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Asymptotic forms of weakly increasing positive solutions of quasilinear ordinary differential equations

(準線型常微分方程式の弱増加正値解の漸近形)

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1 Introduction

This talk is a joint work with Professor Ken-ichi Kamo (Sapporo Medical University, Japan). Let us consider the equations of the form

$$(|u'|^{\alpha-1}u')' + p(t)|u|^{\beta-1}u = 0 \tag{E}$$

under the following conditions:

(A_1) α and β are positive constants satisfying $\alpha < \beta$;
(A_2) $p(t)$ is a C^1−function defined near $+\infty$ satisfying the asymptotic condition $p(t) \sim t^{-\sigma}$ for some $\sigma \in R$ as $t \to \infty$.

By condition (A_2) equation (E) can be rewritten in the form

$$(|u'|^{\alpha-1}u')' + t^{-\sigma}(1+\varepsilon(t))|u|^{\beta-1}u = 0, \tag{E}$$

where $\varepsilon(t) = t^{\sigma}p(t) - 1$ satisfies $\lim_{t\to\infty} \varepsilon(t) = 0$. Of course, here and in what follows the symbol "$f(t) \sim g(t)$ as $t \to \infty$" means that $\lim_{t\to\infty} f(t)/g(t) = 1$. A function u is defined to be a solution of equation (E) if $u \in C^1[t_1, \infty)$ and $|u'|^{\alpha-1}u' \in C^1[t_1, \infty)$ and it satisfies equation (E) on $[t_1, \infty)$ for sufficiently large t_1.

It is easily seen that all positive solutions $u(t)$ of (E) are classified into the following three types according as their asymptotic behavior as $t \to \infty$:
(I) (asymptotically linear solution):

$$u(t) \sim c_1 t \quad \text{for some constant} \quad c_1 > 0;$$

(II) (*weakly increasing solution*):

$$u'(t) \downarrow 0, \quad \text{and} \quad u(t) \uparrow \infty;$$

(III) (asymptotically constant solution):

$$u(t) \sim c_1 \quad \text{for some constant} \quad c_1 > 0.$$

Concerning qualitative properties of positive solutions, the study of asymptotic behavior of asymptotically linear solutions and asymptotically constant solutions are rather easy, because their first approximations are given by definition. On the other hand, we can not easily find how the weakly increasing positive solutions behave except for the case of $\alpha = 1$ ([1, 4]).

In [4, Section 20], equation (E) with $\alpha = 1$ has been considered systematically, and asymptotic forms of weakly increasing positive solutions are given by means of the parameters β and σ. When $\alpha \neq 1$, as far as the authors are aware, there are no works in which asymptotic forms of weakly increasing positive solutions are studied systematically. Motivated by these facts we have been making an attempt to find out asymptotic forms of weakly increasing positive solutions of (E) for the general case $\alpha > 0$.

To gain an insight into our problem, we consider the typical equation

$$(|u'|^{\alpha-1}u')' + t^{-\sigma}|u|^{\beta-1}u = 0, \tag{E$_0$}$$

where $\sigma \in R$ is a constant. Note that equation (E) can be regarded as a perturbed equation of this equation. Equation (E_0) has a weakly increasing positive solution of the form $ct^\rho, (c > 0, 0 < \rho < 1)$ if and only if $\alpha + 1 < \sigma < \beta + 1$. This solution is uniquely given by

$$u_0(t) = \hat{C}t^k, \tag{1}$$

where

$$k = \frac{\sigma - \alpha - 1}{\beta - \alpha} \in (0, 1), \quad \hat{C} = \{\alpha(1-k)k^\alpha\}^{\frac{1}{\beta-\alpha}}. \tag{2}$$

From this simple observation we can see that asymptotic forms of weakly increasing positive solutions of (E) may be strongly affected by that of the coefficient function $p(t)$. Furthermore we conjecture that weakly increasing positive solutions u of (E) behave like $u_0(t)$ given by (1) and (2) if $|\varepsilon(t)|$ is sufficiently small at ∞. It should be noted that the number k appearing in (2) plays important roles in the sequel.

We have shown in [3] that the above conjecture is true in some case as seen from the following theorem:

Theorem 1 *Let* $\alpha \leq 1$ *and* $1/2 < k < 1 (\Leftrightarrow (\alpha + \beta + 2)/2 < \sigma < \beta + 1)$. *Suppose furthermore that either*

$$\int^\infty \frac{\varepsilon(t)^2}{t} dt < \infty \tag{3}$$

or

$$\int^\infty |\varepsilon'(t)| dt < \infty \tag{4}$$

holds. Then, every weakly increasing positive solution u *of* (E) *has the asymptotic form*

$$u(t) \sim u_0(t) \quad as \quad t \to \infty,$$

where u_0 *is given by* (1) *and* (2).

In today's talk we report that our conjecture is still valid for other cases; that is, we will prove the following:

Theorem 2 *Let $\alpha \geq 1$ and $0 < k < 1/2 (\Leftrightarrow \alpha + 1 < \sigma < (\alpha + \beta + 2)/2)$. Suppose furthermore that either* (3) *or* (4) *holds. Then, the same conclusion as in Theorem 1 holds.*

Remark. (i) In Theorems 1 and 2, the differentiability of p is unnecessary when (3) is assumed.

(ii) When $\alpha = 1$ and $\varepsilon(t) \equiv 0$, Theorems 1 and 2 have been obtained by [1] and [4, Corollaries 20.2 and 20.3].

We note that existence results of weakly increasing positive solutions to (E) are not well known. But we can show many concrete examples of those equations that have weakly increasing positive solutions. Some of existence results of weakly increasing solutions for the case $\alpha = 1$ are found in [6,7].

The paper is organized as follows. In Section 2 we give preparatory lemmas employed later. In Section 3 we give the proof of Theorem 2. Other related results are found in [3,5,6,7].

2 Preparatory lemmas

Lemma 3 *Let $w \in C^1[t_0, \infty), w'(t) = O(1)$ as $t \to \infty$, and $w \in L^\lambda[t_0, \infty)$ for some $\lambda > 0$. Then, $\lim_{t\to\infty} w(t) = 0$.*

Proof. We have

$$\begin{aligned} |w(t)|^\lambda w(t) &= |w(t_0)|^\lambda w(t_0) + \int_{t_0}^t (|w(s)|^\lambda w(s))' ds \\ &= |w(t_0)|^\lambda w(t_0) + (\lambda + 1) \int_{t_0}^t |w(s)|^\lambda w'(s) ds. \end{aligned}$$

By our assumptions the last integral converges as $t \to \infty$. Hence $\lim_{t\to\infty} |w(t)|^\lambda w(t) \in R$ exists. Since $w \in L^\lambda[t_0, \infty)$, the limit must be 0. The proof is completed.

Lemma 4 *Let $\sigma \in (\alpha+1, \beta+1)$. Then every weakly increasing positive solution u of* (E) *satisfies $u(t) = O(u_0(t))$ and $u'(t) = O(u_0'(t))$ as $t \to \infty$, where u_0 is the exact solution of* (E_0) *given by* (1) *and* (2).

Proof. We may assume that $u, u' > 0$ for $t \geq t_1$. Since u satisfies for large t

$$u'(t)^\alpha = \int_t^\infty p(s) u(s)^\beta ds, \tag{5}$$

and u is increasing, we have

$$u'(t)^\alpha \geq u(t)^\beta \int_t^\infty p(s) ds,$$

that is

$$u'(t)u(t)^{-\frac{\beta}{\alpha}} \geq \left(\int_t^\infty p(s)ds\right)^{\frac{1}{\alpha}}.$$

An integration of this inequality on the interval $[t, \infty)$ will give

$$u(t) \leq C_1 \left\{ \int_t^\infty \left(\int_s^\infty p(r)dr \right)^{1/\alpha} ds \right\}^{-\alpha/(\beta-\alpha)} \equiv C_2 u_0(t),$$

where C_1 and C_2 are positive constant. Furthermore, by (5) we find that

$$u'(t) = \left(\int_t^\infty p(s)u(s)^\beta ds \right)^{1/\alpha} \leq C_3 \int_t^\infty s^{-\sigma+k\beta} ds = C_4 t^{k-1} = O(u_0'(t)) \quad \text{as} \quad t \to \infty,$$

where C_3 and C_4 are positive constants. This completes the proof.

Lemma 5 *Let $\sigma \in (\alpha+1, \beta+1)$, and u a weakly increasing positive solution of equation* (E). *Put $s = \log u_0(t)$ and $v = u/u_0$. Then*
(i) *$v, \dot{v} = O(1)$ as $s \to \infty$, and $v + \dot{v} > 0$ near ∞, where $\cdot = d/ds$;*
(ii) *$v(s)$ satisfies near ∞ the equation*

$$\ddot{v} - a\dot{v} - bv + b(\dot{v}+v)^{1-\alpha}v^\beta + b\delta(s)(\dot{v}+v)^{1-\alpha}v^\beta = 0, \tag{6}$$

where

$$a = \frac{1}{k} - 2 > 0, \quad b = \frac{1-k}{k} > 0, \quad \textit{and} \quad \delta(s) = \varepsilon(t).$$

Proof. We will prove only (i), because (ii) can be proved by direct computations. We assume that $u, u' > 0$. Since $u = u_0 v$, the boundedness of v follows from Lemma 4. Noting $du/dt = \hat{C}kt^{k-1}(v+\dot{v})$, we have $v + \dot{v} > 0$. On the other hand, since $dt/ds = t/k$, we have

$$|\dot{v}| = \left| \frac{d}{dt}\left(\frac{u}{u_0}\right)\frac{dt}{ds} \right| = \left| \frac{u'u_0 - u_0'u}{u_0^2} \right| \frac{t}{k} \leq C\frac{t^{k-1}t^k t}{t^{2k}} = O(1) \quad \text{as} \quad s \to \infty.$$

This completes the proof.

Lemma 6 *Let the assumptions of Theorem 2 holds, and v be as in Lemma 5. Then $\dot{v} \in L^2[s_0, \infty)$ for sufficiently large s_0.*

Proof. We note that conditions (3) and (4), respectively, are equivalent to

$$\int^\infty \delta(s)^2 ds < \infty \tag{7}$$

and

$$\int^\infty |\dot{\delta}(s)| ds < \infty. \tag{8}$$

We multiply the both sides of (6) by $\dot{v}$. Since $\alpha \geq 1$, we have $(1+\delta(s))(\dot{v}+v)^{1-\alpha}\dot{v} \leq (1+\delta(s))v^{1-\alpha}\dot{v}$; and so we obtain

$$a\dot{v}^2 \leq \dot{v}\ddot{v} - bv\dot{v} + b(1+\delta(s))v^{1-\alpha+\beta}\dot{v}.$$

An integration on the interval $[s_0, s]$ gives

$$a\int_{s_0}^{s} \dot{v}^2 dr \leq \frac{\dot{v}^2}{2} - \frac{bv^2}{2} + \frac{bv^{2-\alpha+\beta}}{2-\alpha+\beta} + \int_{s_0}^{s} \delta(r)v^{1-\alpha+\beta}\dot{v}dr + \text{const}; \tag{9}$$

that is

$$a\int_{s_0}^{s} \dot{v}^2 dr \leq b\int_{s_0}^{s} \delta(r)v^{1-\alpha+\beta}\dot{v}dr + O(1) \quad \text{as} \quad s \to \infty.$$

Here we have employed (i) of Lemma 5. Let the integral condition (3) hold; that is, let (7) hold. By the Schwarz inequality we have

$$a\int_{s_0}^{s} \dot{v}^2 dr \leq C_1 \left(\int_{s_0}^{\infty} \delta(r)^2 dr\right)^{1/2} \left(\int_{s_0}^{s} \dot{v}^2 dr\right)^{1/2} + O(1)$$

for some constant $C_1 > 0$. Therefore $\dot{v} \in L^2[s_0, \infty)$. Next let (4) hold. Using integral by parts, we obtain from (9)

$$a\int_{s_0}^{s} \dot{v}^2 dr \leq \frac{\dot{v}^2}{2} - \frac{b}{2}v^2 + \frac{b[1+\delta(r)]v^{2-\alpha+\beta}}{2-\alpha+\beta} - \frac{b}{2-\alpha+\beta}\int_{s_0}^{s} \dot{\delta}(r)v^{2-\alpha+\beta}dr + \text{const}.$$

As before by noting (i) of Lemma 5, we find that $\dot{v} \in L^2[s_0, \infty)$. This completes the proof.

3 Proof of Theorem 2

We are now in a position to prove the main result Theorem 2:

Proof of Theorem 2. To this end it suffices to show that $\lim_{s\to\infty} v(s) = 1$, where $v(s)$ is the function introduced in Lemma 5. The proof is divided into three steps.

Step 1. We claim that $\liminf_{s\to\infty} v(s) > 0$; namely $\liminf_{t\to\infty} u(t)/u_0(t) > 0$. The proof is done by contradiction.

Suppose to the contrary that $\liminf_{s\to\infty} v(s) = 0$. Firstly, we suppose that $v(s)$ decreases to 0 as $s \to \infty$. This means that $u(t)/u_0(t)$ decreases to 0 as $t \to \infty$. Accordingly we have

$$\begin{aligned} u'(t)^\alpha &= \int_t^\infty p(r)u(r)^\beta dr = \int_t^\infty p(r)u_0(r)^\beta \left(\frac{u(r)}{u_0(r)}\right)^\beta dr \\ &\leq \left(\frac{u(t)}{u_0(t)}\right)^\beta \int_t^\infty p(r)u_0(r)^\beta dr = C_1 t^{1-\sigma}u(t)^\beta, \end{aligned}$$

where $C_1 > 0$ is a constant. Consequently we obtain the differential inequality $u' \leq C_2 t^{(1-\sigma)/\alpha}u^{\beta/\alpha}$for some constant $C_2 > 0$ near ∞. But this differential inequality implies that $u(t)/u_0(t) \equiv v(s) \geq C_3 > 0$ for some constant $C_3 > 0$. This is an obvious contradiction.

Next sppose that $\liminf_{s\to\infty} v(s) = 0$ and $\dot{v}(s)$ changes sign in any neighborhood of ∞. Since $v(s)$ takes local maxima in the region $v \geq (1+\delta(s))^{-1/(\beta-\alpha)}$, there are the following sequences $\{\underline{s}_n\}$ and $\{\overline{s}_n\}$ satisfying

$$\underline{s}_n < \overline{s}_n < \underline{s}_{n+1}, \quad \lim_{n\to\infty} \underline{s}_n = \lim_{n\to\infty} \overline{s}_n = \infty$$

and

$$\dot{v}(\underline{s}_n) = \dot{v}(\overline{s}_n) = 0, \quad \lim_{n\to\infty} v(\underline{s}_n) = 0, \quad v(\overline{s}_n) \geq (1+\delta(\overline{s}_n))^{-1/(\beta-\alpha)}.$$

Now, we decompose α in the form $\alpha = m - \rho$, where $m \in N$ and $\rho > 0$. Although there are infinitely many such choices of decomposition for α, we fix one choice for a moment. We rewrite equation (6) as

$$\ddot{v} - a\dot{v} - bv + b(\dot{v}+v)^{-m+1+\rho}v^\beta + b\delta(s)(\dot{v}+v)^{-m+1+\rho}v^\beta = 0.$$

We multiply the both sides by $(v+\dot{v})^m\dot{v}$ and then integrate the resulting equation on the interval $[\underline{s}_n, \overline{s}_n]$ to obtain

$$\int_{\underline{s}_n}^{\overline{s}_n} \ddot{v}\dot{v}(v+\dot{v})^m dr - a\int_{\underline{s}_n}^{\overline{s}_n} (v+\dot{v})^m\dot{v}^2 dr - b\int_{\underline{s}_n}^{\overline{s}_n} v\dot{v}(v+\dot{v})^m dr$$

$$+b\int_{\underline{s}_n}^{\overline{s}_n} (v+\dot{v})^{1+\rho}\dot{v}v^\beta dr + b\int_{\underline{s}_n}^{\overline{s}_n} \delta(r)(v+\dot{v})^{1+\rho}\dot{v}v^\beta dr = 0. \tag{10}$$

The binomial expansion implies that

$$\sum_{k=0}^{m} c_k \int_{\underline{s}_n}^{\overline{s}_n} \ddot{v}\dot{v}^{k+1}v^{m-k} dr - a\int_{\underline{s}_n}^{\overline{s}_n} (v+\dot{v})^m\dot{v}^2 dr - b\sum_{k=0}^{m} c_k \int_{\underline{s}_n}^{\overline{s}_n} v^{m-k+1}\dot{v}^{k+1} dr$$

$$+b\int_{\underline{s}_n}^{\overline{s}_n} (v+\dot{v})^{1+\rho}\dot{v}v^\beta dr + b\int_{\underline{s}_n}^{\overline{s}_n} \delta(r)(v+\dot{v})^{1+\rho}\dot{v}v^\beta dr = 0,$$

where $c_k = \binom{m}{k}$ are the binomial coefficients. Now, we evaluate each term in the left hand side. For $k \in \{0, 1, ..., m-1\}$ we obtain

$$\int_{\underline{s}_n}^{\overline{s}_n} \ddot{v}\dot{v}^{k+1}v^{m-k} dr = \int_{\underline{s}_n}^{\overline{s}_n} \frac{d}{dr}\left(\frac{\dot{v}^{k+2}}{k+2}\right) v^{m-k} dr$$

$$= -\frac{m-k}{k+2}\int_{\underline{s}_n}^{\overline{s}_n} \dot{v}^{k+3}v^{m-k-1} dr = o(1) \quad \text{as} \quad n\to\infty.$$

For $k = m$ obviously we have $\int_{\underline{s}_n}^{\overline{s}_n} \ddot{v}\dot{v}^{k+1} dr = 0$. Hence the first term of the left hand side of (10) tends to 0 as $n \to \infty$. The second term is dominated by Const$\int_{\underline{s}_n}^{\overline{s}_n} \dot{v}^2 dr$, and hence it tends to zero as $n \to \infty$. Next, we compute the third term. For $k \in \{1, 2, ..., m\}$ we have $|\int_{\underline{s}_n}^{\overline{s}_n} v^{m-k+1}\dot{v}^{k+1}dr| \leq \text{const} \int_{\underline{s}_n}^{\overline{s}_n} \dot{v}^2 dr$. For $k = 0$ we have

$$\int_{\underline{s}_n}^{\overline{s}_n} v^{m+1}\dot{v}dr = \frac{1}{m+2}\left(v(\overline{s}_n)^{m+2} - v(\underline{s}_n)^{m+2}\right) = \frac{v(\overline{s}_n)^{m+2}}{m+2} + o(1) \quad \text{as} \quad n \to \infty.$$

Therefore the third term is equal to

$$o(1) - \frac{bv(\overline{s}_n)^{m+2}}{m+2} \quad \text{as} \quad n \to \infty.$$

To evaluate the fourth term we employ the mean value theorem to obtain

$$(v + \dot{v})^{1+\rho} = v^{1+\rho} + (1+\rho)\left(v + \theta(r)\dot{v}\right)^{\rho}\dot{v},$$

where $\theta(r)$ is a quantity between 0 and 1. Hence we can compute

$$\int_{\underline{s}_n}^{\overline{s}_n} (v + \dot{v})^{1+\rho}\dot{v}v^{\beta}dr = \int_{\underline{s}_n}^{\overline{s}_n} v^{1+\rho+\beta}\dot{v}dr + (1+\rho)\int_{\underline{s}_n}^{\overline{s}_n} \left(v + \theta(r)\dot{v}\right)^{\rho}\dot{v}^2 v^{\beta}dr$$

$$= \frac{v(\overline{s}_n)^{2+\rho+\beta} - v(\underline{s}_n)^{2+\rho+\beta}}{2+\rho+\beta} + (1+\rho)\int_{\underline{s}_n}^{\overline{s}_n} O(1)\dot{v}^2 dr = \frac{v(\overline{s}_n)^{2+\rho+\beta}}{2+\rho+\beta} + o(1) \quad \text{as} \quad n \to \infty.$$

Finally by Schwarz's inequality we find that the last term is dominated by the quantity

$$\text{const}\left(\int_{\underline{s}_n}^{\overline{s}_n} \delta(r)^2 dr\right)^{1/2}\left(\int_{\underline{s}_n}^{\overline{s}_n} \dot{v}^2 dr\right)^{1/2} = o(1) \quad \text{as} \quad n \to \infty.$$

Consequently, from (10) we obtain the formula

$$o(1) - \frac{b}{m+2}v(\overline{s}_n)^{m+2} + \frac{b}{2+\rho+\beta}v(\overline{s}_n)^{2+\rho+\beta} + o(1) = 0 \quad \text{as} \quad n \to \infty.$$

This implies that $\lim_{n\to\infty} v(\overline{s}_n) = [(m+2+\beta-\alpha)/(m+2)]^{1/\beta}$. Since m can be moved arbitrarily, this is an obvious contradiction. Therefore $\liminf_{s\to\infty} v > 0$.

Step 2. We claim that $\lim_{s\to\infty} \dot{v}(s) = 0$. Since $\liminf_{s\to\infty} v(s) > 0$ by Step 1, we find that $\liminf_{t\to\infty} u(t)/u_0(t) > 0$. Integrating equation (5), we further find that $\liminf_{t\to\infty} u'(t)/u_0'(t) > 0$. Since $v + \dot{v} = u'(t)/u_0'(t)$, we obtain $\liminf_{s\to\infty}(v + \dot{v}) > 0$. Recalling equation (6), we find that $\ddot{v}(s) = O(1)$ as $s \to \infty$. Since we have already known that $\dot{v} \in L^2[s_0, \infty)$, Lemma 3 shows that $\lim_{s\to\infty} \dot{v} = 0$.

Step 3. We claim that $\lim_{s\to\infty} v(s) = 1$. To see this, we integrate (6) multiplied by $\dot{v}$:

$$\frac{\dot{v}^2}{2} - a\int_{s_0}^{s} \dot{v}^2 dr - \frac{b}{2}v^2 + b\int_{s_0}^{s} (\dot{v} + v)^{1-\alpha}v^{\beta}\dot{v}dr$$

$$+b\int_{s_0}^{s} \delta(r)(\dot{v} + v)^{1-\alpha}v^{\beta}\dot{v}dr = \text{const}. \tag{11}$$

Suppose that condition (3); namely (7) holds. Since $\dot{v} \in L^2[s_0, \infty)$, the first, and the third integrals in the left hand side of (11) converge as $s \to \infty$. The mean value theorem shows that

$$(v + \dot{v})^{1-\alpha} = v^{1-\alpha} + (1 - \alpha)\,(v + \theta(r)\dot{v})^{-\alpha}\,\dot{v}, \tag{12}$$

where $\theta(r)$ is a quantity satisfying $0 < \theta(r) < 1$. Therefore,

$$\begin{aligned}\int_{s_0}^{s} (\dot{v} + v)^{1-\alpha} v^{\beta} \dot{v}\,dr &= \int_{s_0}^{s} \{v^{1-\alpha+\beta}\dot{v} + (1-\alpha)(v + \theta(r)\dot{v})^{-\alpha} v^{\beta}\dot{v}^2\}dr \\ &= \frac{v(s)^{2+\beta-\alpha}}{2+\beta-\alpha} + \int_{s_0}^{s} O(1)\dot{v}^2 dr + \text{const.}\end{aligned}$$

So we find that the function $-v^2/2 + v^{2+\beta-\alpha}/(2+\beta-\alpha)$ has a finite limit. This fact shows that $\ell = \lim_{s\to\infty} v(s) \in (0, \infty)$ exists. Next suppose that (4); namely (8) holds. We have by (12)

$$\begin{aligned}\int_{s_0}^{s} \delta(r)(\dot{v} + v)^{1-\alpha} v^{\beta} \dot{v}\,dr &= \int_{s_0}^{s} \{\delta(r) v^{1-\alpha+\beta}\dot{v} + \delta(r)(1-\alpha)(v + \theta(r)\dot{v})^{-\alpha} v^{\beta}\dot{v}^2\}dr \\ &= \frac{\delta(s)v^{2+\beta-\alpha}}{2+\beta-\alpha} - \frac{1}{2+\beta-\alpha}\int_{s_0}^{s} \dot{\delta}(r) v^{2+\beta-\alpha} dr + \text{const} + \int_{s_0}^{s} O(1)\dot{v}^2 dr\end{aligned}$$

as $s \to \infty$. Hence, as before we know that the function $-v^2/2 + v^{2+\beta-\alpha}/(2+\beta-\alpha)$ has a finite limit. Therefore $\ell = \lim_{s\to\infty} v(s) \in (0, \infty)$ exists.

Finally, we let $s \to \infty$ in equation (6). Then, we have $\lim_{s\to\infty} \ddot{v}(s) = b(\ell - \ell^{1+\beta-\alpha})$. Since $\dot{v} = o(1)$, we must have $\lim_{s\to\infty} \ddot{v}(s) = 0$, implying $\ell = 1$. The proof of Theorem 2 is completed.

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