数理解析研究所講究録
第 1547 巻 2007 年 59-67

A Representation of Solutions for Periodic Linear Differential Equations

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1 Introduction

The purpose of the present paper is to give a new representation of solutions for the periodic linear differential equation of the form

$$\frac{d}{dt}x(t) = A(t)x(t) + f(t), \quad x(0) = w \tag{1}$$

where $A(t)$ is a τ-periodic continuous $p \times p$ matrix function with period $\tau > 0$ and $f : \mathbb{R} \to \mathbb{C}^p$ a τ-periodic continuous function. In general, we know the variation of constants formula as a representation of solutions for the inhomogeneous linear differential equation. However it is not easy to obtain the asymptotic behavior of solutions by analyzing the integral term of the variation of constants formula. For the case where $A(t)$ is constant, we gave another, new representation of solutions as the sum of exponential like functions and periodic functions in [1]. This representation is powerful to investigate the asymptotic behavior of solutions.

In this paper we will study representations of solutions for the general periodic equation (1) in such a direction. It is closely related to a new representation of solutions of the linear difference equation of the form

$$x_{n+1} = U(\tau, 0)x_n + b_f, \quad x_0 = w, \tag{2}$$

where $U(t, s)$ is a solution operator for the equation (1) with $f(t) \equiv 0$ and

$$b_f = \int_0^\tau U(\tau, s)f(s)ds.$$

2 Linear difference equations

Throughout this paper we make use of the following notations: Let E be the unit $p\times p$ matrix. For a complex $p \times p$ matrix H we denote by $\sigma(H)$ the set of all eigenvalues of H, and by $h_H(\eta)$ the index of $\eta \in \sigma(H)$. Let $M_H(\eta) = \mathcal{N}((H-\eta E)^{h_H(\eta)})$ be the generalized eigenspace corresponding to $\eta \in \sigma(H)$ and $Q_\eta(H) : \mathbb{C}^p \to M_H(\eta)$ the projection corresponding to the direct sum decomposition $\mathbb{C}^p = \sum_{\eta\in\sigma(H)} \oplus M_H(\eta)$.

Consider the linear difference equation of the form

$$x_{n+1} = Bx_n + b, \quad x_0 = w, \tag{3}$$

where B is a complex $p \times p$ matrix and $b \in \mathbb{C}^p$. Denote by $x_n(w, b)$ the solution of the equation (3). Then the solution $x_n := x_n(w, b)$ is given as

$$x_n = B^n w + S_n(B)b,$$

where

$$S_n(B) = \sum_{k=0}^{n-1} B^k, \quad (n \geq 1), \quad S_0(B) = 0.$$

Put $h(\mu) = h_B(\mu)$, $Q_\mu = Q_\mu(B)$ for $\mu \in \sigma(B)$. Clearly, we have

$$Q_\mu x_n(w, b) = B^n Q_\mu w + S_n(B) Q_\mu b.$$

To describe the results, we prepare the following notations. The factorial numbers $(n)_k$ are given as

$$(n)_k = \begin{cases} 1, & (k = 0), \\ n(n-1)(n-2)\cdots(n-k+1), & (k = 1, 2, \cdots, n), \\ 0, & (k = n+1, n+2, \cdots). \end{cases}$$

Set $a(z) = (z-1)^{-1}$, $(z \neq 1)$. Then we have

$$a^{(k)}(z) := \frac{d^k}{dz^k} a(z) = (-1)^k k! (z-1)^{-k-1}.$$

For any $\mu \in \sigma(B)$ such that $\mu \neq 1$, we define a matrix $Z_\mu(B)$ as follows:

$$Z_\mu(B) = Z_\mu(B, h(\mu))$$

where

$$Z_\mu(B, h) = \sum_{k=0}^{h-1} \frac{a^{(k)}(\mu)}{k!}(B - \mu E)^k = -\sum_{k=0}^{h-1} \frac{1}{(1-\mu)^{k+1}}(B - \mu E)^k, \quad (\mu \neq 1)$$

for $h = 1, 2, \cdots, h(\mu)$. For a $\mu \in \sigma(B)$ and a $w \in \mathbb{C}^p$, two vectors $\gamma_\mu(w,b)$ and $\delta(w,b)$ are defined as follows :

$$\gamma_\mu(w,b) := \gamma_\mu(w,b;B) = Q_\mu w + Z_\mu(B)Q_\mu b \quad (\mu \neq 1)$$

and

$$\delta(w,b) := \delta(w,b;B) = (B-E)Q_1 w + Q_1 b \quad (\mu = 1).$$

Theorem 2.1 [3] *Let $\mu \in \sigma(B)$. The component $Q_\mu x_n(w,b)$ of the solution $x_n(w,b)$ of the equation* (3) *is expressed as follows:*

1) *If $\mu \neq 1$, then*

$$Q_\mu x_n(w,b) = B^n \gamma_\mu(w,b) - Z_\mu(B)Q_\mu b.$$

2) *If $\mu = 1$, then*

$$Q_1 x_n(w,b) = \sum_{k=0}^{h(1)-1} \frac{(n)_{k+1}}{(k+1)!}(B-E)^k \delta(w,b) + Q_1 w.$$

Lemma 2.1 *Let $\mu \in \sigma(B)$, $(\mu \neq 1)$. Then the following relation*

$$(B-E)Z_\mu(B)Q_\mu = Q_\mu \tag{4}$$

holds, that is, $Z_\mu(B)Q_\mu$ is a solution of the equation

$$(B-E)X = Q_\mu.$$

Proof For any $b \in \mathbb{C}^p$ the assertion 1) in Theorem 2.1 holds. Setting $w = 0, n = 1$ in Theorem 2.1, we have $Q_\mu x_1(0,b) = BZ_\mu(B)Q_\mu b - Z_\mu(B)Q_\mu b = Q_\mu b$ for all $b \in \mathbb{C}^p$. This implies the relation (4). □

Lemma 2.2 *If $\mu \neq 1, \mu \in \sigma(B)$, then the following relation*

$$(B-E)\gamma_\mu(w,b) = (B-E)Q_\mu w + Q_\mu b$$

holds. In particular, we have

$$\gamma_\mu(w,b) = 0 \Longleftrightarrow (B-E)\gamma_\mu(w,b) = 0.$$

Proof Using Lemma 2.1, we have

$$\begin{aligned}(B-E)\gamma_\mu(w,b) &= (B-E)Q_\mu w + (B-E)Z_\mu(B)Q_\mu b \\ &= (B-E)Q_\mu w + Q_\mu b.\end{aligned}$$

Now, we assume that $(B-E)\gamma_\mu(w,b) = 0$. Then we see that $\gamma_\mu(w,b) \in M_B(1)$. It follows from definition of $\gamma_\mu(w,b)$ that $\gamma_\mu(w,b) \in M_B(\mu)$. Hence $\gamma_\mu(w,b) \in M_B(1) \cap M_B(\mu)$. On the other hand, since $\mu \neq 1$, we get $M_B(1) \cap M_B(\mu) = \{0\}$. Therefore the relation $\gamma_\mu(w,b) = 0$ holds. □

The following result is one of the main result in this paper.

Theorem 2.2 *The solution $x_n(w,b)$ of the equation* (3) *is expressed as follows:*
1)

$$(B-E)x_n(w,b) = B^n((B-E)w+b)-b.$$

2) *Let $\mu \in \sigma(B)$.*

$$(B-E)Q_\mu x_n(w,b) = B^n((B-E)Q_\mu w + Q_\mu b) - Q_\mu b.$$

(1) *If $\mu \neq 1$, then*

$$(B-E)Q_1 x_n(w,b) = B^n(B-E)\gamma_\mu(w,b) - Q_\mu b.$$

(2) *If $\mu = 1$, then*

$$(B-E)Q_1 x_n(w,b) = B^n\delta(w,b) - Q_1 b.$$

Proof Since $x_n(w,b) = B^n w + S_n(B)b$ and $(B-E)S_n(B)b = B^n b - b$, we have

$$\begin{aligned}(B-E)x_n(w,b) &= B^n(B-E)w + (B-E)S_n(B)b \\ &= B^n(B-E)w + B^n b - b \\ &= B^n((B-E)w+b)-b,\end{aligned}$$

which implies the assertion 1). The assertion 2) can be easily obtained by using the assertion 1), Theorem 2.1 and Lemma 2.2. □

3 A representation of solutions of periodic linear differential equations

Denote by $x(t)$ the solution $x(t;0,w)$ of the equation (1). In this section, we give a representation of the solution $x(t)$ to the equation (1). The solution operator $U(t,s)$ is defined as $U(t,s)w = u(t;s,w), w \in \mathbb{C}^p$ by using the unique solution $u(t;s,w)$ of the equation $u'(t) = A(t)u(t)$ with the initial condition $u(s) = w \in \mathbb{C}^p$. Define the well known periodic map $V(t), t \in \mathbb{R}$ by $V(t) = U(t,t-\tau) = U(t+\tau,t)$. Then it is easy to check the following properties : $V(t+\tau) = V(t), V(t)U(t,s) = U(t,s)V(s)$.

Set

$$Q_\mu(t) = Q_\mu(V(t)) \ (\mu \in \sigma(V(0))).$$

We give a representation of the component $Q_\mu(t)x(t)$ of the solution $x(t)$ for the equation (1) by using the method of periodicizing functions, cf.[2]. It is expressed by characteristic multipliers. Hereafter, we set

$$b_f = \int_0^\tau U(\tau,s)f(s)ds$$

and

$$\gamma_\mu(w,b_f)=\gamma_\mu(w,b_f;V(0)),\quad \delta(w,b_f)=\delta(w,b_f;V(0)).$$

Now we consider the problem of finding a solution $z(t) := \Delta_\tau^{-1}(-U(t,0)b_f)$ of the following equation

$$\Delta_\tau z(t) := z(t+\tau)-z(t) = -U(t,0)b_f,\quad (t\in\mathbb{R}). \tag{5}$$

Theorem 3.1

1) *The solution $x(t)$ of the equation* (1) *is expressed as follows:*

$$x(t)=U(t,0)w-\Delta_\tau^{-1}(-U(t,0)b_f)+h(t,b_f),\quad (t\in\mathbb{R}),$$

where

$$h(t,b_f)=\Delta_\tau^{-1}(-U(t,0)b_f)+\int_0^t U(t,s)f(s)ds$$

is a continuous τ-periodic function.

2) *Let $\mu\in\sigma(V(0))$. The component $Q_\mu(t)x(t)$ of the solution $x(t)$ of the equation* (1) *is expressed as follows:*

$$Q_\mu(t)x(t)=U(t,0)Q_\mu(0)w-\Delta_\tau^{-1}(-U(t,0)Q_\mu(0)b_f)+h_\mu(t,b_f),\quad (t\in\mathbb{R}),$$

where

$$h_\mu(t,b_f)=\Delta_\tau^{-1}(-U(t,0)Q_\mu(0)b_f)+\int_0^t U(t,s)Q_\mu(s)f(s)ds$$

is a continuous τ-periodic function.

To get representations of solutions for the equation (1), we will calculate the functions $\Delta_\tau^{-1}(-U(t,0)b_f)$ and $\Delta_\tau^{-1}(-U(t,0)Q_\mu(0)b_f)$ in Theorem 3.1.

Theorem 3.2

1) *The following relation holds.*

$$(V(t)-E)\Delta_\tau^{-1}(-U(t,0)b_f)=-U(t,0)b_f+e(t),\quad (t\in\mathbb{R}) \tag{6}$$

where e(t) *is a τ-periodic function.*

2) *Let $\mu\in\sigma(V(0))$. Then*

$$(V(t)-E)\Delta_\tau^{-1}(-U(t,0)Q_\mu(0)b_f)=-U(t,0)Q_\mu(0)b_f+d(t),\quad (t\in\mathbb{R}) \tag{7}$$

where $d(t)$ is a τ-periodic function.

3) *Let $\mu\in\sigma(V(0))$ such that $\mu\neq 1$. Then*

$$\Delta_\tau^{-1}(-U(t,0)Q_\mu(0)b_f)=-U(t,0)Z_\mu(V(0))Q_\mu(0)b_f+c(t),\quad (t\in\mathbb{R}), \tag{8}$$

where $c(t)$ is a periodic constant.

Proof 1) Let $z(t), t \in \mathbb{R}$, be a continuous solution of the equation (5). Operating $V(t) - E$ to the both sides of the equation (5), we have

$$\begin{aligned}(V(t)-E))(z(t+\tau)-z(t)) &= -(V(t)-E)U(t,0)b_f \\ &= -U(t+\tau,0)b_f + U(t,0)b_f.\end{aligned}$$

Since $V(t+\tau) = V(t)$, the above relation becomes

$$(V(t+\tau)-E)z(t+\tau) + U(t+\tau,0)b_f = (V(t)-E)z(t) + U(t,0)b_f.$$

Thus $e(t) := (V(t)-E)z(t) + U(t,0)b_f$ is a τ-periodic function on $\mathbb{R}$. Therefore the following relation holds true:

$$(V(t)-E)z(t) = -U(t,0)b_f + e(t), \quad (t \in \mathbb{R}).$$

This proves the assertion 1).

2) The relation (7) is easily proved by operating $Q_\mu(t)$ to the both sides of (6).

3) Let $z(t), t \in \mathbb{R}$, be a continuous solution of the equation

$$z(t+\tau) - z(t) = -U(t,0)Q_\mu(0)b_f. \tag{9}$$

Then for any $t \in \mathbb{R}$ and $n \in \mathbb{N}_0$, we have

$$\begin{aligned}z(t+n\tau) &= z(t) - \sum_{k=0}^{n-1} U(t+k\tau,0)Q_\mu(0)b_f \\ &= z(t) - U(t,0)\sum_{k=0}^{n-1} U^k(\tau,0)Q_\mu(0)b_f \\ &= z(t) - U(t,0)Q_\mu(0)x_n(0),\end{aligned}$$

where $x_n(0)$ is the solution of the equation of the type $x_{n+1} = V(0)x_n + b_f,\ x_0 = 0$. It follows from Theorem 2.1 that $Q_\mu(0)x_1(0) = V(0)\gamma - \gamma (= Q_\mu(0)b_f)$, where $\gamma = Z_\mu(V(0))Q_\mu(0)b_f$. Hence we have

$$\begin{aligned}U(t,0)Q_\mu(0)x_1(0) &= U(t,0)(V(0)\gamma - \gamma) \\ &= U(t+\tau,0)\gamma - U(t,0)\gamma,\end{aligned}$$

from which we see that $z(t+\tau) = (z(t) + U(t,0)\gamma) - U(t+\tau,0)\gamma$, that is,

$$z(t+\tau) + U(t+\tau,0)\gamma = z(t) + U(t,0)\gamma.$$

Thus $c(t) := z(t) + U(t,0)\gamma$ is a τ-periodic function on $\mathbb{R}$. This implies that

$$z(t) = \Delta_\tau^{-1}(-U(t,0)Q_\mu(0)b_f) = -U(t,0)\gamma + c(t), \quad (t \in \mathbb{R}).$$

Therefore the proof of the theorem is completed. □

Using Theorem 3.2, we will crystallize Theorem 3.1.

Theorem 3.3 *For the solution $x(t)$ of the equation* (1) *the following representations hold true:*

1)

$$(V(t)-E)x(t) = U(t,0)((V(0)-E)w+b_f)+v(t,b_f), \quad (t \in \mathbb{R}), \tag{10}$$

where

$$\begin{aligned} v(t,b_f) &= (V(t)-E)h(t,b_f) \\ &= -U(t,0)b_f + \int_0^t U(t,s)(V(s)-E)f(s)ds \end{aligned} \tag{11}$$

is a continuous τ-periodic function.

2) *Let $\mu \in \sigma(V(0))$. Then*

$$\begin{aligned} (V(t)-E)Q_\mu(t)x(t) &= U(t,0)[(V(0)-E)Q_\mu(0)w+Q_\mu(0)b_f] \\ &\quad +v_\mu(t,b_f), \quad (t \in \mathbb{R}), \end{aligned}$$

where

$$\begin{aligned} v_\mu(t,b_f) &= (V(t)-E)h_\mu(t,b_f) \\ &= -U(t,0)Q_\mu(0)b_f + \int_0^t U(t,s)(V(s)-E)Q_\mu(s)f(s)ds \end{aligned}$$

is a continuous τ-periodic function.

Proof Since $h(t,b_f)$ given in Theorem 3.1 and $V(t)$ are τ-periodic, $v(t,b_f) := (V(t)-E)h(t,b_f)$ is also τ-periodic. Moreover, (10) and (11) are easily proved by combining Theorem 3.1 with Theorem 3.2. The remainder is obvious. □

We are now in a position to state the main theorem in this paper.

Theorem 3.4 *Let $\mu \in \sigma(V(0))$. For the component $Q_\mu(t)x(t)$ of the solution $x(t)$ of the equation* (1) *the following representations hold true:*

1) *Let $\mu \neq 1$. Then*

$$Q_\mu(t)x(t) = U(t,0)\gamma_\mu(w,b_f)+h_\mu(t,b_f), \quad (t \in \mathbb{R}) \tag{12}$$

where

$$h_\mu(t,b_f) = -U(t,0)Z_\mu(V(0))Q_\mu(0)b_f + \int_0^t U(t,s)Q_\mu(s)f(s)ds$$

is a continuous τ-periodic function.

2) *Let* $\mu = 1$. *Then*

$$(V(t) - E)Q_1(t)x(t) = U(t,0)\delta(w, b_f) + v_1(t, b_f), \quad (t \in \mathbb{R}), \tag{13}$$

where

$$v_1(t, b_f) = -U(t,0)Q_1(0)b_f + \int_0^t U(t,s)(V(s) - E)Q_1(s)f(s)ds$$

is a continuous τ*-periodic function.*

Proof The assertion 1) is easily proved by using Theorem 3.1 and 3) in Theorem 3.2. The assertion 2) is the case where $\mu = 1$ in 2) of Theorem 3.3. □

Corollary 3.1 *If* $\delta(w, b_f) = 0$ *in* 2) *of Theorem* 3.4, *then*

$$\Delta_\tau^{-1}(-U(t,0)Q_1(0)b_f) = U(t,0)Q_1(0)w + e(t), \tag{14}$$

where $e(t)$ *is a periodic constant, and* $Q_1(t)x(t) = h_1(t, b_f)$ *is a* τ*-periodic solution of the equation* (1).

Proof Since $\delta(w, b_f) = 0$, we have $(V(0) - E)Q_1(0)w = -Q_1(0)b_f$. Then for a continuous solution $z(t), t \in \mathbb{R}$, of the equation (9) we have

$$\begin{aligned} z(t+\tau) - z(t) &= U(t,0)(V(0) - E)Q_1(0)w \\ &= U(t+\tau,0)Q_1(0)w - U(t,0)Q_1(0)w, \end{aligned}$$

from which it follows that $e(t) := z(t) - U(t,0)Q_1(0)w$ is a τ-periodic function. Therefore we obtain the relation (14). In view of Theorem 3.1 we have $Q_1(t)x(t) = h_1(t, b_f)$. □

Finally, we consider the case where $1 \notin \sigma(V(0))$. Then we have the following result.

Theorem 3.5 *Let* $1 \notin \sigma(V(0))$. *Then the following results hold.*

1)

$$\Delta_\tau^{-1}(-U(t,0)b_f) = -U(t,0)(V(0) - E)^{-1}b_f + p(t), \quad (t \in \mathbb{R})$$

where $p(t)$ *is a periodic constant.*

2) *For the solution* $x(t)$ *of the equation* (1) *the following representation holds true:*

$$x(t) = U(t,0)(w + (V(0) - E)^{-1}b_f) + \hat{h}(t, b_f), \quad (t \in \mathbb{R})$$

where

$$\hat{h}(t, b_f) = -U(t,0)(E - V(0))^{-1}b_f + \int_0^t U(t,s)f(s)ds$$

is a τ*-periodic solution of the equation* (1).

Proof Let $z(t), t \in \mathbb{R}$, be a continuous solution of the equation (5). Since $1 \notin \sigma(V(0))$, we have

$$\begin{aligned} z(t+\tau) &= z(t) - U(t,0)b_f \\ &= z(t) - U(t,0)(E - V(0))(E - V(0))^{-1}b_f. \end{aligned}$$

Thus we get

$$z(t+\tau) - U(t+\tau,0)(E - V(0))^{-1}b_f = z(t) - U(t,0)(E - V(0))^{-1}b_f.$$

This means that $z(t) = U(t,0)(E-V(0))^{-1}b_f + p(t), (t \in \mathbb{R})$, where $p(t)$ is a periodic constant. The remainder is obvious. □

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