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転移点と特異点を持つ２階線形常微分方程式について

On asymptotics of a second order linear O.D.E with
a turning-regular singular point

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§1. Introduction.

1.1. The differential equation studied is

$$\varepsilon^2 \frac{d^2y}{dx^2} - \left(x^m - \frac{\varepsilon}{x}\right) y = 0, \tag{1.1}$$

$$x, y \in \mathbb{C};\ \ 0 < |x| \le x_0,\ \ 0 < \varepsilon \le \varepsilon_0;\ \ m \in \mathbb{N},$$

where x_0 and ε_0 are constants. This differential equation has a turning point and a regular singular point, both of which are situated at the origin. We do not have a one-step-method to obtain an asymptotic approximation to the solution as $\varepsilon \to 0$ in the whole domain $\mathsf{D} = \{x : 0 < |x| \le x_0\}$, so we split (1.1) into two different types of the differential equation whose solutions are obtained separately (§2) and then we connect them by a so-called matching matrix in a common domain as shown in §4.

1.2. The differential equation (1.1) is represented in the matrix form:

$$\varepsilon \frac{dY}{dx} = \begin{bmatrix} 0 & 1 \\ x^m - \varepsilon/x & 0 \end{bmatrix} Y, \tag{1.2}$$

where Y is a 2-by-2 matrix. (1.2) has the first two terms of

$$\varepsilon \frac{dY}{dx} = \left\{ \begin{bmatrix} 0 & 1 \\ x^m & 0 \end{bmatrix} + \varepsilon \begin{bmatrix} 0 & 0 \\ -1/x & 0 \end{bmatrix} + O(\varepsilon^2) \right\} Y. \tag{1.3}$$

If $O(\varepsilon^2)$ is small for $x \in \mathsf{D}$ and ε, then a solution of (1.3) is a regular perturbation of one of (1.2) with respect to a small ε. In this sense (1.2) is dominant to (1.3)

Our aim is to get two types of the formal solution of (1.1) and match them as $\varepsilon \to 0$. In order to do it, analyzing Stokes curve configuration is important (§3). The case of $m = 1$

has been studied in Nakano [5].

Remark: We do not show any proofs or illustrations as they would take many pages.

§2. The reduced equations.

2.1. The differential equation (1.2) is written in the form

$$x^{(m+1)/2}(x^{-m-1}\varepsilon)\frac{dZ}{dx} = \left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + (x^{-m-1}\varepsilon)\begin{bmatrix} 0 & 0 \\ -1 & -mx^{m/2}/2 \end{bmatrix}\right) Z, \tag{2.1}$$

where $Y := \operatorname{diag}[1,\ x^{m/2}]\ Z$. This differential equation is called an *outer equation* of (1.2) and it should be analyzed when $x^{-m-1}\varepsilon \to 0$, that is, for x in a sub-domain $\mathsf{S} := \{x :\ K\varepsilon^{1/(m+1)} \le |x| \le x_0\}$ (K = large constant) of the whole domain D. A solution of (2.1) is called an *outer solution* of (1.2).

Theorem 2.1. *The formal outer solution* $\tilde{Y}_{out}$ *of* (1.2) *is given by*

$$\tilde{Y}_{out} := x^{\frac{m}{4}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} e^{\alpha\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}, \tag{2.2}$$

$$\alpha := \frac{2}{m+2}\frac{1}{\varepsilon}x^{(m+2)/2} + \frac{1}{m}\frac{1}{x^{m/2}},$$

or

$$\tilde{Y}_{out} := \begin{bmatrix} x^{-m/4} & 0 \\ 0 & x^{m/4} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{\alpha} & 0 \\ 0 & e^{-\alpha} \end{bmatrix}, \tag{2.3}$$

which is the leading term of an asymptotic expansion of a true outer solution of (1.2), *namely, there exists a true outer solution* Y_{out} *such that*

$$Y_{out} \sim \tilde{Y}_{out} \qquad (x^{-m-1}\varepsilon \to 0) \tag{2.4}$$

in an outer domain, i.e., in a sector

$$\mathsf{S}_m := \left\{x \ :\ K\varepsilon^{1/(m+1)} \le |x| \le x_0,\ -\frac{\pi}{m+2} < \arg x < \frac{3\pi}{m+2}\right\}. \tag{2.5}$$

Notice that the arguments of x in the above sector S_m correspond to the arguments of the boundaries of a canonical domain C_m^{∞} (cf. (2.10)). $\tilde{Y}_{out}$ is an outer *WKB approximation* to the solution of (1.2) of a matrix form.

2.2. We reduce (1.2) to another form in the complement $\mathsf{C} := \{x : 0 < |x| < K\varepsilon^{1/(m+1)}\}$ of the sub-domain S, i.e., $\mathsf{D} = \mathsf{C} \cup \mathsf{S}$. Let $x := \varepsilon^{1/(m+1)}t$ (a *stretching transform*) and $Y := \operatorname{diag}\left[1,\ \varepsilon^{m/2(m+1)}\right] U$, then (1.2) becomes a form such as

$$\varepsilon^{m/2(m+1)}\frac{dU}{dt} = \begin{bmatrix} 0 & 1 \\ p(t) & 0 \end{bmatrix} U \quad \left(p(t) := t^m - \frac{1}{t}\right), \tag{2.6}$$

which has a very similar form to (1.2) but lacks a term of ε and is called an *inner equation* of (1.2). The origin $t = 0$ is a regular singular point and zeros of $p(t)$ are turning points of (2.6), which are called *secondary turning points* of (1.2). A solution of (2.6) is called an *inner solution* of (1.2).

Theorem 2.2. *The formal inner solution $\tilde{Y}_{in}$ of* (1.2) *is given by*

$$\tilde{Y}_{in} := \begin{bmatrix} 1 & 0 \\ 0 & \varepsilon^{m/2(m+1)} \end{bmatrix} p^{1/4\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} e^{\beta\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}, \tag{2.7}$$

$$\beta := \frac{1}{\varepsilon^{m/2(m+1)}}\int^t \sqrt{p}\, dt,$$

or

$$\tilde{Y}_{in} := \begin{bmatrix} 1 & 0 \\ 0 & \varepsilon^{m/2(m+1)} \end{bmatrix} \begin{bmatrix} p^{-1/4} & 0 \\ 0 & p^{1/4} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{\beta} & 0 \\ 0 & e^{-\beta} \end{bmatrix}, \tag{2.8}$$

which is the leading term of the asymptotic expansion of a true inner solution of (1.2), *namely, there exists a true inner solution Y_{in} of* (1.2) *such that*

$$Y_{in} \sim \tilde{Y}_{in} \quad as \quad \begin{cases} \varepsilon \to 0 \\ t \to \infty \end{cases} \tag{2.9}$$

in a canonical domain

$$\mathsf{C}_m^{\infty} := \left\{t \ :\ 0 < |t| < \infty, -\frac{\pi}{m+2} < \arg t < \frac{3\pi}{m+2} \ \ near\ \ t = \infty\right\}. \tag{2.10}$$

$\tilde{Y}_{in}$ is an inner *WKB approximation* to the solution of (1.2) of a matrix form. The property (2.9) is called the *double asymptotic property* (Fedoryuk [2]).

§3. Stokes curves and the canonical domains.

3.1. A Stokes curve for (2.6) is, by definition, a set of points t's given by

$$\{t \ : \ \Re\xi(a,t) = 0\}, \tag{3.1}$$

where

$$\xi(a,t) := \int_a^t \sqrt{p}\, dt \quad (p(a) = 0). \tag{3.2}$$

An anti-Stokes curve of (2.6) is defined by an equation

$$\Im\xi(a,t) = 0 \quad (p(a) = 0). \tag{3.3}$$

These curves are particular level curves defined by $\Re\xi(a,t)$ = const. and $\Im\xi(a,t)$ = const., namely, they are the curves of level zero.

The global property of Stokes curve configuration for a general rational function $p(t)$ is well known in Evgrafov-Fedoryuk [1], Fedoryuk [2] and Nakano [6]-[7], and Fukuhara [3], Hukuhara [4] and Paris-Wood [8] for a local property of Stokes cueves. The outline of the Stokes curve configuration for (2.6) is as follows:

Theorem 3.1. *The Stokes and anti-Stokes curves for* (2.6) *possess the following properties*:

(*i*) *The origin $t = 0$ is a regular singular point from which one Stokes curve and one anti-Stokes curve emerge.*

When m =odd, two lines $t < -1$, $0 < t < 1$ on the real axis are Stokes curves, and two lines $-1 < t < 0$, $1 < t$ are anti-Stokes curves.

When m =even, a line $0 < t < 1$ on the real axis is a Stokes curve and two lines $t < 0$, $1 < t$ on the real axis are anti-Stokes curves.

(*ii*) *The point at infinity $t = \infty$ is an irregular singular point and $m+3$ Stokes curves emerge from* (*or tend to*) *$t = \infty$ at angles $\pm\dfrac{\pi}{m+2}, \pm\dfrac{3\pi}{m+2}, \pm\dfrac{5\pi}{m+2}, \cdots$.*

Also, $m+3$ anti-Stokes curves emerge from (*or tend to*) *$t = \infty$ at middle angles between neighboring two Stokes curves.*

(*iii*) *All the zero $t = e^{2k\pi i/(m+1)}$ $(k = 0, 1, 2, 3, \cdots)$ of $p(t)$ are situated on the unit circle $|t| = 1$ symmetrically with respect to the real axis and they are simple secondary* *turning points. From a turning point* $t = e^{2k\pi i/(m+1)}$ *three Stokes curves emerge at angles* $\pm\frac{\pi}{3} + \frac{4k\pi}{3(m+1)}, \pi + \frac{4k\pi}{3(m+1)}$.

Three anti-Stokes curves emerge from every zero at middle angles between neighboring two Stokes curves.

(*iv*) *There is a Stokes curve connecting* $\alpha := e^{2k\pi i/(m+1)}$ *and* $\alpha^* := e^{2\pi i - 2k\pi i/(m+1)}$. *This Stokes curve crosses the anti-Stokes curve* $-1 < t < 0$ *and can not cross lines* $t < -1$ *or* $0 < t < 1$.

(*v*) *There is an anti-Stokes curve connecting* $\alpha := e^{2k\pi i/(m+1)}$ *and* $\bar{\alpha} := e^{-2k\pi i/(m+1)}$. *This anti-Stokes curve crosses only the Stokes curve* $0 < t < 1$.

(*vi*) *Any Stokes curve* (*resp., any anti-Stokes curve*) *can not cross other Stokes curves* (*resp., anti-Stokes curves*) *except for at turning points or at* $t = \infty$.

(*vii*) *A Stokes curve and an anti-Stokes curve emerging from a turning point tend to another turning point or to* $t = \infty$.

(*viii*) *Any Stokes curve or any anti-Stokes curve can not cross itself.*

(*ix*) *When a point* $t = \alpha$ *is a turning point or a simple pole, there are no* (*sums of*) *Stokes or anti-Stokes curves homotopic to a circle surrounding* α. *Therefor there are no circle-like Stokes or anti-Stokes curves for* (6.1).

3.2. A canonical domain on the t-plane (or the Riemann surface) is, by definition, a simply connected domain bounded by Stokes curves which is mapped by $\xi = \xi(a, t)$ onto the whole ξ-plane except several slits. Refering Theorem 3.1 we can get several canonical domains whose illustration is omitted here.

§4. A matching matrix.

Existence domains S_m and C_m^∞ of the outer and the inner solutions have a common part where two solutions relate linearly. This linear relation is represented by a so-called *matching matrix*. The matrcing matrix $\mathsf{M}:=[\mathsf{m}_{ij}]$ between Y_{out} and Y_{in} is defined by the equality $Y_{out}\mathsf{M} = Y_{in}$, i.e.,

$$\tilde{Y}_{out}\mathsf{M} \sim \tilde{Y}_{in} \quad (\varepsilon \to 0). \tag{4.1}$$

Theorem 4.1. *The matching matrix defined by* (4.1) *is given by*

$$\mathsf{M} \sim \varepsilon^{m/4(m+1)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (\varepsilon \to 0). \tag{4.2}$$

§5. The main theorem.

Summing up the results so far, we can get

The main theorem. *The differential equation* (1.1) (*or* (1.2)) *posseses a formal outer solution* (*an outer WKB approximation*) (2.2) (*or* (2.3)) *which is an asymptotic expansion of the true outer solution in a sector* (*i.e., an outer domain*) (2.5) *as* $x^{-m-1}\varepsilon \to 0$. *The differential equation* (1.1) *possesses a formal inner solution* (*an inner WKB approximation*) (2.7) (*or* (2.8)) *which is an asymptotic expansion of the true inner solution in a canonical domain* (*i.e., an inner domain*) *as* $\varepsilon \to 0$ *or* $t \to \infty$. *The arguments of the outer domain's boundaries are* $-\pi/(m+2)$ *and* $3\pi/(m+2)$, *and those of the inner domain's boundaries are identical for a large* t, *and two domains have a common part in which the outer and the inner solutions are related by the matching matrix* (4.2).

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