{-\sigma_k}^{-\tau_k} |\dot{\eta}(s)|\, ds \leq r + M_r(\sigma_k - \tau_k)$$

for some $M_r > 0$ not depending on $k \in \mathbb{N}$. Thus, by setting

$$I_t := \{s \in [\tau_1, t]\,;\, \eta(-s) \in B(z, 3r)\}, \qquad t \in [\tau_1, +\infty],$$

we see

$$m(I_\infty) = \lim_{t\to+\infty} m(I_t) \geq \sum_{k=1}^{N} (\sigma_k - \tau_k) \geq \frac{3rN}{M_r} \qquad \text{for all } N \in \mathbb{N}.$$

On the other hand,

$$\begin{aligned}
\phi(\eta(-\tau_1)) - \phi(\eta(-t)) &= \int_{-t}^{-\tau_1} D\phi(\eta(s))\, \dot{\eta}(s)\, ds\\
&\leq \int_{-t}^{-\tau_1} \{L_f(\eta(s), \dot{\eta}(s)) + H_f(\eta(s), D\phi(\eta(s)))\}\, ds\\
&\leq C_f - \delta\, m(I_t).
\end{aligned}$$

By letting $t \to +\infty$, we obtain

$$3M_r^{-1} rN \leq m(I_\infty) \leq \delta^{-1}(C_f + 2|\phi|_\infty) < +\infty.$$

Since N is arbitrary, we get the contradiction. Hence $\tau < +\infty$. □

Remark 4.4. This proposition shows that Assumption 2 is crucial for the property $|\eta(-t)| \longrightarrow +\infty$ as $t \to +\infty$.

For $x, y \in \mathbb{R}^n$, we set

$$d_f(x, y) := \inf\left\{ \int_0^t L_f(\gamma(s), \dot{\gamma}(s))\, ds \;\Big|\; t > 0,\ \gamma \in \mathcal{C}([0, t]; y, x) \right\}. \tag{16}$$

It can be checked that the right-hand side of (16) is finite for all $x, y \in \mathbb{R}^n$. By Proposition A.2 (e) in Appendix, $d_f(\,\cdot\,, y)$ is a subsolution of (13) in $\mathbb{R}^n$ and is a supersolution in $\mathbb{R}^n \setminus \{y\}$. Moreover, By Lemma A.3, d_f is lower bounded on $\mathbb{R}^n \times \mathbb{R}^n$ since there exists a bounded subsolution of (13).

Lemma 4.5. *Let $\eta \in AC((-\infty,0];\mathbb{R}^n)$ be such that*

$$\lim_{k\to\infty}\int_{-t_k}^{0} L_f(\eta,\dot\eta)\,ds < +\infty \tag{17}$$

for some diverging sequence $\{t_k\}_{k\in\mathbb{N}}$. Then, there exists a subsequence $\{t_{k_l}\}_{l\in\mathbb{N}}$ such that $\{y_l\}_{l\in\mathbb{N}} := \{\eta(-t_{k_l})\}_{l\in\mathbb{N}}$ satisfies the following:

$$\lim_{k\to+\infty}\lim_{l\to+\infty} d_f(y_k,y_l) = 0. \tag{18}$$

Proof. We set $c_k := \int_{-t_k}^{0} L_f(\eta,\dot\eta)\,ds$. Then, for every $\varepsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that

$$d_f(\eta(-t_k),\eta(-t_{k+m})) \le \int_{-t_{k+m}}^{-t_k} L_f(\eta,\dot\eta)\,ds = c_{k+m} - c_k < \varepsilon$$

for all $k \ge k_0$ and $m \in \mathbb{N}$.

Now, we fix any $\mathbb{Z}^n$-periodic subsolution ϕ of (14) and take a subsequence $\{t_{k_l}\}_{l\in\mathbb{N}}$ so that $\{z_l\}_{l\in\mathbb{N}} := \{\phi(\eta(-t_{k_l}))\}_{l\in\mathbb{N}}$ forms a Cauchy sequence. Then, there exists $l_0 \in \mathbb{N}$ such that

$$-\varepsilon < \phi(z_l) - \phi(z_{l+m}) \le d_f(z_l, z_{l+m}) < \varepsilon$$

for all $l \ge l_0$ and $m \in \mathbb{N}$. Hence, we have completed the proof. $\square$

5 Uniqueness set.

In this section, we seek for a uniqueness set for (13). As is pointed out in the introduction, the asymptotic behavior of solutions to (13) as $|x| \to +\infty$ has an important role to specify their structure.

We first consider the equation (14) under $\mathbb{Z}^n$-periodic setting and define

$$\mathcal{A} := \{y \in \mathbb{R}^n\,;\ \text{there is no } \mathbb{Z}^n\text{-periodic subsolution of (14) strict at } y\} \ne \emptyset.$$

Remark that $\mathcal{A}$ is nothing but the $\mathbb{Z}^n$-periodic extension of the Aubry set for the following equation in the unit torus $\mathbb{T}^n$:

$$H(x,Du(x)) = 0 \qquad \text{in } \mathbb{T}^n. \tag{19}$$

See [10] for the precise definition of the Aubry set for (19). In particular, $\mathcal{A}$ is $\mathbb{Z}^n$-periodic, namely $\mathcal{A} = \mathcal{A} + e := \{y + e\,;\ y \in \mathcal{A}\}$ for all $e \in \mathbb{Z}^n$.

Proposition 5.1. *Let D be any open set satisfying* $\mathrm{supp}(f) \subset D$. *Then, for every bounded solution u of (13), the following formula is valid:*

$$u(x) = \inf_{y\in\mathcal{A}\setminus D}\big(d_f(x,y) + u(y)\big). \tag{20}$$

Proof. We divide the proof into several steps.

1. We denote the right-hand side of (20) by $v(x)$ and show $u = v$ on $\mathbb{R}^n$. By Proposition A.4 in Appendix, we see that $u \le v$ on $\mathbb{R}^n$ and $u = v$ on $\mathcal{A} \setminus D$. So, it remains to prove that $u = v$ outside $\mathcal{A} \setminus D$.

2. Suppose that $v(y) - u(y) =: 4\beta > 0$ for some $\beta > 0$ and $y \notin \mathcal{A} \setminus D$. Then, there exists $\rho_0 > 0$ such that

$$y \notin K_\rho^D := \{x \in \mathbb{R}^n\, ;\, \mathrm{dist}(x, \mathcal{A} \setminus D) \le \rho\}$$

for all $0 < \rho \le \rho_0$. We fix $\rho > 0$ so that $\mathrm{supp}(f) \cap K_\rho^D = \emptyset$ and $\rho < (2M)^{-1}\beta$, where $M > 0$ denotes the universal Lipschitz constant for subsolutions of (13).

3. We set $K_\rho := \{x \in \mathbb{R}^n\, ;\, \mathrm{dist}(x, \mathcal{A}) \le \rho\}$. Then, from Section 6 of [10], we can construct a $\mathbb{Z}^n$-periodic subsolution $\phi_1 \in BC(\mathbb{R}^n) \cap C^1(\mathbb{R}^n \setminus K_\rho)$ of (14) satisfying the strict subsolution property:

$$H(x, D\phi_1(x)) \le -\delta_1 \quad \text{in } \mathbb{R}^n \setminus K_\rho \quad \text{for some } \delta_1 > 0. \tag{21}$$

On the other hand, by Lemma 3.5, there exist $\delta_2 > 0$ and a bounded subsolution ϕ_2 of (13) such that

$$H_f(x, D\phi_2(x)) \le -\delta_2 \quad \text{in } \overline{D}. \tag{22}$$

4. Let $\psi \in C_c^\infty(\mathbb{R}^n)$ be such that $\mathrm{supp}(\psi) \subset B(0,1)$ and $\int_{\mathbb{R}^n} \psi(x)\, dx = 1$. We set $\psi_\varepsilon(x) := \varepsilon^{-n}\psi(\varepsilon^{-1}x)$. For $\lambda_1, \lambda_2 \in (0,1)$ satisfying $\lambda_1 + \lambda_2 < 1$, we define $w \in C^1(\mathbb{R}^n)$ by

$$w(x) := \lambda_1\phi_1(x) + \lambda_2(\phi_2 * \psi_\varepsilon)(x) + (1 - \lambda_1 - \lambda_2)(v * \psi_\varepsilon)(x),$$

and for $\alpha > 0$ we set $w_\alpha(x) := w(x) - \alpha(|x - y|^2 + 1)^{1/2}$, where $(\phi_2 * \psi_\varepsilon)(\cdot)$ and $(v * \psi_\varepsilon)(\cdot)$ stand for mollified functions of ϕ_2 and v by ψ_ε, respectively. Since v is Lipschitz continuous with Lipschitz constant $M > 0$, we have $|v * \psi_\varepsilon - v|_\infty \le M\varepsilon$. Thus,

$$\begin{aligned}
|w - v|_\infty &\le \lambda_1|\phi_1|_\infty + \lambda_2|\phi_2|_\infty + (\lambda_1 + \lambda_2)|v|_\infty + |v * \psi_\varepsilon - v|_\infty \\
&\le \lambda_1|\phi_1|_\infty + \lambda_2|\phi_2|_\infty + (\lambda_1 + \lambda_2)|v|_\infty + M\varepsilon \\
&=: \omega_1(\varepsilon, \lambda_1, \lambda_2).
\end{aligned}$$

We choose ε, λ_1 and λ_2 so that $\omega_1(\varepsilon, \lambda_1, \lambda_2) < \beta$. Then, for $\alpha < \beta$, we have

$$w_\alpha(y) = w(y) - \alpha \ge v(y) - \omega_1(\varepsilon, \lambda_1, \lambda_2) - \alpha > u(y) + 2\beta. \tag{23}$$

5. In view of the convexity of H in p, there exists a constant $C > 0$ such that

$$\big|H(x,p) - H(x,q)\big| \le C\,|p - q| \qquad \text{for all } x \in \mathbb{R}^n, \quad p, q \in B(0, M+1).$$

Then, we have

$$\begin{aligned}
H_f(x, Dw_\alpha(x)) &\le H_f(x, Dw(x)) + C\alpha \\
&\le \lambda_1 H_f(x, D\phi_1(x)) + \lambda_2 H_f(x, D(\phi_2 * \psi_\varepsilon)(x)) \\
&\qquad + (1 - \lambda_1 - \lambda_2) H_f(x, D(v * \psi_\varepsilon)(x)) + C\alpha \\
&=: \lambda_1 I_1(x) + \lambda_2 I_2(x) + (1 - \lambda_1 - \lambda_2) I_3(x) + C\alpha.
\end{aligned}$$

6. By taking into account that $f \equiv 0$ in $\mathbb{R}^n \setminus \overline{D}$, we can show in combination with (21) that $I_1(x) \le |f|_\infty$ in $\overline{D} \cup K_\rho$ and $I_1(x) \le -\delta_1$ in $(\mathbb{R}^n \setminus \overline{D}) \cap (\mathbb{R}^n \setminus K_\rho)$. The convexity of H in p and (22) yield

$$
\begin{aligned}
I_2(x) &\le \int_{B(x,\varepsilon)} \psi_\varepsilon(x-z) H_f(z, D\phi_2(z))\, dz \\
&\qquad + \sup_{z \in B(x,\varepsilon)} \left| H_f(x, D\phi_2(z)) - H_f(z, D\phi_2(z)) \right| \\
&\le \begin{cases} -\delta_2 + \omega_{H_f}(\varepsilon) & \text{in } \overline{D}, \\ \omega_{H_f}(\varepsilon) & \text{in } \mathbb{R}^n \setminus \overline{D}, \end{cases}
\end{aligned}
$$

where $\omega_{H_f}(\cdot)$ denotes the modulus of continuity for H_f with respect to x, that is,

$$
\left| H_f(x,p) - H_f(x',p) \right| \le \omega_{H_f}(|x-x'|) \qquad \text{for all } x,\, x' \in \mathbb{R}^n, \quad p \in B(0, M+1).
$$

Similarly, we can prove $I_3(x) \le \omega_{H_f}(\varepsilon)$ for all $x \in \mathbb{R}^n$.

7. By collecting estimates in Steps 5 and 6, we can conclude that

$$
\begin{aligned}
&H_f(x, Dw_\alpha(x)) \\
&\le \begin{cases} \lambda_1 |f|_\infty - \delta_2 \lambda_2 + \omega_{H_f}(\varepsilon) + C\alpha & \text{in } \overline{D}, \\ -\delta_1 \lambda_1 + \omega_{H_f}(\varepsilon) + C\alpha & \text{in } (\mathbb{R}^n \setminus \overline{D}) \cap (\mathbb{R}^n \setminus K_\rho). \end{cases}
\end{aligned}
$$

Remark that $(\mathbb{R}^n \setminus K_\rho^D) \subset \overline{D} \cup ((\mathbb{R}^n \setminus \overline{D}) \cap (\mathbb{R}^n \setminus K_\rho))$.

We now take sufficiently small ε, α and $\lambda_1 > 0$ so that

$$
H_f(x, Dw_\alpha(x)) < 0 \qquad \text{in } \mathbb{R}^n \setminus K_\rho^D. \tag{24}
$$

Note that the estimate (23) is still valid even if we replace ε, α and $\lambda_1 > 0$ with smaller ones.

8. Let y' be any maximum point of $w_\alpha - u$ in $\mathbb{R}^n$. Remark that such a point exists since u is bounded and $w_\alpha(x) \to -\infty$ as $|x| \to +\infty$. Moreover, we can show $y' \in \mathbb{R}^n \setminus K_\rho^D$. Indeed, let us take any $x \in K_\rho^D$. Then, by the definition of K_ρ^D and the Lipschitz continuity of u and v, we see

$$
u(x) + 2\beta > u(x) + 2M\rho + \beta \ge v(x) + \beta \ge w(x) \ge w_\alpha(x),
$$

which implies in view of (23) that any $x \in K_\rho^D$ cannot be a maximum point. Therefore, $w_\alpha(\cdot)$ is a C^1-subtangent to u at y'. Since u is a supersolution of (13), we have $H_f(y', Dw_\alpha(y')) \ge 0$. But, this contradicts the strict subsolution property (24). Hence, β must be zero and we have $u = v$ in $\mathbb{R}^n$. □

Corollary 5.2. *Let D be any bounded open set such that* $\mathrm{supp}(f) \subset D$. *Then, two bounded solutions of (13) equating on $\mathcal{A} \setminus D$ coincide on $\mathbb{R}^n$.*

For a diverging sequence $\mathbf{y} = \{y_k\}_{k \in \mathbb{N}}$ in $\mathcal{A}$, we say $\mathbf{y} \in \Lambda$ if and only if (18) holds, that is, for every $\varepsilon > 0$, there exists a number $k_0 \in \mathbb{N}$ such that

$$
-\varepsilon < d_f(y_k, y_{k+m}) < \varepsilon \quad \text{for all } k \ge k_0 \text{ and } m \in \mathbb{N}.
$$

The next proposition shows that Λ is not empty.

Proposition 5.3. *For every $y \in \mathcal{A}$, there exists a divergent sequence $\mathbf{e} = \{e_k\}_{k\in\mathbb{N}} \subset \mathbb{Z}^n$ such that $\mathbf{y} := \{y - e_k\}_{k\in\mathbb{N}} \in \Lambda$.*

Proof. Fix $y \in \mathcal{A}$. By one of the equivalent definition of the Aubry set $\mathcal{A}$ for (19) (see Section 5 of [10] or Proposition 5.10 of [14]), for each $k \in \mathbb{N}$, we can find $e'_k \in \mathbb{Z}^n$, $t_k > 0$ and $\gamma_k \in \mathcal{C}([-t_k, 0]; y - e'_k, y)$ such that

$$0 \leq \int_{-t_k}^{0} L(\gamma_k(s), \dot{\gamma}_k(s))\, ds < 2^{-k}.$$

We define $T_k > 0$ and $e_k \in \mathbb{Z}^n$ inductively by $T_0 := 0$, $T_k := t_k + T_{k-1}$ and $e_k := \sum_{i=1}^{k} e'_i$, respectively. We next define $\eta \in \mathcal{C}((-\infty, 0]; y)$ by

$$\eta(t) := \gamma_k(t + T_{k-1}) - e_{k-1} \quad \text{for } t \in (-T_k, -T_{k-1}], \quad k \in \mathbb{N}.$$

Then, by the $\mathbb{Z}^n$-periodicity of $L(x,\xi)$ in x, we see

$$\int_{-T_k}^{0} L(\eta, \dot{\eta})\, ds = \sum_{i=1}^{k} \int_{-T_i}^{-T_{i-1}} L(\eta, \dot{\eta})\, ds = \sum_{i=1}^{k} \int_{-t_i}^{0} L(\gamma_i, \dot{\gamma}_i)\, ds \leq \sum_{i=1}^{k} 2^{-i} < 1,$$

which shows that η satisfies (a) with $S = -\infty$ and $T = 0$. Indeed, fix any bounded subsolution ϕ of (14). Then, for every $-\infty < -T_k \leq a < b \leq 0$,

$$\phi(\eta(0)) - \phi(\eta(b)) + \int_a^b L(\eta, \dot{\eta})\, ds + \phi(\eta(a)) - \phi(\eta(-T_k)) \leq \int_{-T_k}^{0} L(\eta, \dot{\eta})\, ds < 1.$$

Since ϕ is bounded, letting $k \to +\infty$ yields (a).

Thus, we can apply Lemma 4.2 and Proposition 4.3 to see $|\eta(-t)| \longrightarrow +\infty$ as $t \to +\infty$. In particular, there exists $k_0 \in \mathbb{N}$ such that $\eta(-t) \notin \operatorname{supp}(f)$ for all $t \in (-\infty, T_{k_0}]$, and for all $k \geq k_0$ and $m \in \mathbb{N}$, we obtain

$$\begin{aligned}
d_f(y - e_k, y - e_{k+m}) &= d_f(\eta(-T_k), \eta(-T_{k+m})) \\
&\leq \int_{-T_{k+m}}^{-T_k} L_f(\eta, \dot{\eta})\, ds = \int_{-T_{k+m}}^{-T_k} L(\eta, \dot{\eta})\, ds \\
&= \sum_{i=k+1}^{k+m} \int_{-t_i}^{0} L(\gamma_i, \dot{\gamma}_i)\, ds \leq \sum_{i=k+1}^{k+m} 2^{-i} \\
&= 2^{-k}(1 - 2^{-m}).
\end{aligned}$$

Hence, $\{y_k\}_{k\in\mathbb{N}} := \{y - e_k\}_{k\in\mathbb{N}}$ satisfies

$$\limsup_{k\to\infty} \limsup_{l\to\infty} d_f(y_k, y_l) \leq 0.$$

On the other hand, fix any bounded subsolution ϕ of (13) and take a subsequence $\{y_{k_m}\}_{m\in\mathbb{N}}$ so that $\{\phi(y_{k_m})\}_{m\in\mathbb{N}}$ forms a Cauchy sequence. Then,

$$\liminf_{m\to\infty} \liminf_{l\to\infty} d_f(y_{k_m}, y_{k_l}) \geq \lim_{m\to\infty} \phi(y_{k_m}) - \lim_{l\to\infty} \phi(y_{k_l}) = 0.$$

Hence, $\mathbf{y} := \{y_{k_m}\}_{m\in\mathbb{N}} \in \Lambda$ and we have completed the proof. $\square$

Proposition 5.4. *Let w be any bounded solution of (13). Then,*

$$w(x) = \inf_{\mathrm{y}\in\Lambda} \liminf_{l\to+\infty} \left(d_f(x, y_l) + w(y_l)\right) \qquad \textit{for all } x \in \mathbb{R}^n. \tag{25}$$

In particular, if two solutions w_1, w_2 of (13) satisfy

$$\lim_{k\to+\infty} (w_1 - w_2)(y_k) = 0 \qquad \textit{for all } \mathrm{y} \in \Lambda, \tag{26}$$

then, $w_1 = w_2$ on $\mathbb{R}^n$.

Proof. We denote the right-hand side of (25) by $\tilde{w}(x)$ and show $w = \tilde{w}$ on $\mathbb{R}^n$. Since w is a subsolution of (13), we have $w \le \tilde{w}$ on $\mathbb{R}^n$ by virtue of Lemma A.3. Thus, it remains to prove $w \ge \tilde{w}$ on $\mathbb{R}^n$.

Fix any $x \in \mathbb{R}^n$ and $\delta > 0$. By (20), there exists $z_1 \in \mathcal{A}$ such that

$$w(x) + 2^{-1}\delta > d_f(x, z_1) + w(z_1).$$

Similarly, there exists $z_2 \in \mathcal{A}$ such that

$$w(z_1) + 2^{-2}\delta > d_f(z_1, z_2) + w(z_2).$$

Inductively, we can choose a sequence $\mathbf{z} := \{z_k\}_{k\in\mathbb{N}}$ in $\mathcal{A}$ so that

$$w(x) + \delta \sum_{j=1}^{k} 2^{-j} > \sum_{j=1}^{k} d_f(z_{j-1}, z_j) + w(z_k) \quad \text{for all } k \in \mathbb{N},$$

where we have set $z_0 := x$. Remark that $\mathbf{z}$ can be taken so that $|z_k| \to +\infty$ as $k \to +\infty$ since the bounded set D in (20) is arbitrarily chosen.

Now, let us take $\eta \in C((-\infty, 0]; x)$ such that $\eta(-t_k) = z_k$ and

$$d_f(z_{k-1}, z_k) > \int_{-t_k}^{-t_{k-1}} L_f(\eta, \dot{\eta})\, ds - 2^{-k}\delta \quad \text{for all } k \in \mathbb{N}$$

for some diverging sequence $\{t_k\}_{\mathbb{N}}$. Then, we have

$$w(x) + 2\delta > \int_{-t_k}^{0} L_f(\eta, \dot{\eta})\, ds + w(z_k) \quad \text{for all } k \in \mathbb{N}, \tag{27}$$

which yields that η satisfies (17) since w is bounded. Thus, in view of Lemma 4.5, $\mathbf{z}$ belongs to Λ and

$$\begin{aligned} w(x) + 2\delta &> \liminf_{k\to+\infty} \left(d_f(x, z_k) + w(z_k)\right) \\ &\ge \inf_{\mathrm{y}\in\Lambda} \liminf_{k\to+\infty} \left(d_f(x, y_k) + w(y_k)\right) = \tilde{w}(x). \end{aligned}$$

Since $\delta > 0$ is arbitrary, we can conclude that $w \ge \tilde{w}$ on $\mathbb{R}^n$. Hence, we obtain (25).

Now, let w_1 and w_2 be bounded solutions of (13) satisfying (26). Then, for any $x \in \mathbb{R}^n$,

$$\begin{aligned} w_1(x) &= \inf_{\mathrm{y}\in\Lambda} \liminf_{l\to+\infty} \left(d_f(x, y_l) + w_1(y_l)\right) \\ &= \inf_{\mathrm{y}\in\Lambda} \liminf_{l\to+\infty} \left(d_f(x, y_l) + w_2(y_l)\right) = w_2(x). \end{aligned}$$

Hence, we have completed the proof. □

Corollary 5.5. *Let w be any bounded solution of (13). Then, for any $\delta > 0$ and $x \in \mathbb{R}^n$, there exists $\eta \in C((-\infty, 0]; x)$ such that*

$$w(x) + \delta > \int_{-t}^{0} L_f(\eta, \dot{\eta})\, ds + w(\eta(-t)) \qquad \text{for all } \ t > 0.$$

Proof. In view of (27), there exists $\eta \in C((-\infty, 0]; x)$ such that for any given $t > 0$ and $t_k \geq t$, we see

$$\begin{aligned} w(x) + \delta &> \int_{-t}^{0} L_f(\eta, \dot{\eta})\, ds + \int_{-t_k}^{-t} L_f(\eta, \dot{\eta})\, ds + w(\eta(-t_k)) \\ &\geq \int_{-t}^{0} L_f(\eta, \dot{\eta})\, ds + d_f(\eta(-t), \eta(-t_k)) + w(\eta(-t_k)) \\ &\geq \int_{-t}^{0} L_f(\eta, \dot{\eta})\, ds + w(\eta(-t)), \end{aligned}$$

where we have used Lemma A.3 to show the last inequality. □

For $v_0 \in BC(\mathbb{R}^n)$, we define $v : \mathbb{R}^n \longrightarrow \mathbb{R}$ by

$$v(x) := \inf_{\mathbf{y} \in \Lambda} \liminf_{l \to \infty} \big(d_f(x, y_l) + v_0(y_l)\big). \tag{28}$$

Lemma 5.6. *v is a bounded function on $\mathbb{R}^n$.*

Proof. For $x \in \mathbb{R}^n$, we can find $y \in \mathcal{A}$ such that $x - y \in [0, 1)^n$. In particular, $|x - y| \leq \sqrt{n}$. By Proposition 5.3, there exists $\{y_l\}_{l \in \mathbb{N}} \in \Lambda$ and $C_f > 0$ such that $d_f(y, y_l) \leq C_f$ for all $l \in \mathbb{N}$. Then,

$$\begin{aligned} d_f(x, y_l) + v_0(y_l) &\leq d_f(x, y) + d_f(y, y_l) + v_0(y_l) \\ &\leq C\sqrt{n} + C_f + |v_0|_\infty \end{aligned}$$

for some $C > 0$. In particular, we have

$$v(x) \leq \liminf_{l \to \infty} (d_f(x, y_l) + v_0(y_l)) \leq C\sqrt{n} + C_f + |v_0|_\infty.$$

Thus, v is upper bounded on $\mathbb{R}^n$. It is clear that v is lower bounded since d_f and v_0 are lower bounded. Hence, v is bounded. □

Proposition 5.7. *Let v be the function defined by (28). Then,*
(a) v is the maximal subsolution of (13) satisfying

$$\limsup_{k \to +\infty} (v - v_0)(y_k) \leq 0 \qquad \text{for all } \ \mathbf{y} \in \Lambda. \tag{29}$$

Moreover, if v_0 is a bounded subsolution of (13), then v satisfies

$$\lim_{k \to +\infty} (v - v_0)(y_k) = 0 \qquad \text{for all } \ \mathbf{y} \in \Lambda. \tag{30}$$

(b) v is a supersolution of (13).

Proof. Fix any $x, z \in \mathbb{R}^n$ and $\delta > 0$, and take $\mathbf{y}' = \{y'_k\} \in \Lambda$ so that

$$v(z) + \delta > \liminf_{l\to\infty} \left(d_f(z, y'_l) + v_0(y'_l)\right).$$

Then,

$$\begin{aligned} v(x) - v(z) - \delta &\le \lim_{k\to\infty} \left\{ \inf_{l\ge k} \left(d_f(x, y'_l) + v_0(y'_l)\right) - \inf_{l\ge k} \left(d_f(z, y'_l) + v_0(y'_l)\right) \right\} \\ &\le \lim_{k\to\infty} \sup_{l\ge k} \left(d_f(x, y'_l) - d_f(z, y'_l)\right) \le d_f(x, z). \end{aligned}$$

Since $\delta > 0$ is arbitrary, we obtain

$$v(x) - v(z) \le d_f(x, z) \qquad \text{for all } x, z \in \mathbb{R}^n.$$

Thus, v is a subsolution of (13) in view of Lemma A.3 in Appendix. We also see from this inequality that v is continuous on $\mathbb{R}^n$.

We next show (29). Fix $\varepsilon > 0$ and $\mathbf{y} \in \Lambda$ arbitrarily. Then, there exists $k_0 \in \mathbf{N}$ such that for all $k \ge k_0$ and $m \ge k+1$,

$$\inf_{l\ge m} \left(d_f(y_k, y_l) + v_0(y_l)\right) \le \varepsilon + \inf_{l\ge m} v_0(y_l).$$

Letting $m \to +\infty$ yields $v(y_k) \le \varepsilon + \liminf_{l\to+\infty} v_0(y_l)$. Thus, we obtain

$$\limsup_{k\to+\infty} (v - v_0)(y_k) \le \varepsilon + \liminf_{l\to+\infty} v_0(y_l) - \liminf_{k\to+\infty} v_0(y_k) = \varepsilon.$$

Since ε is arbitrary, we get (29).

To prove the maximality of v, let ϕ be any bounded subsolution of (13) satisfying (29) with ϕ in place of v. Then, for every $x \in \mathbb{R}^n$,

$$\begin{aligned} \phi(x) &\le \inf_{\mathbf{y}\in\Lambda} \liminf_{l\to+\infty} \left(d_f(x, y_l) + \phi(y_l)\right) \\ &\le \inf_{\mathbf{y}\in\Lambda} \liminf_{l\to+\infty} \left(d_f(x, y_l) + v_0(y_l)\right) + \sup_{\mathbf{y}\in\Lambda} \limsup_{l\to+\infty} (\phi - v_0)(y_l) \\ &\le v(x). \end{aligned}$$

Suppose now that v_0 is a bounded subsolution of (13). Then, for every $x \in \mathbb{R}^n$,

$$\begin{aligned} v(x) &= \inf_{\mathbf{y}\in\Lambda} \liminf_{l\to\infty} \left(d_f(x, y_l) + v_0(y_l)\right) \\ &\ge \inf_{\mathbf{y}\in\Lambda} \liminf_{l\to\infty} \left(v_0(x) - v_0(y_l) + v_0(y_l)\right) = v_0(x). \end{aligned}$$

In particular, (30) holds.

We next show (b). Suppose that there exist a point $z \in \mathbb{R}^n$ and a strict C^1-subtangent ϕ to v at z such that $H_f(z, D\phi(z)) < 0$. Fix $r > 0$ so that $H_f(x, D\phi(x)) < 0$ for all $x \in B(z, r)$. Then, we can find $\varepsilon > 0$ such that $v(x) - \phi(x) > \varepsilon$ for all $x \in \partial B(z, r)$ since ϕ is a strict subtangent. Now, we define a new function $\psi \in C(\mathbb{R}^n)$ by

$$\psi(x) := \begin{cases} \max\{\phi(x) + \varepsilon, v(x)\} & \text{if } \ x \in B(z, r) \\ v(x) & \text{otherwise.} \end{cases}$$

Then, ψ is a subsolution of (13) satisfying $\psi \geq v$ on $\mathbb{R}^n$ and $\psi(z) > v(z)$.

Now, fix any $\mathbf{y} \in \Lambda$. Since $|y_k| \longrightarrow +\infty$ as $k \to +\infty$, there exists $k_0 \in \mathbb{N}$ such that $v(y_k) = \psi(y_k)$ for all $k \geq k_0$. Thus,

$$\lim_{k\to+\infty} \sup_{l\geq k}(\psi - v_0)(y_l) = \lim_{k\to+\infty} \sup_{l\geq k}(v - v_0)(y_k) \leq 0.$$

But, this contradicts the maximality of v. Hence, v is a supersolution of (13). □

Remark 5.8. If we set $d_f(x, \mathbf{y}) := \lim_{l\to+\infty} d_f(x, y_l)$ and $v_0(\mathbf{y}) := \liminf_{l\to+\infty} v_0(y_l)$ for $\mathbf{y} \in \Lambda$, then, (28) can be rewritten as

$$v(x) = \inf_{\mathbf{y}\in\Lambda} \big(d_f(x, \mathbf{y}) + v_0(\mathbf{y})\big), \qquad x \in \mathbb{R}^n.$$

6 Representation and Convergence.

This section is devoted to proving Theorem 2.4 as well as getting a representation formula for asymptotic solutions.

We first give the proof of Proposition 2.3 which we postponed. Since uniform continuity is standard, we only check boundedness. Let u be the unique solution of Cauchy problem (1) with an initial function $u_0 \in BUC(\mathbb{R}^n)$. Notice that H has been normalized so that $c = 0$. Let ϕ be any bounded solution of (13). Since u_0 is bounded, we can take $A > 0$ so that $\phi(x) - A \leq u_0(x) \leq \phi(x) + A$ for all $x \in \mathbb{R}^n$. Remark also that $\phi + A$ and $\phi - A$ are solutions of (1) with initial data $\phi + A$ and $\phi - A$, respectively. Then, the standard comparison theorem for (1) infers that $\phi(x) - A \leq u(x,t) \leq \phi(x) + A$ for all $(x,t) \in \mathbb{R}^n \times [0, +\infty)$. In particular, u is bounded on $\mathbb{R}^n \times [0, +\infty)$ and we have completed the proof of Proposition 2.3.

Let us denote by $\{S(t)\}_{t\geq 0}$ the semi-group of mappings on $BUC(\mathbb{R}^n)$ defined by $(S(t)u_0)(x) := u(x,t)$, where u_0 is a given initial function and u is the unique solution of (1). We next define $v^+, v^- \in BUC(\mathbb{R}^n)$ by

$$v^+(x) := \limsup_{t\to+\infty}(S(t)u_0)(x) = \limsup_{t\to+\infty}{}^* u(x,t),$$
$$v^-(x) := \liminf_{t\to+\infty}(S(t)u_0)(x) = \liminf_{t\to+\infty}{}_* u(x,t).$$

Note that from the general theory of viscosity solutions, v^+ and v^- are sub- and supersolutions of (13), respectively. Moreover, the convexity of $H(x,\cdot)$ implies that v^- is a subsolution of (13) (see [5]). In particular, v^- is a bounded solution of (13).

We try to obtain a representation formula for v^-.

Lemma 6.1. *v^- satisfies*

$$\limsup_{k\to+\infty} (v^- - u_0)(y_k) \leq 0 \qquad \textit{for all } \mathbf{y} \in \Lambda. \tag{31}$$

Proof. Take $\mathbf{y} \in \Lambda$ and $\delta > 0$ arbitrarily. We fix $k \in \mathbb{N}$ so that $d_f(y_k, y_{k+m}) < \delta$ for all $m \in \mathbb{N}$. Similarly as in the proof of Proposition 5.4, we can find $\eta \in C((-\infty, 0]; y_k)$ such that $y_{k+m} = \eta(-t_m)$ for some diverging sequence $\{t_m\}_{m\in\mathbb{N}}$ and

$$\int_{-t_m}^{0} L_f(\eta, \dot{\eta})\, ds < d_f(y_k, y_{k+m}) + \delta < 2\delta.$$

Thus,

$$u(y_k, t_m) \le \int_{-t_m}^{0} L_f(\eta, \dot{\eta})\, ds + u_0(\eta(-t_m)) \le 2\delta + u_0(y_{k+m}),$$

and we have

$$v^-(y_k) = \liminf_{t\to+\infty} u(y_k, t) \le \liminf_{m\to+\infty} u(y_k, t_m) \le 2\delta + \liminf_{l\to+\infty} u_0(y_l).$$

In particular,

$$\limsup_{k\to+\infty} (v^- - u_0)(y_k) \le 2\delta + \liminf_{l\to+\infty} u_0(y_l) - \liminf_{k\to+\infty} u_0(y_k) = 2\delta.$$

Since δ is arbitrary, we obtain (31). □

Lemma 6.2. *Suppose that u_0 is a subsolution of (13). Then, the solution u of (1) converges in $C(\mathbb{R}^n)$ to the function*

$$\tilde{v}(x) := \inf_{\mathbf{y}\in\Lambda} \liminf_{l\to\infty} \big(d_f(x, y_l) + u_0(y_l)\big).$$

Proof. Since u_0 is a subsolution of (13), we can see $u_0 \le \tilde{v}$ on $\mathbb{R}^n$. Moreover, since u_0 and $\tilde{v}$ are sub- and supersolutions of (1), respectively, the comparison theorem for (1) yields that for all $t > 0$,

$$u_0 \le S(t)u_0 \le S(t)\tilde{v} = \tilde{v} \qquad \text{in } \mathbb{R}^n.$$

In particular, $u_0 \le v^- \le \tilde{v}$, and in view of Lemma 6.1 and Proposition 5.7, we have

$$\lim_{l\to+\infty} (\tilde{v} - v^-)(y_l) = \lim_{l\to+\infty} (\tilde{v} - u_0)(y_l) - \lim_{l\to+\infty} (v^- - u_0)(y_l) = 0$$

for all $\mathbf{y} \in \Lambda$. Thus, we can apply Proposition 5.4 to conclude that $\tilde{v} = v^- = v^+$ on $\mathbb{R}^n$. Hence, we have completed the proof. □

Proposition 6.3. *Let $u_0 \in BUC(\mathbb{R}^n)$ be any initial function. Then, we have the following formula:*

$$v^-(x) = \inf_{\mathbf{y}\in\Lambda} \liminf_{l\to\infty} \big(d_f(x, y_l) + v_0(y_l)\big),$$

where $v_0(x) := \inf_{y\in\mathbb{R}^n}(d_f(x, y) + u_0(y))$.

Proof. We denote the right-hand side by $v(x)$. Since v_0 is a subsolution of (13) and $v_0 \le u_0$ on $\mathbb{R}^n$ by Proposition A.4, we have $S(t)v_0 \le S(t)u_0$ for all $t \ge 0$. By the comparison theorem for (1) and Lemma 6.2, we see

$$v(x) = \lim_{t\to\infty} S(t)v_0 \le \liminf_{t\to\infty} S(t)u_0 = v^-(x).$$

Hence, it suffices to show that $v^- \leq v$ on $\mathbb{R}^n$.

Fix any $\delta > 0$, $x \in \mathbb{R}^n$ and $\mathbf{y} := \{y_l\}_{l\in\mathbb{N}} \in \Lambda$. We construct a curve $\eta \in C((-\infty,0];x)$ such that $y_j = \eta(-s_j)$ for some positive sequence $\{s_j\}_{j\in\mathbb{N}}$ and

$$\int_{-s_j}^{0} L_f(\eta(s),\dot\eta(s))\,ds < d_f(x,y_j) + \delta, \qquad j \in \mathbb{N}.$$

For each $l \in \mathbb{N}$, we fix $z_l \in \mathbb{R}^n$ so that

$$d_f(y_l,z_l) + u_0(z_l) < \inf_{y\in\mathbb{R}^n} (d_f(y_l,y) + u_0(y)) + \delta.$$

We also take $t_l > 0$ and $\gamma_l \in C([-s_l - t_l, -s_l]; z_l, y_l)$ such that

$$\int_{-s_j-t_l}^{-s_l} L_f(\gamma_l(s),\dot\gamma_l(s))\,ds < d_f(y_l,z_l) + \delta.$$

Now, we define $\eta_l \in C([-s_l - t_l, 0]; x)$ by

$$\eta_l(s) = \begin{cases} \eta(s) & \text{if } s \in [-s_l, 0] \\ \gamma_l(s) & \text{if } s \in [-s_l - t_l, -s_l]. \end{cases}$$

Then, in view of (8), we see

$$\begin{aligned} u(x, s_l + t_l) &\leq \int_{-s_j-t_l}^{0} L_f(\eta_l(s),\dot\eta_l(s))\,ds + u_0(\eta_l(-s_l - t_l)) \\ &\leq \int_{-s_j}^{0} L_f(\eta(s),\dot\eta(s))\,ds + d_f(y_l,z_l) + u_0(z_l) + \delta \\ &\leq d_f(x,y_l) + \inf_{z\in\mathbb{R}^n} \big(d_f(y_l,z) + u_0(z)\big) + 3\delta \\ &= d_f(x,y_l) + v_0(y_l) + 3\delta \qquad \text{for all } l \in \mathbb{N}. \end{aligned}$$

Since $|y_l| \longrightarrow +\infty$ as $l \to \infty$, we have $s_l \to +\infty$, and therefore $s_l + t_l \to +\infty$. Thus,

$$v^-(x) \leq \liminf_{l\to\infty} u(x, s_l + t_l) \leq \liminf_{l\to\infty} \big(d_f(x,y_l) + v_0(y_l)\big) + 3\delta.$$

By considering the infimum over all $\mathbf{y} \in \Lambda$ and letting $\delta \downarrow 0$, we obtain $v^-(x) \leq v(x)$ on $\mathbb{R}^n$. □

We finally prove our main theorem. Fix any $u_0 \in BUC(\mathbb{R}^n)$ satisfying (u1) of Assumption 1 for some $\mathbb{Z}^n$-periodic function $\hat u_0 \in BUC(\mathbb{R}^n)$, and let $u(x,t)$ and $\hat u(x,t)$ be solutions of Cauchy problems (1) and (3) with initial data u_0 and $\hat u_0$, respectively. Remark that $\hat u(\cdot,t)$ is $\mathbb{Z}^n$-periodic for all $t > 0$.

Lemma 6.4. *For every $\delta \in (0,1)$ and $t > 0$, there exists $R = R(\delta,t) > 0$ such that*

$$u(x,t) < \hat u(x,t) + \delta \qquad \textit{for all } x \in \mathbb{R}^n \setminus B(0,R).$$

Proof. Fix $\delta \in (0,1)$ and $t > 0$, and take any $\eta \in C([-t,0];x)$ such that

$$\hat u(x,t) + \delta/2 > \int_{-t}^{0} L(\eta,\dot\eta)\,ds + \hat u_0(\eta(-t)). \tag{32}$$

Then, by Lemma 4.1 with $f = 0$, there exists a constant $C > 0$ not depending on (x, t) such that

$$\int_{-t}^{0} |\dot{\eta}(s)|\, ds \leq C(1+t).$$

Let $R_0 > 0$ be a number which satisfies $\mathrm{supp}(f) \subset B(0, R_0)$ and $\sup_{|x|\geq R_0} |u_0(x) - \hat{u}_0(x)| < \delta/2$. We choose a sufficiently large $R > R_0$ so that $R - R_0 > C(1+t)$. Then, for every $x \in \mathbb{R}^n \setminus B(0, R)$ and $\eta \in \mathcal{C}([-t, 0]; x)$ satisfying (32), we see $\eta([-t, 0]) \cap \mathrm{supp}(f) = \emptyset$ and $|\eta(-t)| \geq R_0$. Therefore,

$$\begin{aligned} u(x,t) &\leq \int_{-t}^{0} L_f(\eta, \dot{\eta})\, ds + u_0(\eta(-t)) \\ &< \int_{-t}^{0} L(\eta, \dot{\eta})\, ds + \hat{u}_0(\eta(-t)) + \delta/2 < \hat{u}(x,t) + \delta. \end{aligned}$$

Hence, we have completed the proof. □

Proof of Theorem 2.4. It suffices to show $v^+ = v^-$ on $\mathbb{R}^n$. Fix any $\delta > 0$ and $x \in \mathbb{R}^n$. Take a diverging sequence $\{t_j\}_{j\in\mathbb{N}}$ such that $u(x, t_j)$ converges to $v^+(x)$. Then, in view of (9) and Corollary 5.5, there exists $\eta \in \mathcal{C}((-\infty, 0]; x)$ such that

$$\begin{aligned} u(x, t_j) &\leq \int_{-t}^{0} L_f(\eta, \dot{\eta})\, ds + u(\eta(-t), t_j - t) \\ &< v^-(x) - v^-(\eta(-t)) + \delta + u(\eta(-t), t_j - t) \end{aligned}$$

for all $j \in \mathbb{N}$ and $t \in [0, t_j]$. We know from Lemma 6.4 that for each $k \in \mathbb{N}$, there exists $R_k > 0$ such that $u(z, k) < \hat{u}(z, k) + \delta$ for every $z \in \mathbb{R}^n \setminus B(0, R_k)$. Since $|\eta(-t)| \to +\infty$ as $t \to +\infty$ by Proposition 4.3, we can find $j(k) \in \mathbb{N}$ such that $|\eta(-t_{j(k)} + k)| > R_k$ for all $k \in \mathbb{N}$. In particular, by setting $s_k := t_{j(k)} - k$, we have $u(\eta(-s_k), k) < \hat{u}(\eta(-s_k), k) + \delta$, and therefore

$$u(x, t_{j(k)}) < v^-(x) - v^-(\eta(-s_k)) + \hat{u}(\eta(-s_k), k) + 2\delta.$$

Thus, letting $k \to +\infty$ yields

$$v^+(x) = \lim_{k\to+\infty} u(x, t_{j(k)}) < v^-(x) - \limsup_{k\to+\infty} v^-(\eta(-s_k)) + \liminf_{k\to+\infty} \hat{u}(\eta(-s_k), k) + 2\delta.$$

Since $\hat{u}(\cdot, t)$ converges uniformly in $\mathbb{R}^n$ (or equivalently in $\mathbb{T}^n$) to $\hat{v}(\cdot)$ and $\hat{v} \leq v^-$ on $\mathbb{R}^n$, we finally obtain

$$v^+(x) < v^-(x) - \limsup_{k\to+\infty} v^-(\eta(-s_k)) + \liminf_{k\to+\infty} \hat{v}(\eta(-s_k)) + \delta \leq v^-(x) + 2\delta,$$

which infers $v^+(x) \leq v^-(x)$ after letting $\delta \downarrow 0$. Since $v^- \leq v^+$ on $\mathbb{R}^n$, we get $v^+ = v^-$ and the proof of Theorem 2.4 has been completed. □

Final Remarks. Throughout this paper, the strict convexity of H is used only to guarantee the convergence of $\hat{u}(\,\cdot\,, t)$ as $t \to +\infty$. Thus, if it converges under the assumption that

H is merely convex, then Theorem 2.4 is also valid without assuming the strict convexity of H.

Concerning condition (u1) of Assumption 1, we do not have to assume that $u_0 \geq \hat{u}_0$ if $a_f < 0$, where a_f is the critical eigenvalue for (11) (see also [2]). Indeed, let $u^{(1)}$ and $u^{(2)}$ be solutions of Cauchy problem (1) with $\mathbb{Z}^n$-periodic initial function $\hat{u}_0$ and its perturbation u_0 such that $\lim_{R\to+\infty} \sup_{|x|\geq R} |u_0(x) - \hat{u}_0(x)| = 0$, respectively. Fix $\delta > 0$, $(x,t) \in \mathbb{R}^n \times [0,+\infty)$ and take $\gamma^{(t)} \in \mathcal{C}([-t,0];x)$ so that

$$u^{(2)}(x,t) + \delta > \int_{-t}^{0} L_f(\gamma^{(t)}(s), \dot{\gamma}^{(t)}(s))\, ds + u_0(\gamma^{(t)}(-t)). \tag{33}$$

Then, in view of (8), we see

$$u^{(1)}(x,t) - u^{(2)}(x,t) < \hat{u}_0(\gamma^{(t)}(-t)) - u_0(\gamma^{(t)}(-t)) + \delta.$$

We claim here that $|\gamma^{(t)}(-t)| \to +\infty$ as $t \to +\infty$. To show this, suppose that $\sup_j |\gamma_j(-t_j)| < +\infty$ for some sequence $\gamma_j := \gamma^{(t_j)} \in \mathcal{C}([-t_j,0];x)$ satisfying (33) with $t = t_j$, $j \in \mathbb{N}$. Then, for any subsolution ϕ of (11) with $a = a_f$, we have

$$\begin{aligned}\phi(\gamma_j(0)) - \phi(\gamma_j(-t_j)) &\leq \int_{-t_j}^{0} \{L_f(\gamma_j(s), \dot{\gamma}_j(s)) + H_f(\gamma_j(s), D\phi(\gamma_j(s)))\}\, ds \\ &\leq |u_0|_\infty + \sup_{j\in\mathbb{N}} |u^{(2)}(\,\cdot\,, t_j)|_\infty + \delta + a_f t_j.\end{aligned}$$

Since $a_f < 0$, we get the contradiction by letting $j \to +\infty$. Thus, we obtain

$$\limsup_{t\to+\infty} (u^{(1)}(x,t) - u^{(2)}(x,t)) \leq \delta.$$

Similarly, we also have

$$\limsup_{t\to+\infty} (u^{(2)}(x,t) - u^{(1)}(x,t)) \leq \delta.$$

Remark that the convergence is uniform on any compact subset of $\mathbb{R}^n$. Hence, $u^{(1)}(\,\cdot\,,t) - u^{(2)}(\,\cdot\,,t)$ converges to zero in $C(\mathbb{R}^n)$.

If $\mathcal{A}$ contains an equilibrium point or a closed loop of critical curve, then we can see that $a_f < 0$. However, we do not know if $\mathcal{U}_f = \emptyset$ implies $a_f < 0$ in general cases.

We also remark that Theorem 2.4 is still valid if $\lim_{l\to+\infty} (\hat{v} - v_0)(y_l) = 0$ for all $\mathbf{y} \in \Lambda$ even in the case where $\mathcal{U}_f = \emptyset$, $a_f = 0$ and $u_0(x) < \hat{u}_0(x)$ for some $x \in \mathbb{R}^n$. The last claim is clear from the proof of Theorem 2.4.

A Fundamental facts.

We collect some properties of $d_f(x,y)$ defined by (16) (cf. [10, 14]).

Lemma A.1. *There exists $\varepsilon > 0$ and $C > 0$ such that $L_f(x,\xi) \leq C$ for all $(x,\xi) \in \mathbb{R}^n \times B(0,\varepsilon)$.*

Proof. This lemma is a slight modification of Proposition 2.1 in [14] by taking into account that L is $\mathbb{Z}^n$-periodic in x. □

Proposition A.2.

(a) $d_f(x,z) \le d_f(x,y) + d_f(y,z)$ *for all* $x,y,z \in \mathbb{R}^n$.

(b) $d_f(y,y) = 0$ *for all* $y \in \mathbb{R}^n$.

(c) $d_f(\cdot\,,y)$ *is Lipschitz continuous on* $\mathbb{R}^n$ *uniformly in* $y \in \mathbb{R}^n$.

(d) $d_f(x,\cdot\,)$ *is Lipschitz continuous on* $\mathbb{R}^n$ *uniformly in* $x \in \mathbb{R}^n$.

(e) $d_f(\cdot\,,y)$ *is a subsolution of (13) in* $\mathbb{R}^n$ *and is a supersolution in* $\mathbb{R}^n \setminus \{y\}$.

(f) $-d_f(y,\cdot\,)$ *is a subsolution of (13) in* $\mathbb{R}^n$ *and is a supersolution in* $\mathbb{R}^n \setminus \{y\}$.

Proof. One can easily show (a) by the definition of d_f. (b) is also easily checked since $d_f(y,y) \ge 0$ by (a) and one can see $d_f(y,y) \le 0$ by taking a convergent sequence $t_n \downarrow 0$ and $\gamma_n \equiv y \in C([0,t_n];y,y)$ in (16).

To show (c), fix any x, $y \in \mathbb{R}^n$, $\delta > 0$ and set $T := \varepsilon^{-1}(\delta + |x-y|)$ and $\xi := T^{-1}(x-y) \in B(0;\varepsilon)$, where $\varepsilon > 0$ is taken so that Lemma A.1 holds. Next, we define the curve $\gamma \in C([0,T];y,x)$ by $\gamma(s) := y + s\xi$. Then, we get

$$d_f(x,y) \le \int_0^T L_f(\gamma(s),\dot\gamma(s))\,ds = \int_0^T L_f(y+s\xi,\xi)\,ds \le CT \le \varepsilon^{-1}C(\delta + |x-y|).$$

Letting $\delta \downarrow 0$ yields $d_f(x,y) \le \varepsilon^{-1}C|x-y|$, which implies in particular that d_f is a continuous function on $\mathbb{R}^n \times \mathbb{R}^n$. By using (a), we can show that

$$|d_f(x,y) - d_f(z,y)| \le \varepsilon^{-1}C|x-z| \qquad \text{for all} \quad x,y,z \in \mathbb{R}^n.$$

Hence, $d_f(\cdot\,,y)$ is Lipschitz continuous uniformly in $y \in \mathbb{R}^n$. The assertion (d) is now trivial from the proof of (c).

We prove (e). Since $d_f(x,y)$ is continuous with respect to x on $\mathbb{R}^n$, we can apply Theorems A.1 and A.2 of [14] to show that $d_f(\cdot\,,y)$ is a subsolution of (13) in $\mathbb{R}^n$ and is a supersolution of (13) in $\mathbb{R}^n \setminus \{y\}$.

To show (f), remark first that $d_f(y,x)$ can be represented as

$$d_f(y,x) := \inf\left\{ \int_0^t \tilde L_f(\gamma(s),\dot\gamma(s))\,ds \;\middle|\; t>0,\ \gamma \in C([0,t];y,x) \right\},$$

where $\tilde L_f(x,\xi) := \tilde L(x,\xi) + f(x)$ and $\tilde L(x,\xi) = L(x,-\xi)$. Since $\tilde L$ is the convex conjugate of $\tilde H(x,p) := H(x,-p)$ and $\tilde H$ satisfies (H1)-(H4) in place of H, we can apply Appendix A.1 in [14] to deduce that $d_f(y,\cdot\,)$ is a subsolution of $\tilde H(x,Du) - f(x) = 0$ in $\mathbb{R}^n$. Thus, $-d_f(y,\cdot\,)$ is a subsolution of (13) in $\mathbb{R}^n$. $\square$

Lemma A.3. *A function* $u \in C(\mathbb{R}^n)$ *(which is possibly unbounded) is a subsolution of (13) if and only if the following formula is valid:*

$$u(x) - u(y) \le d_f(x,y) \qquad \textit{for all} \quad x,y \in \mathbb{R}^n. \tag{34}$$

In particular, $d_f(\cdot\,,y)$ *and* $-d_f(y,\cdot\,)$ *are the maximal and minimal subsolutions of (13) equating* 0 *at* y*, respectively.*

Proof. The "only if" part is a direct consequence of Proposition 2.5 in [14]. Now, we assume (34). Fix any $x \in \mathbb{R}^n$ and let ϕ be a C^1-supertangent to u at x such that $\phi(x) = 0$. Then, by (34),

$$\phi(y) \geq u(y) - u(x) \geq -d_f(x, y) \qquad \text{for all } y \in \mathbb{R}^n,$$

and $\phi(x) = -d_f(x, x) = 0$. Thus, ϕ is also a C^1-supertangent to $-d_f(x, \cdot)$ at x. By Proposition A.2 (f), we have $H_f(x, D\phi(x)) \leq 0$, which implies the subsolution property of u. □

Proposition A.4. *Let C be any subset of $\mathbb{R}^n$ and $u_0 \in BC(\mathbb{R}^n)$. Then, the function $u \in C(\mathbb{R}^n)$ defined by*

$$u(x) := \inf_{y \in C} \left(d_f(x, y) + u_0(y)\right) \tag{35}$$

is the maximal subsolution of (13) not exceeding u_0 on C, and it is a solution in $\mathbb{R}^n \setminus \overline{C}$. Moreover, suppose that u_0 is a subsolution of (13). Then, $u \equiv u_0$ on C.

Proof. By the previous lemma, d_f is lower bounded since (13) has a bounded subsolution. Fix any $x, y \in \mathbb{R}^n$, $\delta > 0$ and take a point $y_\delta \in \mathbb{R}^n$ such that

$$d_f(x, y_\delta) + u_0(y_\delta) < \inf_{z \in C} \left(d_f(x, z) + u_0(z)\right) + \delta.$$

Then, we see that

$$\begin{aligned} u(x) - u(y) &< d_f(x, y_\delta) + u_0(y_\delta) - d_f(y, y_\delta) - u_0(y_\delta) + \delta \\ &\leq d_f(x, y) + \delta, \end{aligned}$$

where we have used the triangle inequality for d_f. Since $\delta > 0$ is arbtrary, we obtain the subsolution property of u.

Let us take any subsolution $\phi \in C(\mathbb{R}^n)$ of (13) not exceeding u_0 on C. Then,

$$\phi(x) \leq \inf_{z \in C} \left(d_f(x, z) + \phi(z)\right) \leq \inf_{z \in C} \left(d_f(x, z) + u_0(z)\right) = u(x),$$

which implies the maximality of u.

We next show the supersolution property of u in $\mathbb{R}^n \setminus \overline{C}$. Suppose that there exist a point $z \in \mathbb{R}^n \setminus \overline{C}$ and a strict C^1-subtangent ϕ to u at z such that $H_f(z, D\phi(z)) < 0$. Fix $r > 0$ so that $B(z, r) \cap \overline{C} = \emptyset$ and $H_f(x, \phi(x)) < 0$ for all $x \in B(z, r)$. Then, we can find $\varepsilon > 0$ such that $u(x) - \phi(x) > \varepsilon$ for all $x \in \partial B(z, r)$ since ϕ is a strict subtangent. Now, we define a new function $\psi \in C(\mathbb{R}^n)$ by

$$\psi(x) := \begin{cases} \max\{\phi(x) + \varepsilon, u(x)\} & \text{if } \ x \in B(z, r) \\ u(x)\ . & \text{otherwise.} \end{cases}$$

Then, it is clear that ψ is a subsolution of (13) in $\mathbb{R}^n$ not exceeding u_0 on C and $\psi(z) > u(z)$. But, this contradicts the maximality of u. The last assertion can also be proved by the maximality of u. □

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