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RECENT ADVANCES IN THE THEORY OF ARONSSON EQUATIONS

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ABSTRACT. These notes, consisting of two separate but related parts, are a crash course introduction to some recent advances in the theory of Aronsson equations. In the first part, we review the known results concerning the mutual relations between the viscosity solutions of the Aronsson equations, absolute minimizers of supremum functionals, and functions enjoying comparison with (generalized) cones. In the second part, we verify the conjecture of Barron, Evans and Jensen [8] regarding the characterization of viscosity solutions of general Aronsson equations in terms of the properties of associated forward and backwards Hamilton-Jacobi flows; this is based on a joint work with Eero Saksman [29]. In both cases the proofs are given only in the special case of the infinity Laplacian.

1. INTRODUCTION

The origin of the Aronsson equations, as well as the reason for their name, goes back to a series of papers [2]-[6] written by Gunnar Aronsson in the 1960's. In those papers he was investigating variational problems involving functionals of the form

$$S(u,\Omega) = \operatorname*{ess\,sup}_{x\in\Omega} H(x,u(x),Du(x)). \tag{1.1}$$

Aronsson was particularly interested in the special case $H(x,r,p)=\frac{1}{2}|p|^2$ that corresponds to the minimal Lipschitz extension problem, but by his work set guidelines for the theory of more general problems as well. In particular, he was able to associate an Euler-Lagrange equation, nowadays known as the Aronsson equation,

$$-\frac{d}{dx}(H(x,u(x),Du(x))\cdot H_p(x,u(x),Du(x))=0 \tag{1.2}$$

to (1.1). Here H_p denotes the derivative of H with respect to the gradient variable.

After the fundamental paper of Jensen [26], in which viscosity solutions were incorporated to the field, the theory of Aronsson equations and the associated minimization problems have enjoyed a wide-spread interest among researchers of partial differential equations and calculus of variations. Our goal in this paper is to review the current status of selected aspects of the theory, including some very recent advances, and provide the reader with up-to-date references.

In the first part of the paper, we review the known results concerning the mutual relations between the viscosity solutions of the Aronsson equations, absolute minimizers of supremum functionals, and functions enjoying comparison with (generalized) cones. Moreover, we show how the Aronsson equation can be derived from various standpoints, including the random turn tug-of-war game approach of Peres, Schramm, Sheffield and Wilson [32]. In the second part, following the outline of a recent joint paper with Eero Saksman [29], we verify the conjecture of Barron, Evans and Jensen [8] regarding the characterization of viscosity solutions of general Aronsson equations in terms of the properties

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of associated forward and backwards Hamilton-Jacobi flows. In both parts the proofs are given only in the special case of the infinity Laplacian.

A more thorough and extensive introduction to the topic, and, in particular, to the theory of infinite harmonic functions, is available in [7] and [16]. However, after the completion of these two surveys, important new tools have been introduced and several new results obtained, and therefore there is a need for an update.

The Aronsson equations and the supremum functionals have found their way to many applications in rather surprising contexts. For example, they have been used (directly or via duality) in image interpolation [13], in the weak KAM theory [22], brain and surface warping [30], shape metamorphism [20], in the study of Riemannian submersions [31], landslide modeling [25], modeling of the dielectric breakdown of a composite conductor [24] and in the interpolation of terrain elevation maps obtained from satellites [1]. The vast range of these applications shows that the Aronsson equations and the associated variational problems are of interest beyond mere mathematical curiosity.

2. Aronsson equations - a quick introduction

In this section, we try to explain the very basics of the theory as well as offer a glimpse, accompanied with adequate references, to its more sophisticated features. For simplicity, we will approach the general Aronsson equation

$$-\frac{d}{dx}(H(x,u(x),Du(x))\cdot H_p(x,u(x),Du(x))=0$$

via the special case of the infinity Laplacian

$$-\Delta_\infty u(x) = -\sum_{i,j=1}^{n} u_{x_i}u_{x_j}u_{x_ix_j} = 0 \tag{2.1}$$

that corresponds[1] to the choice $H(x,r,p)=\frac{1}{2}|p|^2$. The two auxiliary notions we need to discuss right away are 'absolute minimizers' and 'comparison with cones'.

2.1. Absolute minimizers.

A fundamental difference between the classical integral functionals of calculus of variations and the supremum functionals (or L^∞ functionals) of the form

$$S(u,\Omega)=\operatorname*{ess\,sup}_{x\in\Omega} H(x,u(x),Du(x))$$

is that the latter are not set additive. This means, in particular, that while a function u that minimizes an integral functional

$$I(v,\Omega)=\int_\Omega f(x,u(x),Du(x))\,dx$$

with given boundary data automatically also minimizes $I(\cdot,V)$, subject to its own boundary values, for every subdomain $V\subset\Omega$, the same is not true for $S(\cdot,\Omega)$. Since it further turns out, see [9], [26], that only imposing this "locality" leads to a satisfactory class of solutions, we are led to the concept of absolute minimizers, introduced by Aronsson [2]-[4]:

Definition 2.1. A locally Lipschitz continuous function $u:\Omega\to\mathbb{R}^m$, $m\geq 1$, is called an *absolute minimizer* of $S(\cdot,\Omega)$, if

$$S(u,V)\leq S(v,V)$$

for every $V\subset\subset\Omega$ and $v\in W^{1,\infty}(V)\cap C(\overline{V}))$ such that $v|_{\partial V}=u|_{\partial V}$.

[1]Since minimizing the functional $\operatorname{ess\,sup}|Du|^2$ is equivalent to minimizing $\operatorname{ess\,sup}|Du|$, we could have used, instead of (2.1), the singular and one-homogeneous version $-\frac{1}{|Du|^2}\sum u_{x_i}u_{x_j}u_{x_ix_j}=0$ of the infinity Laplacian.

The results obtained by Barron, Jensen and Wang [9], [10] and others clearly indicate that the theory of supremum functionals is indeed related to the classical calculus of variations, but that it has a character of its own. For example, in the scalar case the main condition guaranteeing the weak lower semicontinuity of the functional I_p is the convexity of the kernel $f(x, u, Du)$ in its last variable, whereas for the supremum functional $S(\cdot, \Omega)$ the right requirement seems to be *level-convexity* of H, that is,

$$H(x, t, \lambda\eta + (1-\lambda)\xi) \leq \max\{H(x, t, \eta), H(x, t, \xi)\}$$

for all $x \in \Omega$, $t \in \mathbb{R}$, $\eta, \xi \in \mathbb{R}^n$ and $\lambda \in (0,1)$. From now on, we will restrict our attention to the scalar case $m = 1$ and refer the reader to [9], [10] for what is known about the vector valued case.

2.2. **Comparison with cones.** In the case of the infinity Laplacian, the cone functions $C(x) = a|x - x_0| + b$ are solutions of the equation in $\mathbb{R}^n \setminus \{x_0\}$. It is remarkable that with the aid of these "fundamental solutions" one can characterize all (sub)solutions and obtain all the known estimates. A key notion is the following:

Definition 2.2 (Crandall, Evans, Gariepy [17]). A function $u \in C(\Omega)$ enjoys *comparison with cones from above* if, whenever $V \subset\subset \Omega$ is open, $x_0 \notin V$, $a, b \in \mathbb{R}$, $a > 0$, are such that

$$u(x) \leq C(x) := a|x - x_0| + b \qquad \text{on } \partial V,$$

we have

$$u(x) \leq C(x) \qquad \text{in } V.$$

A function $v \in C(\Omega)$ enjoys *comparison with cones from below* if $-v$ enjoys comparison with cones from above.

Finally, $u \in C(\Omega)$ enjoys *comparison with cones* if it enjoys comparison with cones both from above and below.

For a general Hamiltonian $H = H(x, u, Du)$ one has to figure out the correct definition of a "cone function". Roughly speaking, if H is level-convex in the gradient variable, then the functions

$$C_{k,r}(x, x_0) = \inf\left(\int_0^T L(\dot{\xi}, \xi, k)\, dt : \xi \in \text{path}(x, r)\right),$$

where

$$L(p, x, k) = \max\{q \cdot p : H(q, x) \leq k\}$$

do the job. See e.g. [14], [35], [19]. If H depends only on the gradient variable, then things simplify considerably, see [23].

2.3. **Equivalences.** It is important to observe that in the case of a supremum functional, the derivation of the Euler-Lagrange equation (as well as any regularity properties for the solutions) require genuinely new techniques. Namely, there is a basic problem with finding the Gateaux derivative, and consequently the first variation, of a non-smooth functional such as $S(u, \Omega) = \text{ess sup}\, |Du|^2$, especially when u is only Lipschitz continuous. This complicates significantly the relationship between absolute minimizers and the solutions of the formal Euler-Lagrange equation.

These questions have been and still are under intense research. For the infinity Laplacian the situation is nowadays more or less clear:

Theorem 2.3. *Let $u : \Omega \to \mathbb{R}$ be a locally Lipschitz continuous function. Then the following are equivalent:*

1. *u is an absolute minimizer of the functional $S(v) = \text{ess sup}\, |Du|$.*
2. *u is an AMLE (=absolutely minimizing Lipschitz extension): for every $V \subset\subset \Omega$ we have $Lip(u, V) = Lip(u, \partial V)$.*

(3) *u is a viscosity solution of the infinity Laplacian.*
(4) *u enjoys comparison with cones.*

Proof. **(1) implies (2):** Let us fix $V \subset\subset \Omega$. Since the lower McShane-Whitney extension

$$\Lambda(x) = \sup_{z \in \partial V} \{u(z) - Lip(u, \partial V)|x - y|\}$$

satisfies $Lip(\Lambda, V) = Lip(u, \partial V)$ and $u = \Lambda$ on ∂V, we have

$$\operatorname*{ess\,sup}_V |Du| \leq \operatorname*{ess\,sup}_V |D\Lambda| \leq Lip(\Lambda, V) = Lip(u, \partial V).$$

Hence

$$Lip(u, V) = \max\{\operatorname*{ess\,sup}_V |Du|, Lip(u, \partial V)\} = Lip(u, \partial V).$$

(2) implies (3): Suppose that there exists $\varphi \in C^2(\Omega)$ such that $u - \varphi$ has a local maximum at $\hat{x} \in \Omega$ and

$$\left(D^2\varphi(\hat{x})D\varphi(\hat{x})\right) \cdot D\varphi(\hat{x}) < 0.$$

Then by the proof of Proposition 3.5 below, there exists a cone function $C(x) = k|x-x_0|+b$, $x_0 \neq \hat{x}$, such that $C(\hat{x}) = \varphi(\hat{x})$, $DC(\hat{x}) = D\varphi(\hat{x})$ and $D^2C(\hat{x}) > D^2\varphi(\hat{x})$. Choose $\varepsilon > 0$ so small that $x_0 \notin B_\varepsilon(\hat{x})$ and $C > \varphi$ on $\partial B_\varepsilon(\hat{x})$. Let also

$$\delta = \min_{\partial B_\varepsilon(\hat{x})} (C - u) > 0.$$

Then for $\tilde{C} := C - \frac{\delta}{2}$ we have $\tilde{C} > u$ on $\partial B_\varepsilon(\hat{x})$ and $\tilde{C}(\hat{x}) < u(\hat{x})$. Denote by W the connected component of $\{u > \tilde{C}\}$ containing $\hat{x}$ and note that $W \subset B_\varepsilon(\hat{x})$. Since $u = \tilde{C}$ on ∂W and $x_0 \notin W$, we have $Lip(u, W) = Lip(u, \partial W) = k$ by the AMLE property. But it is easy to check that this can happen only if $u \equiv \tilde{C}$ in W, which contradicts the definition of W.

(3) implies (4): Fix $V \subset\subset \Omega$ and a cone function $C(x) = k|x - x_0| + b$, $k > 0$, such that $x_0 \notin V$ and $u \leq C$ on ∂V. If

$$0 < u(\hat{x}) - C(\hat{x}) = \sup_{x \in \overline{V}} \left(u(x) - C(x)\right)$$

for some $\hat{x} \in V$, then also the function $u - \phi_\varepsilon \circ C$, where $\phi_\varepsilon(t) = t - \varepsilon t^2$, has an interior local maximum at some point $x_\varepsilon \in V$ for $\varepsilon > 0$ small enough. Hence, as u is a viscosity subsolution,

$$0 \geq -\Delta_\infty(\phi_\varepsilon \circ C)(x_\varepsilon) = 2\varepsilon(1 - 2\varepsilon C(x_\varepsilon))k^4 > 0,$$

a contradiction. Here the equality is a result of a straightforward computation, that uses only the fact $-\Delta_\infty C(x) = 0$ in V.

(4) implies (2): Since

$$u(z) - Lip(u, \partial V)|x - z| \leq u(x) \leq u(z) + Lip(u, \partial V)|x - z|$$

holds for $x, z \in \partial V$ and $V \subset\subset \Omega$ and u enjoys comparison with cones from above and below, the inequalities hold also if $x \in V$. Thus

$$Lip(u, \partial V) = Lip(u, \partial(V \setminus \{x\}))$$

for any fixed $x \in V$. Using this twice, we have

$$Lip(u, \partial V) = Lip(u, \partial(V \setminus \{x, y\})) \quad \text{for } x, y \in V.$$

Hence $|u(x) - u(y)| \leq Lip(u, \partial V)|x - y|$, which shows that $Lip(u, V) = Lip(u, \partial V)$.

(3) implies (1): A rigorous proof of this implication is beyond the scope of this article, and for it we refer the reader to [17], [7] or [16]. Let us, however, indicate the main ideas.

Suppose that $u \in C^2(\Omega)$ satisfies $-\Delta_\infty u(x) = 0$ in Ω and that there exists $V \subset\subset \Omega$, $x_0 \in V$ and $v \in W^{1,\infty}(\Omega) \cap C(\overline{\Omega})$ such that $u = v$ on ∂V and

$$|Du(x_0)| > \operatorname*{ess\,sup}_V |Dv|. \tag{2.2}$$

Let γ be the unit speed integral curve of the vector field $x \mapsto Du(x)$ that passes through x_0. Since

$$0 = \Delta_\infty u(x) = \frac{1}{2} D(|Du(x)|^2) \cdot Du(x)$$

in Ω, the function $x \mapsto |Du(x)|^2$ is constant along γ. By the boundedness of u in $\overline{V}$ this implies that there exist $y, z \in \partial V \cap \gamma$, that is, the curve goes from boundary to boundary. Now we have

$$|u(y) - u(z)| = |\int_\gamma Du(\gamma(t)) \cdot \dot{\gamma}(t)\, dt| = \int_\gamma |Du(\gamma(t))|\, dt = |Du(x_0)| \int_\gamma dt,$$

while

$$|v(y) - v(z)| \le \operatorname*{ess\,sup}_V |Dv| \int_\gamma dt.$$

Combining these two inequalities with (2.2) yields $|u(y) - u(z)| > |v(y) - v(z)|$, which contradicts $u = v$ on ∂V. □

For a general Hamiltonian H the exact relationship between absolute minimizers and the solutions of the Aronsson equation is not yet fully clear. Conditions ensuring that an absolute minimizer is a solution to the Aronsson equation have been found in [9] and [15], and very recently in [19], whose main result we next quote:

Theorem 2.4. *Suppose that either*

$$H(x, \xi) \in C^1(\Omega \times \mathbb{R}^n) \ \textit{and} \ \ \xi \mapsto H(x, \xi) \ \textit{is level convex}$$

or

$$H(x, r, \xi) \in C^1(\Omega \times \mathbb{R} \times \mathbb{R}^n) \ \textit{and} \ \ \xi \mapsto H(x, r, \xi) \ \textit{is convex.}$$

Then any absolute minimizer of the functional (1.1) is a viscosity solution of (1.2).

The converse implication is perhaps less well understood. In this respect, Yu [35] has recently proved the following

Theorem 2.5. *Let $H(x, \xi) \in C^2(\Omega \times \mathbb{R}^n)$ be convex in ξ and*

$$\lim_{|\xi| \to \infty} H(x, \xi) = +\infty \qquad \textit{uniformly in } \Omega.$$

Then any viscosity solution $u \in C(\Omega)$ of (1.2) in Ω is an absolute minimizer of the functional (1.1).

Moreover, Yu has shown by counterexamples that level convexity alone is not sufficient for absolute minimality.

2.4. **Derivation of the Aronsson equation.** Theorem 2.3 shows that being a viscosity solution of the infinity Laplacian is equivalent to being an absolute minimizer. However, the proof does not explain very well "where the equation comes from".

There are, however, several ways, ranging from rigorous to purely formal, to derive the Aronsson equation. Below we discuss briefly four different approaches to this question in the special case of the infinity Laplacian; as usual, references to papers containing more details and generalizations are given.

p-Laplacian approximation. This is the way Aronsson originally derived the equation. The idea is to approximate the Lipschitz extension problem by the problems of minimizing the p-Dirichlet integral

$$\int_\Omega |Du|^p\,dx$$

with given boundary values. The Euler-Lagrange equation of this problem is the p-Laplace equation

$$-\operatorname{div}(|Du|^{p-2}Du) = 0,$$

which can be written as

$$-(p-2)|Du|^{p-4}\left(\frac{|Du|^2}{p-2}\Delta u + \Delta_\infty u\right) = 0.$$

If $Du \neq 0$, this implies

$$-\Delta_\infty u = \frac{|Du|^2}{p-2}\Delta u,$$

and thus letting $p \to \infty$ we recover the infinity Laplace equation $-\Delta_\infty u = 0$.

This formal argument can be easily made rigorous, cf. [12], [28], [7]. Moreover, the analogous result holds true for Hamiltonians $H(x, u, Du)$ that are convex in the gradient variable; here the convexity is used to guarantee that the associated L^p-problem has a solution.

Derivation from cone comparison. Next we derive the infinity Laplace equation from the property of comparison with cones, which in turn was shown to be equivalent to being an absolute minimizer in Theorem 2.3.

So let $u\colon \Omega \to \mathbb{R}$ enjoy comparison with cones from above. Then, in particular,

$$u(x) \le u(y) + \sup_{\{w:|w-y|=r\}} \left(\frac{u(w)-u(y)}{r}\right)|x-y| \tag{2.3}$$

whenever $y \in \Omega$ and $|x-y| \le r < \operatorname{dist}(y, \partial\Omega)$.

Suppose that u is twice differentiable at y in the sense that there is a $p \in \mathbb{R}^n$ and a symmetric $n \times n$ matrix X such that

$$u(x) = u(y) + p\cdot(x-y) + \frac{1}{2}X(x-y)\cdot(x-y) + o(|x-y|^2) \tag{2.4}$$

as $x \to y$. We claim that then

$$Xp\cdot p \ge 0. \tag{2.5}$$

Without loss of generality, we may assume that $y = 0$ and $p \neq 0$. Using (2.4) in (2.3) and writing $w = r\omega$ where $|\omega| = 1$ leads to

$$p\cdot x + \frac{1}{2}Xx\cdot x + o(|x|^2) \le \left(\sup_{\{\omega:|\omega|=1\}} \left(p\cdot\omega + \frac{r}{2}X\omega\cdot\omega + o(r)\right)\right)|x|.$$

Dividing both sides by $|x|$, writing $\hat{x} = x/|x|$ and using $|x| \le r$ gives

$$p\cdot\hat{x} + |x|\frac{1}{2}X\hat{x}\cdot\hat{x} \le \sup_{\{\omega:|\omega|=1\}} \left(p\cdot\omega + \frac{r}{2}X\omega\cdot\omega\right) + o(r).$$

Letting $x \to 0$ with $\hat{x} = \hat{p}$ we find

$$|p| \le \sup_{\{\omega:|\omega|=1\}} \left(p\cdot\omega + \frac{r}{2}X\omega\cdot\omega\right) + o(r).$$

When $r = 0$ the max on the right is achieved only at $\omega = \hat{p}$. Thus any maximum point ω_r for $r > 0$ satisfies $\omega_r \to \hat{p}$ as $r \to 0$. Then we have

$$\begin{aligned}|p| &\le p \cdot \omega_r + \frac{r}{2} X\omega_r \cdot \omega_r + o(r) \\ &\le |p| + \frac{r}{2} X\hat{p} \cdot \hat{p} + o(r).\end{aligned}$$

It follows that $Xp \cdot p \ge 0$. Similarly, comparison with cones from below implies that $Xp \cdot p \le 0$.

The above argument is due to Crandall and taken from [7]. It can be quite easily made rigorous in the setting of viscosity solutions [16] and it applies to the case of more general Hamiltonians [19] under the optimal assumption that $H \in C^1$.

First variation of the functional. In the classical calculus of variations, the Euler-Lagrange equation of an integral functional is usually derived by using the first variation. This method relies on the fact that one can, under suitable assumptions, change the order between integration and differentiation. For supremum functionals the analogue of this is not true, but things can be salvaged with the aid of Danskin's theorem [11].

Suppose that $u \in C^2(\Omega)$ is an absolute minimizer in Ω, and that $\varphi \in C_0^2(B_r(x_0))$, $B_r(x_0) \subset\subset \Omega$. Since

$$t \mapsto F(u + t\varphi) := \operatorname*{ess\,sup}_{B_r} |D(u + t\varphi)|^2 = \max_{B_r} |D(u + t\varphi)|^2$$

attains its minimum at $t = 0$, Danskin's theorem implies that

$$0 \le D_\varphi F(u) = \max_{x \in S(u)} Du(x) \cdot D\varphi(x).$$

Here $D_\varphi F(u)$ is the directional derivative F in the direction of φ at u, i.e.,

$$D_\varphi F(u) = \lim_{t \to 0+} \frac{F(u + t\varphi) - F(u)}{t}$$

and $S(u) = \{x \in \overline{B}_r(x_0) : F(u) = \max_{x \in \overline{B}_r(x_0)} |Du(x)|^2\}$. By choosing

$$\varphi_r(x) = \frac{1}{2}|x - x_0|^2 - \frac{1}{2}\varepsilon^2 \in C_0^2(B_r),$$

we have

$$0 \le Du(x_r) \cdot (x_r - x_0) \tag{2.6}$$

for some $x_r \in \overline{B}_r(x_0)$ for which $|Du(x_r)|^2 = \max_{|x - x_0| \le r} |Du(x)|^2$. It is easy to check that

$$\lim_{r \to 0+} \frac{x_r - x_0}{r} = \frac{\frac{d}{dx}|Du(x_0)|^2}{|\frac{d}{dx}|Du(x_0)|^2|};$$

notice that we may assume $\frac{d}{dx}|Du(x_0)|^2 = D^2u(x_0)Du(x_0) \neq 0$ since otherwise $\Delta_\infty u(x_0) = 0$ automatically. Thus, after dividing by r and then letting $r \to 0$, (2.6) implies

$$0 \le Du(x_0) \cdot \left(D^2u(x_0)Du(x_0)\right).$$

By repeating the same argument with φ replaced by $-\varphi$ we obtain the reverse inequality, and hence $-\Delta_\infty u(x_0) = -Du(x_0) \cdot \left(D^2u(x_0)Du(x_0)\right) = 0$.

The above argument is taken from [10], where further details can be found.

Dynamic programming principle and Tug-of-war. In [32], the following zero-sum two player stochastic game called tug-of-war is considered: fix $x = x_0 \in \Omega$ and step-size $\varepsilon > 0$. At the k^{th} turn, the players toss a coin and winner chooses an x_k with $|x_k - x_{k-1}| < \varepsilon$. The game ends when $x_k \in \partial\Omega$, and player I's payoff is $g(x_k)$, where $g : \partial\Omega \to \mathbb{R}$ is a given (continuous) function.

Suppose that the value function $u_\varepsilon(x)$ of the above game is continuous for each $\varepsilon > 0$ and that $u_\varepsilon \to u$ uniformly in Ω. We will show that u is a viscosity solution of the infinity Laplacian.

The dynamic programming principle (DPP) (which is easy to believe, but perhaps not so easy to verify rigorously) reads

$$u_\varepsilon(x) = \frac{1}{2}\big(\max_{B_\varepsilon(x)} u_\varepsilon + \min_{B_\varepsilon(x)} u_\varepsilon\big).$$

Suppose that $u - \varphi$, where $\varphi \in C^2(\Omega)$, has a strict local maximum at x. Then $u_\varepsilon - \varphi$ has a local maximum at some point x_ε and $x_\varepsilon \to x$ as $\varepsilon \to 0$. Using the (DPP) we have

$$\varphi(x_\varepsilon) \le \frac{1}{2}\big(\max_{B_\varepsilon(x_\varepsilon)} \varphi + \min_{B_\varepsilon(x_\varepsilon)} \varphi\big). \tag{2.7}$$

For ε small, we have

$$\max_{B_\varepsilon(x_\varepsilon)} \varphi = \varphi(x_\varepsilon) + \varepsilon|D\varphi(x_\varepsilon)| + \frac{\varepsilon^2}{2} D^2\varphi(x_\varepsilon)\frac{D\varphi(x_\varepsilon)}{|D\varphi(x_\varepsilon)|} \cdot \frac{D\varphi(x_\varepsilon)}{|D\varphi(x_\varepsilon)|} + o(\varepsilon^2)$$

and

$$\min_{B_\varepsilon(x_\varepsilon)} \varphi = \varphi(x_\varepsilon) - \varepsilon|D\varphi(x_\varepsilon)| + \frac{\varepsilon^2}{2} D^2\varphi(x_\varepsilon)\frac{D\varphi(x_\varepsilon)}{|D\varphi(x_\varepsilon)|} \cdot \frac{D\varphi(x_\varepsilon)}{|D\varphi(x_\varepsilon)|} + o(\varepsilon^2).$$

Substituting these to (2.7), canceling terms and then dividing by ε^2 and letting $\varepsilon \to 0$, we conclude

$$0 \le D^2\varphi(x)\frac{D\varphi(x)}{|D\varphi(x)|} \cdot \frac{D\varphi(x)}{|D\varphi(x)|}.$$

A more detailed proof, based on (DPP), and a vast number of generalizations of the theme can be found in [8]. The original tug-of-war article of Peres, Schramm, Sheffield and Wilson [32] does not rely on (DPP) - instead they argue directly and use the equivalence of viscosity solutions and comparison with cones property.

3. Hamilton-Jacobi flows and Aronsson equations

In this section, we restrict our attention to convex Hamiltonians depending only on the gradient variable, $H = H(Du)$, whence the Aronsson equation simplifies to

$$\Delta_\infty^H u := \sum_{i,j=1}^{n} H_{p_i}(Du)H_{p_j}(Du)u_{ij} = \big(D^2u\, DH(Du)\big) \cdot DH(Du) = 0. \tag{3.1}$$

To be more precise, we assume that the Hamiltonian $H : \mathbb{R}^n \to \mathbb{R}$ satisfies the following conditions:

(i) $H \in C^2(\mathbb{R}^n)$, $H(0) = 0$ and $H(p) \ge 0$ for all $p \in \mathbb{R}^n$.
(ii) H is uniformly convex, i.e., $D^2H(p) \ge cI$ for some $c > 0$.
(iii) H is superlinear:

$$\liminf_{|p|\to\infty} \frac{H(p)}{|p|} = \infty.$$

For a Lipschitz continuous function $u : \mathbb{R}^n \to \mathbb{R}$, let $w(x,t)$ and $v(x,t)$ be the unique viscosity solutions to the Hamilton-Jacobi flow equations

$$\begin{cases} w_t - H(Dw) = 0 & \text{in } \mathbb{R}^n\times]0,\infty[, \\ w = u & \text{for } t = 0, \end{cases} \tag{3.2}$$

and

$$\begin{cases} v_t + H(Dv) = 0 & \text{in } \mathbb{R}^n\times]0,\infty[, \\ v = u & \text{for } t = 0. \end{cases} \tag{3.3}$$

Theorem 3.1. *A locally Lipschitz continuous function $u : \mathbb{R}^n \to \mathbb{R}$ is a viscosity subsolution of $\Delta_\infty^H \varphi = 0$ in $\mathbb{R}^n$ if and only if the function $t \mapsto w(x,t)$ is convex for all $x \in \mathbb{R}^n$. Similarly, u is a viscosity supersolution of $\Delta_\infty^H \varphi = 0$ in $\mathbb{R}^n$ if and only if the function $t \mapsto v(x,t)$ is concave for all $x \in \mathbb{R}^n$.*

This characterization of subsolutions and supersolutions of (3.1) was conjectured by Barron, Evans and Jensen in [8]. Let us briefly recall its formal derivation. By differentiating w in (3.2) (assume w is smooth) with respect to t and x_k, we obtain

$$w_{tt} - \sum_{i=1}^{n} H_{p_i}(Dw) w_{x_i t} = 0$$

and

$$w_{tx_k} - \sum_{i=1}^{n} H_{p_i}(Dw) w_{x_i x_k} = 0.$$

Substituting the second equation to the first yields

$$w_{tt} - \Delta_\infty^H w = 0.$$

Using the fact that the flow (3.2) preserves subsolutions of (3.1), i.e., if $-\Delta_\infty^H u \le 0$ then $-\Delta_\infty^H w(\cdot,t) \le 0$ for all $t > 0$ (see Lemma 3.2 below), this implies $w_{tt} \ge 0$. Hence $t \mapsto w(x,t)$ is convex for every x. The reasoning in the supersolution case is analogous, with the equation (3.2) being replaced by (3.3).

The idea for the proof of Theorem 3.1 is fairly simple and is based on the observation that the evolution of a generalized cone function with non-negative slope under the backwards flow (3.2) is affine in time, see Proposition 3.3 below. Thus it suffices to show that the comparison principle with respect to generalized cones that characterizes the viscosity subsolutions of (3.1) translates to comparison with affine functions that in turn characterizes convexity, and vice versa.

Although Theorem 3.1 is stated for functions defined in $\mathbb{R}^n$, it will be clear from the proofs that the characterization is indeed local and therefore also applies to functions defined on any subdomain $\Omega \subset \mathbb{R}^n$. We will elaborate on this in after proving Theorem 3.1.

As in the previous section, we will prove Theorem 3.1 only in the case of the infinity Laplacian, that is, assuming that $H(p) = \frac{1}{2}|p|^2$. For the more general situation, see [29].

3.1. **Preliminaries.** It is well-known (see e.g. [21]) that the functions w and v in (3.2) and (3.3), respectively, are given by the Hopf-Lax formulas

$$\begin{aligned} w(x,t) &= \sup_{y\in\mathbb{R}^n} \left(u(y) - \frac{1}{2t}|y-x|^2 \right) \\ v(x,t) &= \inf_{y\in\mathbb{R}^n} \left(u(y) + \frac{1}{2t}|y-x|^2 \right). \end{aligned} \tag{3.4}$$

If u is Lipschitz continuous, it is enough to take the sup and inf in (3.4) over those y's that satisfy

$$\frac{1}{2t}|y-x|^2 < \mathrm{Lip}(u)\frac{|y-x|}{t}. \tag{3.5}$$

This set is contained in the ball $B_{Mt}(x)$, where $M = 2\,\mathrm{Lip}(u)$.

It is also well known that these inf- and sup-convolutions provide semiconcave viscosity supersolutions and semiconvex subsolutions, respectively, for translation invariant equations.

Lemma 3.2. *A continuous function $u : \mathbb{R}^n \to \mathbb{R}$ is a viscosity subsolution of $\Delta_\infty\varphi = 0$ in $\mathbb{R}^n$ if and only if the function $x \mapsto w(x,t)$ is viscosity subsolution of $\Delta_\infty\varphi = 0$ for every $t > 0$.*

Along with Theorem 2.3, our main tool in proving Theorem 3.1 is the following Proposition:

Proposition 3.3. *Let $W(x,t)$ be the unique viscosity solution to*

$$\begin{cases} W_t - \frac{1}{2}|DW|^2 = 0 & \text{in } \mathbb{R}^n\times]0,\infty[, \\ W = C_k & \text{for } t = 0, \end{cases} \tag{3.6}$$

where $C_k(x) = k|x| + b$, $k > 0$. Then $W(x,t) = C_k(x) + \frac{1}{2}k^2t$. In particular, $t \mapsto W(x,t)$ is affine for all $x \in \mathbb{R}^n$. Moreover,

$$W(x,t) = \max_{y\in\mathbb{R}^n}\left(C_k(y) - \frac{1}{2t}|x-y|^2\right), \tag{3.7}$$

and

1. *for $x \neq 0$, $t > 0$ the maximum is achieved if and only if $y = (|x| + kt)\frac{x}{|x|}$.*
2. *for $x = 0$, $t > 0$ the maximum is achieved if and only if y satisfies $|y| = kt$.*

Proof. It is easy to verify directly that $(x,t) \mapsto C_k(x)+\frac{1}{2}k^2t$ is a viscosity solution (3.6) and its uniqueness follows from a standard comparison principle, see e.g. [18]. Alternatively, one can use the facts that this function satisfies the equation a.e. in the classical sense and is (semi)convex, and then apply the results in [21].

The rest of the proof is a calculus exercise that we leave to the reader. □

Remark 3.4. Due to the semiconvexity requirement for the solution of (3.6), Proposition 3.3 does not hold for cones with negative slope, see [21], Section 3.3.3.

3.2. **Sufficiency.**

Proposition 3.5. *Let $u \in C(\mathbb{R}^n)$ and suppose that u is not a viscosity subsolution of $-\Delta_\infty u = 0$. Then for each $\varepsilon > 0$ small enough and $N > 1$ there exists a cone $C(x) = k|x - x_0| + u(x_0)$, $k > 0$ such that*

1. *the set $\{x \in B_{N\varepsilon}(x_0) : u(x) > C(x)\}$ is non-empty and contained in $B_\varepsilon(x_0)$.*
2. *$\sup_{B_{M\varepsilon}(x_0)}(u - C) \leq \varepsilon$.*

Proof. Since u is not a viscosity subsolution of $-\Delta_\infty u = 0$, there exists $\varphi \in C^2(\mathbb{R}^n)$ and $\hat{x} \in \mathbb{R}^n$ such that $0 = u(\hat{x}) - \varphi(\hat{x}) > u(x) - \varphi(x)$ for $x \neq \hat{x}$ and

$$(D^2\varphi(\hat{x})D\varphi(\hat{x}))\cdot D\varphi(\hat{x}) < 0. \tag{3.8}$$

In order to prove our claim, it suffices to show that there exists $a > 0$, $b \in \mathbb{R}$ and $y_0 \in \mathbb{R}^n$ such that the function $\varphi - (a|x - y_0| + b)$ has a strict local zero maximum at $\hat{x}$. Indeed, if this is the case, by first decreasing b slightly and then translating the vertex y_0 to a point at which u and the dropped cone agree (and adjusting b again appropriately), we obtain (1) and (2).

Let $p = D\varphi(\hat{x})$ and $a = |p| > 0$. For $z = \delta\frac{p}{|p|} \neq 0$, where $\delta > 0$ will be chosen below, and $y_0 = \hat{x} - z$, we define

$$C_0(x) = a|x - y_0| + b,$$

where $b = \varphi(\hat{x}) - a|\hat{x} - y_0|$. Then $C_0(\hat{x}) = \varphi(\hat{x})$ and $DC_0(\hat{x}) = a\frac{\hat{x}-y_0}{|\hat{x}-y_0|} = D\varphi(\hat{x})$, and thus is suffices to check that $D^2C_0(\hat{x}) > D^2\varphi(\hat{x})$, i.e.,

$$D^2C_0(\hat{x})\xi \cdot \xi > D^2\varphi(\hat{x})\xi \cdot \xi \qquad \text{for all } \xi \neq 0. \tag{3.9}$$

We observe first that since $|DC_0(x)| = a$ for $x \neq y_0$, it follows by taking derivatives that $D^2C_0(\hat{x})DC_0(\hat{x}) = 0$. Thus, using (3.8), we see that (3.9) is valid for $\xi = z$. On the other hand, since

$$D^2(a|x|) = \frac{a}{|x|}\left(I - \tfrac{x}{|x|} \otimes \tfrac{x}{|x|}\right),$$

we have for any $\xi \neq 0$ such that $\xi \cdot z = 0$

$$D^2C_0(\hat{x})\xi \cdot \xi = \frac{a}{|z|}|\xi|^2 > D^2\varphi(\hat{x})\xi \cdot \xi$$

provided that $\delta = |z| = |\hat{x} - y_0|$ is small enough. This completes the proof. □

Let us now prove the sufficiency part of Theorem 3.1. Let $u : \mathbb{R}^n \to \mathbb{R}$ be Lipschitz continuous and let $w \in C(\mathbb{R}^n \times [0, \infty[)$ be the unique viscosity solution to

$$\begin{cases} w_t - \frac{1}{2}|Dw| = 0 & \text{in } \mathbb{R}^n \times]0, \infty[, \\ w = u & \text{for } t = 0. \end{cases}$$

If u is not a subsolution of $-\Delta_\infty u = 0$, we infer from Proposition 3.5 that there exists a cone $C(x) = k|x - x_0| + u(x_0)$ such that (1) and (2) hold, with $N \geq 1$ and $\varepsilon > 0$ chosen below. Without loss of generality, we may assume that $x_0 = 0$ and $C(0) = u(0) = 0$. Let $\Psi \in C(\mathbb{R}^n \times [0, \infty[)$ be the unique viscosity solution to

$$\begin{cases} \Psi_t - \frac{1}{2}|D\Psi| = 0 & \text{in } \mathbb{R}^n \times]0, \infty[, \\ \Psi = C & \text{for } t = 0. \end{cases}$$

By Proposition 3.5 (1), we can find arbitrarily small $t > 0$ for which there exist $y_t \in \mathbb{R}^n$ such that $|y_t| = kt$ and $u(y_t) > C(y_t)$. Thus, for such t's, we obtain using Proposition 3.3 that

$$\Psi(0, t) = C(y_t) - \frac{1}{2t}|y_t|^2 < u(y_t) - \frac{1}{2t}|y_t|^2 \leq w(0, t). \tag{3.10}$$

On the other hand, we claim that if we choose

$$N \geq 4\,\mathrm{Lip}(u)\left(1 + \frac{1}{k}\right),$$

then

$$\Psi(0, t) \geq w(0, t) \qquad \text{for all } \left(1 + \frac{1}{k}\right)\varepsilon \leq t \leq 2\left(1 + \frac{1}{k}\right)\varepsilon. \tag{3.11}$$

Indeed, for such t's we have $B_{2\mathrm{Lip}(u)t}(0) \subset B_{N\varepsilon}(0)$ and thus, in view of (3.5) and Proposition 3.5, the reverse inequality $\Psi(0, t) < w(0, t)$ can hold only if there exists $z \in B_\varepsilon(0)$ such that $u(z) > C(z)$ and $u(z) - \frac{1}{2t}|z|^2 > \Psi(0, t)$. Since $u(z) \leq C(z) + \varepsilon$ and $\Psi(0, t) = kt$, we must then have

$$C(z) + \varepsilon - \frac{1}{2t}|z|^2 > kt.$$

But this is impossible, because

$$C(z)+\varepsilon-\frac{1}{2t}|z|^2\leq C(z)+\varepsilon\leq(1+k)\varepsilon\leq kt$$

by the choice of t. By combining (3.10) and (3.11) with the equality $\Psi(0,0)=C(0)=u(0)=w(0,0)$, we conclude that $t\mapsto w(0,t)$ is not convex. □

3.3. **Necessity.** For a viscosity subsolution u of $-\Delta_\infty v=0$, let

$$w(x,t)=\sup_{y\in\mathbb{R}^n}\left(u(y)-\frac{1}{2t}|y-x|^2\right)$$

and suppose that $f(t)=w(0,t)$ is not convex. Then there exists $0\leq t_1<t_2$ and an affine function $l(t)=at+b$ such that

$$l(t_i)=f(t_i),\ i=1,2,\quad\text{and}\quad l(t)<f(t)\quad\text{for all } t_1<t<t_2. \tag{3.12}$$

Notice that since $f(0)=u(x)\leq w(x,t)=f(t)$ for $t>0$, we may assume without loss of generality that $a>0$. Moreover, by Lemma 3.2, we may assume that $t_1=0$ and thus $b=u(0)$.

Let $C(x)=\sqrt{2a}|x|+u(0)$ and let $W(x,t)$ be the unique solution to

$$\begin{cases} W_t-\frac{1}{2}|DW|^2=0 & \text{in } \mathbb{R}^n\times]0,\infty[,\\ W=C & \text{for } t=0,\end{cases}$$

given by the Hopf-Lax formula

$$W(x,t)=\sup_{y\in\mathbb{R}^n}\left(C(y)-\frac{1}{2t}|y-x|^2\right).$$

By Proposition 3.3, $W(x,t)=C(x)+at$. In particular, $W(0,t)=l(t)$ for all $t\geq 0$.

For $t_1<t<t_2$, $W(0,t)<w(0,t)$ by (3.12), and this is possible only if

$$\{x:u(x)>C(x)\}\cap B(0,t)\setminus\{0\}\neq\emptyset\quad\text{for all } t_1<t<t_2.$$

On the other hand, by Proposition 3.3, $W(0,t_2)=w(0,t_2)$ implies that $u\leq C$ on the sphere $|y|=\sqrt{2a}t_2$. We have thus shown that the generalized cone comparison principle is violated, contrary to the assumption that u is a viscosity subsolution of $-\Delta_\infty v=0$.

Remark 3.6. Above it was assumed that u is a Lipschitz function defined in the whole $\mathbb{R}^n$. This rendered the proofs quite transparent, but, in view of the existing literature on the Aronsson equations, it is also natural to try to consider more general situations. Fortunately, since all our arguments are local in nature, it turns out that we need neither the Lipschitz continuity nor the fact that the domain is the entire $\mathbb{R}^n$ to obtain a characterization for the sub- and supersolutions. The reader can easy verify this by analyzing the arguments used above; alternatively, the paper [29] could be consulted.

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