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A NOTE ON GT-ADMISSIBLE VARIETIES

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1. Contents

In this paper, we introduce a notion of GT-admissible varieties. Roughly speaking, it is a (semi-)simplicial oject in the category of varieties $\mathcal{M}_{0,n}, \Delta^*, \dots$ and their products with tangential points. We introduce a cohomology theory of GT-admissible varieties. To obtain patching varieties, we construct higher homotopy for complexes in the sence of Hanamura. It seems very possible to reconstruct this cohomology theory using a formulation of A_∞-category. All constructions in this paper can be done in the setting of more general Tannakian category.

2. Category (*Basic*)

2.1. Tangential points. In this subsection, we define a category (*Basic*) to define the set of associator and Grothendieck-Teichmüller group. The object of (*Basic*) consists of three object Δ^*, $\mathcal{M}_{0,4}$ and $\mathcal{M}_{0,5}$. Formally speaking, Δ^*, $\mathcal{M}_{0,4}$ and $\mathcal{M}_{0,5}$ are finite set, called tangential points. For example, the most easiest one Δ^* consists of two tangential points $\{+,-\}$. The tangential points of $\mathcal{M}_{0,4}$ consists of $\{\overrightarrow{01}, \overrightarrow{0\infty}, \cdots\}$ (all together 6 points). The tangential points of $\mathcal{M}_{0,5}$ consists of plane trivalent tree with five end points up to mirror. (Plane tree is equipped with a cyclic ordering for each vertex.) Therefore the number of tangential points is 60. Tangential points are expressed geometrically as follows.

The symbol $\mathcal{M}_{0,5}$ is usually used as a moduli space of genus zero curves with marked 5 points. Mumford and Knudsen defined the stable compactification $\overline{\mathcal{M}_{0,5}}$ of the moduli space $\mathcal{M}_{0,5}$ by adding a normal crossing divisor (=1dimensional) in $\overline{\mathcal{M}_{0,.5}}$. There exist 15 crossing points in the boundary divisor. Let p be a crossing point and U_p be a small neighbourhood of p. The intersection $U_p \cap \mathcal{M}_{0,5}(\mathbf{R})$ consists of 4 connected components. Each component corresponds one to one to tangential point of $\mathcal{M}_{0,5}$. In the following context, $\mathcal{M}_{0,5}$ represents the set of these 60 tangental points and forget the geometric meaning of $\mathcal{M}_{0,5}$.

2.2. Two theories of fundamental groups. There exist two theories of fundamental groups, topological fundamental group and de Rham fundamental group. For $X \in$ (*Basic*) and $p \in X$, the topological fundamental group $\pi_1(X,p)$ can be defined purely conbinatorially. For example, the group $\pi_1(\Delta^*,+)$ is generated by a positive generator θ. More generally, for $p, q \in X$, we can define the set of path $\pi_1(X,p,q)$ connecting p and q up to homotopy. We can define the $\mathbf{Q}$-linear span $\mathbf{Q}[\pi_1(X,p,q)]$ of $\pi_1(X,p,q)$, which is a left $\mathbf{Q}[\pi_1(X,q)]$ (resp. right $\mathbf{Q}[\pi_1(X,p)]$) module of rank one and the completion of $\mathbf{Q}[\pi_1(X,p,q)]$ with respect to the augmentation ideal of $\mathbf{Q}[\pi_1(X,q)]$ is denoted as $\mathcal{U}^B(X,p,q)$. There exist "multiplication"

$$\mathcal{U}^B(X,q,r)\otimes\mathcal{U}^B(X,p,q)\to\mathcal{U}^B(X,p,r)$$

which satisfies the axiom of algebroid, i.e. associative multiplication $\gamma\eta$ is defined only if the starting point of γ is equal to the ending point of η.

There is a theory of de Rham fundamental group $\pi_1^{DR}(X,p)$ of X with a base point p. For $p,q\in X$, there is a canonical isomorphism

$$\pi_1^{DR}(X,p)\simeq\pi_1^{DR}(X,q)$$

in de Rham fundamental group theory, which is a different point from the Betti theory. The group $\pi_1^{DR}(X,p)$ is defined as the set of group like elements of a Hopf algebra $\mathcal{U}^{DR}(X,p)$. There exists a canonical isomorphism

$$\mathcal{U}^{DR}(X,p)\simeq\mathcal{U}^{DR}(X,q)$$

as Hopf algebras. For example, we have $\mathcal{U}(\Delta^*,+)=\mathbf{Q}[[e]]$, where $e=Res_0$.

2.3. Functoriality. We introduce morphisms in $(Basic)$. Morphisms cosist of

1. An inclusion $:\Delta^*\to\mathcal{M}_{0,4}$. The tangential points $+$ and $-$ goes to $\vec{ab}$ and $\vec{ac}$, where $\{a,b,c\}=\{0,1,\infty\}$. (Therefore altogether, 6 morphism of this type.)
2. An infinitesimal inclusion:$\mathcal{M}_{0,4}\to\mathcal{M}_{0,5}$. We will not write down precisely here. There are 12 connected components of $\mathcal{M}_{0,5}(\mathbf{R})$ and each connected component is bounded by 5 divisors, i,e. it is a pentagon. There exists a unique infinitesimal inclusion by which a connected component of $\mathcal{M}_{0,4}(\mathbf{R})$ goes to an edge of pentagon.
3. A composite of type (1) and type (2).

The category of pointed objects and two pointed objects in $(Basic)$ are written as $(*Basic)$ and $(**Basic)$, respectively.

Let $\#=B,DR$. The correspondence $(X,p)\to\mathcal{U}^\#(X,p)$ and $(X,p,q)\to\mathcal{U}^\#(X,p,q)$ form functors

$$\mathcal{U}^\#:(*Basic)\to(Vec_{\mathbf{Q}})\quad\text{and}\quad\mathcal{U}^\#:(**Basic)\to(Vec_{\mathbf{Q}}).$$

Moreover these functors give rise to a functor

$$\mathcal{U}^\#:(Basic)\to(Hopf_{\mathbf{Q}})$$

form the category $(Basic)$ to the category of Hopf algebroids over $\mathbf{Q}$.

Theorem 2.1 (Drinfeld). *There exists a functorial isomorphism*

$$\rho:\mathcal{U}^B\otimes\mathbf{C}\simeq\mathcal{U}^{DR}\otimes\mathbf{C}$$

where

(2.1) $\rho(\log\theta)=2\pi ie$ *for* $\rho:\mathcal{U}^B(\Delta,+)\otimes\mathbf{C}\simeq\mathcal{U}^{DR}(\Delta,+)\otimes\mathbf{C}$.

Hodge theory gives a functorial isomorphism. Less trivial part is the compatiblitity for infinitesimal inclusions.

2.4. Associator, Grothendieck-Teichmüller group.

Definition 2.2. 1. *The set of functorial isomorphisms of* $\mathbf{C}$*-Hopf algebroids*

$$Ass = Isom_{Hopf_{\mathbf{C}}}(\mathcal{U}^B, \mathcal{U}^{DR})$$

is called the set of associators. We define $Ass_{2\pi i}$ *as*

$$Ass_{2\pi i} = \{\rho \in Ass \mid \rho \text{ satisfies the condition } (2.1)\}$$

2. *Let* $\# = B, DR$. *The set of functorial isomorphisms of* $\mathbf{Q}$*-Hopf algebroids*

$$GT^{\#} = Isom_{Hopf_{\mathbf{Q}}}(\mathcal{U}^{\#}, \mathcal{U}^{\#})$$

is called the $\#$*-Grothendieck-Teichmüller group. We define* $GT_1^{\#}$ *as*

$$GT_1^{\#} = \{g \in GT^{\#} \mid g(\log\theta) = \log\theta \ (\text{if } \# = B)\ g(e) = e \ (\text{if } \# = DR) \\ \text{for } g : \mathcal{U}^{\#}(\Delta^*, +) \to \mathcal{U}^{\#}(\Delta^*, +)\}$$

We have an exact sequence

$$1 \to GT_1^{\#} \to GT^{\#} \to \mathbf{G}_m \to 1.$$

The set Ass and $Ass_{2\pi i}$ is a left (resp. right) principal homogoeneous space under the group $GT^{DR}(\mathbf{C})$ and $GT_1^{DR}(\mathbf{C})$, ($GT^B(\mathbf{C})$ and $GT_1^B(\mathbf{C})$), respectively. The group $GT_1^{\#}$ is a nilpotent Lie group and its Lie algebra is denoted as $\mathcal{GT}_1^{\#}$. By the definition of GT^B, $\mathcal{U}^B(\mathcal{M}_{0,4})$ and $\mathcal{U}^B(\mathcal{M}_{0,5})$ are representations of GT^B.

2.5. Category $(Fund)$**.** We define a category $(Fund)$. The object of the category $(Fund)$ consists of

1. $\mathcal{M}_{0,i}$ for $i \geq 4$,
2. Δ^n − (big diagonal), $(\Delta^*)^n$ − (big diagonal), and
3. their products.

The morphisms consist of inclusions, infinitesimal inclusions and certain projections. We define categories $(*Fund)$ and $(**Fund)$ by pointed and two pointed objects of $(Fund)$. As in $(Basic)$ case, we can define two functors $\mathcal{U}^B$ and $\mathcal{U}^{DR}$. (For an object $(X,p) \in (*Func)$, $\mathcal{U}^B(X,p)$ is the completion of the group algebra of the fundamental group with respect to the augmentation ideal.) The following theorem is due to Ihara and Lochak.

Theorem 2.3. *Let* $\# = B$ *or* DR. *The action of* $GT^{\#}$ *on the fuctor* $\mathcal{U}^{\#}$: $(**Basic) \to (Hopf_{\mathbf{Q}})$ *extends uniquely to the action on the functor* $\mathcal{U}^{\#}$: $(**Fund) \to (Hopf_{\mathbf{Q}})$.

This principle is known as MacLane coherence principle.

3. Differential graded algebra

3.1. From Hopf algebroid to differential graded algebra. Let $\mathcal{U}$ be a $\mathbf{Q}$-Hopf algebroid with an augmentation $\epsilon : \mathcal{U} \to \mathbf{Q}$ over a set X, which is complete for the topology defined by the augmentation ideal $I = Ker(\epsilon)$. In this paper, the coproduct Δ of the Hopf algebroid is always cocommutative and coassociative. Therefore the groupoid like element in $\mathcal{U}$ forms a pro-unipotent groupoid. We introduce linear topology on $\mathcal{U}$ and assume that $\mathcal{U}$ is compact for the topology. In particular, $\mathcal{U}/I^n$ is a finite dimensional Hopf algebroid. Functors Hom, $\otimes$ are always considered in the category of locally compact vector spaces.

Definition 3.1. 1. *Let $p, q \in X$. We define a complex $K^\bullet(\mathcal{U})_{p,q}$ by*

$$\begin{array}{ccccccc}
\cdots & \to & \mathcal{U}_{p,q} \otimes \mathcal{U}_{p,q} \otimes \mathcal{U}_{p,q} & \to & \mathcal{U}_{p,q} \otimes \mathcal{U}_{p,q} & \to & \mathcal{U}_{p,q} \\
 & & a \otimes b \otimes c & \mapsto & \begin{pmatrix} \epsilon(a)b \otimes c - \epsilon(b)a \otimes c \\ +\epsilon(c)a \otimes b \end{pmatrix} & & \\
 & & & & a \otimes b & \mapsto & \epsilon(a)b - \epsilon(b)a
\end{array}$$

Degree of the complex is given by $K^{-i}(\mathcal{U}) = \mathcal{U}^{\otimes i+1}$.

2. *We define the n-coproduct $\Delta^{(n)}$ by the composite*

$$(1^{\otimes^{n-2}} \otimes \Delta) \circ \cdots (1 \otimes \Delta) \circ \Delta : \mathcal{U}_{p,q} \to \mathcal{U}_{p,q} \otimes \cdots \otimes \mathcal{U}_{p,q}.$$

Via this coproduct, $\mathcal{U} \otimes \cdots \otimes \mathcal{U}$ is a two sided $\mathcal{U}$ module. Via this $\mathcal{U}$ structures, the differentials in the complex $K^\bullet(\mathcal{U})$ are homomorphisms of two sided $\mathcal{U}$ modules.

Proposition 3.2. *The cohomologies of the complex is*

$$H^{-i}(K^\bullet(\mathcal{U})_{p,q}) = \begin{cases} \mathbf{Q} & \text{if } i = 0 \\ 0 & \text{if } i \neq 0 \end{cases}$$

Proof. Let $p \in X$ and G_p be the group like element in $\mathcal{U}_p$. We define a simplicial complex Y as follows: the set of n-simplices is $\{(g_0, \dots, g_n) \mid g_i \in G\}$. Since G_p is a pro-$\mathbf{Q}$-unipotent group, the completion of the chain complex of Y is canonically isomorphic to $K^\bullet(\mathcal{U})_p$ and the completion of the cohomologies is isomorphic to the cohomology of the completion for the chain complex of Y. □

Definition 3.3 (Differential graded algebra). 1. *Let $p, q \in X$. We define*

$$\Omega^i(\mathcal{U})_p = Hom_{(left)\mathcal{U}_q}(K^{-i}(\mathcal{U}_{p,q}), \mathbf{Q}).$$

Note that this complex does not depend on the choice of q. Using the transpose of the differential of $K^\bullet$, we have a complex

$$\Omega^\bullet(\mathcal{U})_p : 0 \to \Omega^0(\mathcal{U})_p \to \Omega^1(\mathcal{U})_p \to \Omega^2(\mathcal{U})_p \to \cdots$$

There exists a left action of $\mathcal{U}_p$ on $\Omega^\bullet$ arising from the right action of $\mathcal{U}_p$ on $K^\bullet(\mathcal{U})_{p,q}$.

2. *We introduce a product structure on* $\Omega^\bullet(\mathcal{U})_p$. *Let* ω *and* η *be elements of* $\Omega^i(\mathcal{U})_p$ *and* $\Omega^j(\mathcal{U})_p$, *respectively. We define* $\omega \cdot \eta \in \Omega^{i+j}(\mathcal{U})_p$ *by*
$$(\omega \cdot \eta)(a_0 \otimes \cdots \otimes a_{i+j}) = (\omega \otimes \eta)(a_0 \otimes \cdots \otimes \Delta(a_i) \otimes \cdots \otimes a_{i+j}).$$
Then this multiplication is associative and we have $d(a \cdot b) = da \cdot b + (-1)^{deg(a)} a \cdot db$.

Proposition 3.4. *The cohomology* $H^i(\Omega^\bullet(\mathcal{U})_p)$ *of* $\Omega^\bullet(\mathcal{U})_p$ *is equal to* $H^i(G, \mathbf{Q})$. *The induced right action of* $\mathcal{U}_p$ *is trivial. i.e. the action of* $\mathcal{U}_p$ *factors through* $\epsilon : \mathcal{U}_p \to \mathbf{Q}$.

Proof. Let Y be the simplical complex defined as before. Since the group G is pro-$\mathbf{Q}$-nipotent, the action of G on Y is fixed point free. Therefore the quotient space $G\backslash Y$ is $K(G,1)$ space. The cochain complex of $G\backslash Y$ is equal to $\Omega^\bullet(\mathcal{U})_p$. The triviallity of the action of $\mathcal{U}_p$ will be explained later. □

3.2. **Differential graded algebroid structure on** $K^\bullet$. Let $\mathcal{U}$ be a Hopf algebroid over a set X, p,q two points in X and $K^\bullet(\mathcal{U})_{p,q}$ be the complex defined in 3.1. We introduce an associative product structure
$$K^\bullet(\mathcal{U})_{q,r} \otimes K^\bullet(\mathcal{U})_{p,q} \to K^\bullet(\mathcal{U})_{p,r},$$
for $p, q, r \in X$.

Definition 3.5. *Let* a, b *be elements in* $\mathbf{N}$ *and set* $k = a + b$. *Minimal path connecting* $(0,0)$ *and* (a,b) *is a sequence of* $(a_i, b_i) \in \mathbf{N}^2$ *(i=0, ... ,k) such that*

1. *either* $a_{i+1} = a_i + 1, b_{i+1} = b_i$, *or* $a_{i+1} = a_i, b_{i+1} = b_i + 1$, *and*
2. $a_0 = 0, b_0 = 0$ *and* $a_k = a, b_k = b$.

for $0 \le i \le k-1$. *The set of minimal path connecting* $(0,0)$ *and* (a,b) *is denoted by* $MP_{a,b}$. *For a minimal path* $p = (a_i, b_i)_i \in MP_{ab}$, *the number* $\sum_{i=0}^{a} a_i - (a(a+1)/2)$ *is called the volume of the path and denoted by vol(p). We define the signature sign(p) of a path p by* $(-1)^{vol(p)}$.

Let p, q, r be elements in X. We define a dot product
$$* \cdot * : K^a(\mathcal{U})_{q,r} \otimes K^b(\mathcal{U})_{p,q} \to K^{a+b}(\mathcal{U})_{p,r}.$$
Let $g_0, \dots, g_a, h_0, \dots, h_b$ be group(oid) like elements in $\mathcal{U}$. Using these elements the product is defined by
$$\begin{aligned}&(g_0 \otimes \cdots \otimes g_a) \cdot (h_0 \otimes \cdots \otimes h_b) \\ = & \sum_{p=(a_i,b_i) \in MP_{a,b}} sign(p)(g_{a_0} h_{b_0} \otimes g_{a_1} h_{b_1} \otimes \cdots \otimes g_{a_k} h_{b_k})\end{aligned}$$
We can describe the above product using Δ and the multiplication of $\mathcal{U}$. Then we have
$$d(v \cdot w) = dv \cdot w + (-1)^a v \cdot dw$$
for $v \in K^a(\mathcal{U})_{q,r}, w \in K^b(\mathcal{U})_{p,q}$. The product is associative, i.e. for $x \in K^\bullet(\mathcal{U})_{r,s}, y \in K^\bullet(\mathcal{U})_{q,r}, z \in K^\bullet(\mathcal{U})_{p,q}$, we have $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.

We define an action of $K^\bullet(\mathcal{U})_{p,q}$ from $\Omega^\bullet(\mathcal{U})_p$ to $\Omega^\bullet(\mathcal{U})_q$, using the dot product structure on $K^\bullet(\mathcal{U})$ as follows.

$$\begin{array}{cccc} *\cdot*: & K^\bullet(\mathcal{U})_{p,q}\otimes\Omega^\bullet(\mathcal{U})_p & \to & \Omega^\bullet(\mathcal{U})_q \\ & a\otimes\varphi & \mapsto & a\cdot\varphi=(b\to\varphi(b\cdot a)) \end{array}$$

This homomorphism is a homomorphism of complex. The the associativity for "$\cdot$" implies the associativity of the action of $K^\bullet$ on $\Omega^\bullet$.

4. Patching by Cech complex of complexes

4.1. Cubic flag and n-homotopy. In this subsection, we define an n-homotopy $h(S)$ for a finite totally ordered set S, which will be used in the next subsection to defined n-homotopies for patching complexes. Let S be a totally ordered set with finite n elements. Cube $\square^S$ over S is defined by $\{0,1\}^S$. If $S=\{1,\dots,n\}$, the cube over S is denoted as $\square^n$. A minimal path from $(0,\dots,0)\in\square^S$ to $(1,\dots,1)\in\square^S$ (minimal path of S for simiplicity) is defined by a permutation $\sigma=(\sigma_1,\dots,\sigma_n)$ of S. A minimal path can be regarded as a sequence $v_0,\dots,v_n$ of $\square^S$ defined by

1. $v_0=(0,\cdots,0)$,
2. $v_i=v_{i-1}+e_{\sigma_i}$, where e_σ is the elementary unit vector for the σ component.

Let $\Sigma(S)$ be the set of minimal path of S expressed as permutations of S and $W(S)$ be the vector space spanned by $\Sigma(S)$ over $\mathbf{Q}$.

Definition 4.1 (Dot product). *Let T_1 be a subset in S and $T_2=S-T_1$. For elemenets g and h in $\Sigma(T_1)$ and $\Sigma(T_2)$, $(gh)\in\Sigma(S)$ denotes the composite of words g and h. We define a product*

$$\begin{array}{ccc} W(T_1)^{\otimes(p+1)}\otimes W(T_2)^{\otimes(q+1)} & \to & W(S)^{\otimes(p+q+1)} \\ (g_0,\dots,g_p)\otimes(h_0,\dots,h_q) & \mapsto & g\cdot h \end{array}$$

by

$$g\cdot h=\sum_{p=(a_i,b_i)\in MP(p,q)} sign(p)(g_{a_0}h_{b_0})\otimes\cdots\otimes(g_{a_{p+q}}h_{b_{p+q}}),$$

where g_i and h_j are elements in $\Sigma(T_1)$ and $\Sigma(T_2)$.

We define $d: W(S)^{\otimes n}\to W(S)^{\otimes(n-1)}$ by

$$\begin{array}{ccc} W(S)^{\otimes n} & \to & W(S)^{\otimes(n-1)} \\ (v_1\otimes\cdots\otimes v_n) & \mapsto & \sum_{k=1}^n(-1)^{k-1}v_1\otimes\cdots\otimes v_{k-1}\otimes v_{k+1}\otimes\cdots\otimes v_n, \end{array}$$

for $v_1,\dots,v_n\in\Sigma(S)$.

Definition 4.2 (n-Homotopy for an ordered set S). *Let S be a finite totally ordered set with $\#S=n\geq 1$. We set $tot(S)\in\Sigma(S)$ by $tot(S)=(g_1,\dots,g_n)$, where $S=\{g_1,\dots,g_n\}$ and $g_1<\cdots<g_n$. We define n-homotopy $h(S)$ of S by the induction on $\#S$.*

1. *If $S=\{a\}$, then we define*

$$h(S)=(a).$$

2. *If* $S = \{a, b\}$ *and* $a < b$, *then we define*
$$h(S) = (ab) \otimes (ba).$$
3. *If* $\#S \geq 3$, *then we define*
$$h(S) = tot(S) \otimes \Big(\sum_{\emptyset \neq T \subsetneq S} (-1)^{\#T} sign(S,T) h(T) \cdot h(S-T) \Big).$$

where $sign(S,T)$ *is the signature of the permutation* $\begin{pmatrix} tot(S) \\ tot(T), tot(S-T) \end{pmatrix}$.

Proposition 4.3. *We have*
$$d\Big(\sum_{\emptyset \neq T \subsetneq S} (-1)^{\#T} sign(S,T) h(T) \cdot h(S-T) \Big) = 0$$
and
$$d(h(S)) = \sum_{\emptyset \neq T \subsetneq S} (-1)^{\#T} sign(S,T) h(T) \cdot h(S-T).$$

4.2. **First step for patching differential graded algebras.** Let $\mathcal{K}$ be a finite simplicial complex with a total order on the set of vertices. Then $\mathcal{K}$ is a category whose objects and morphisms are given by simplices and their inclusions. Let X be a contravariant functor from the complex $\mathcal{K}$ to the category of Hopf algebroid, i.e. for a simplex σ in $\mathcal{K}$, X_σ is a finite set and $\mathcal{U}(\sigma)$ is a Hopf algebroid over the set X_σ. For a face $\tau < \sigma$ of σ, we have a map of finite set $X_\sigma \to X_\tau$ and a homomorphism of Hopf algebroid $\mathcal{U}(\sigma) \to \mathcal{U}(\tau)$ compatible with the composite for inclusions $\sigma_1 < \sigma_2 < \sigma_3$ of simplices.

Then we have a differential graded algebroid $K^\bullet(\mathcal{U}_\sigma)$ over X_σ and a family of differential graded algebra $\Omega^\bullet(\mathcal{U}_\sigma)_x$ for $x \in X_\sigma$. The action of $K^\bullet(\mathcal{U}_\sigma)$ on $\Omega^\bullet(\mathcal{U}_\sigma)$ is introduced in the last section. A set of base points $B = (b_\sigma)_\sigma$ with $b_\sigma \in X_\sigma$ is called a system of base points of X. In this subsection, we define a complex $\Omega^\bullet(X) = \Omega^\bullet(X)_B$ for a system of base points B.

Let σ be a simplex of $\mathcal{K}$. We define a full subcategory $\mathcal{K}_\sigma$ of $\mathcal{K}$ defined by
$$ob(\mathcal{K}_\sigma) = \{\tau \mid \sigma < \tau\}.$$
All the morphisms in $\mathcal{K}_\sigma$ of codimension one is denoted as $Mor(\sigma)$, i.e.
$$Mor(\sigma) = \{(\alpha < \beta) \mid \sigma < \alpha < \beta, \dim \alpha + 1 = \dim \beta\}.$$
We introduce a total order in $Mor(\sigma)$. For example lexico graphic order for (α, β). For a morphism $(\alpha < \beta) \in Mor(\sigma)$, the complex $K^\bullet(\mathcal{U}(\alpha))_{b_\alpha, b_\beta | X_\alpha}$ is denoted by $K^\bullet_{\alpha,\beta}$ for short. Then we have a natural homomorphism $\mu_{\alpha,\beta}$ of complexes:
$$\mu_{\alpha,\beta} : K^\bullet_{\alpha,\beta} \otimes \Omega^\bullet(\mathcal{U}(\alpha))_{b_\alpha} \to \Omega^\bullet(\mathcal{U}(\alpha))_{b_\beta|\alpha} \to \Omega^\bullet(\mathcal{U}(\beta))_{b_\beta}.$$

Definition 4.4. 1. *For a simplex* σ *in* $\mathcal{K}$, *we define a complex* $\widetilde{\Omega}^\bullet(\sigma)$ *by*
$$\widetilde{\Omega}^\bullet(\sigma) = \Big(\bigotimes_{(\alpha<\beta) \in Mor(\sigma)} K^\bullet_{\alpha,\beta} \Big) \otimes \Omega^\bullet(\mathcal{U}(\sigma))_{b_\sigma}.$$

The differential of $\widetilde{\Omega}^\bullet(\sigma)$ is given as the differential on the tensor product. As usual, using an isomorphism

$$\begin{array}{ccc} K^\bullet \otimes L^\bullet & \simeq & L^\bullet \otimes K^\bullet \\ a \otimes b & \mapsto & (-1)^{\deg a \cdot \deg b} b \otimes a, \end{array}$$

the differential is independent of the order of tensor product.

2. *Let $\sigma < \tau$ be a homomorphism in $\mathcal{K}$ of codimension one. We define $\delta_{\sigma,\tau} : \widetilde{\Omega}^\bullet(\sigma) \to \widetilde{\Omega}^\bullet(\tau)$:*

$$\begin{array}{c} \bigotimes_{(\alpha<\beta)\in Mor(\sigma)} K^\bullet_{\alpha,\beta} \otimes \Omega^\bullet(\mathcal{U}(\sigma))_{b_\sigma} \\ \| \\ \bigotimes_{(\alpha<\beta)\in Mor(\tau)} K^\bullet_{\alpha,\beta} \otimes \bigotimes_{\substack{(\alpha<\beta)\in Mor(\sigma)-Mor(\tau) \\ (\alpha<\beta)\neq(\sigma<\tau)}} K^\bullet_{\alpha,\beta} \otimes K^\bullet_{\sigma,\tau} \otimes \Omega^\bullet(\mathcal{U}(\sigma))_{b_\sigma} \\ \downarrow \\ \bigotimes_{(\alpha<\beta)\in Mor(\tau)} K^\bullet_{\alpha,\beta} \otimes \Omega^\bullet(\mathcal{U}(\tau))_{b_\tau} \end{array}$$

by

$$\begin{array}{c} (\bigotimes_{\alpha,\beta} a_{\alpha,\beta} \otimes \bigotimes_{\alpha,\beta} b_{\alpha,\beta} \otimes c_{\sigma,\tau} \otimes \omega_{b_\sigma}) \\ \downarrow \\ \bigotimes_{\alpha,\beta} a_{\alpha,\beta} \cdot \prod_{\alpha,\beta} \epsilon(b_{\alpha,\beta}) \cdot (c_{\sigma,\tau} \cdot \omega_{b_\sigma}). \end{array}$$

The following proposition is a direct consequence from the definition of $\delta_{\sigma,\tau}$.

Proposition 4.5. *Let $\sigma < \gamma_1 < \tau$ and $\sigma < \gamma_2 < \tau$ be distinct sequence of codimensition one simplices in $\mathcal{K}$. We define ∂ as*

$$\begin{array}{cccc} \partial : K^\bullet_{\gamma_1,\tau} \otimes K^\bullet_{\sigma,\gamma_1} \otimes K^\bullet_{\gamma_2,\tau} \otimes K^\bullet_{\sigma,\gamma_2} & \to & K^\bullet_{\sigma,\tau} \\ x \otimes y \otimes z \otimes w & \mapsto & \epsilon(z)\epsilon(w) x \cdot y - \epsilon(x)\epsilon(y) z \cdot w \end{array}$$

Then $\delta_{\gamma_1,\tau}\delta_{\sigma,\gamma_1} - \delta_{\gamma_2,\tau}\delta_{\sigma,\gamma_2}$ is equal to

$$\begin{array}{c} \bigotimes_{(\alpha<\beta)\in Mor(\sigma)} K^\bullet_{\alpha,\beta} \otimes \Omega^\bullet(\mathcal{U}(\sigma))_{b_\sigma} \\ \| \\ \bigotimes_{(\alpha<\beta)\in Mor(\tau)} K^\bullet_{\alpha,\beta} \otimes \bigotimes_{\substack{(\alpha<\beta)\in Mor(\sigma)-Mor(\tau) \\ (\alpha<\beta)\neq(\sigma<\gamma_1),(\gamma_1<\tau), \\ (\sigma<\gamma_2),(\gamma_2<\tau)}} K^\bullet_{\alpha,\beta} \\ \otimes K^\bullet_{\gamma_1,\tau} \otimes K^\bullet_{\sigma,\gamma_1} \otimes K^\bullet_{\gamma_2,\tau} \otimes K^\bullet_{\sigma,\gamma_2} \otimes \Omega^\bullet(\mathcal{U}(\sigma))_{b_\sigma} \\ \downarrow \\ \bigotimes_{(\alpha<\beta)\in Mor(\tau)} K^\bullet_{\alpha,\beta} \otimes \Omega^\bullet(\mathcal{U}(\tau))_{b_\tau} \end{array}$$

is given by

$$\begin{array}{c} (\bigotimes_{\alpha,\beta} a_{\alpha,\beta} \otimes \bigotimes_{\alpha,\beta} b_{\alpha,\beta} \otimes x \otimes y \otimes z \otimes w \otimes \omega_{b_\sigma}) \\ \downarrow \\ \bigotimes_{\alpha,\beta} a_{\alpha,\beta} \cdot \prod_{\alpha,\beta} \epsilon(b_{\alpha,\beta}) \cdot (\partial(x \otimes y \otimes z \otimes w) \cdot \omega_{b_\sigma}). \end{array}$$

This proposition shows that the following diagram does not commute in general.

$$\begin{array}{ccc} \widetilde{\Omega}^\bullet(\sigma) & \xrightarrow{\delta_{\sigma,\gamma_1}} & \widetilde{\Omega}^\bullet(\gamma_1) \\ \delta_{\sigma,\gamma_2} \downarrow & & \downarrow \delta_{\gamma_1,\tau} \\ \widetilde{\Omega}^\bullet(\gamma_2) & \xrightarrow[\delta_{\gamma_2,\tau}]{} & \widetilde{\Omega}^\bullet(\tau) \end{array}$$

In the next subsection, we show that this diagram commutes up to homotopy. Moreover we show the existence of highter homotopy to define a total complex of $\{\widetilde{\Omega}^\bullet(\sigma)\}_{\sigma\in\mathcal{K}}$.

4.3. **Higher homotopy for differential graded algebras.** Let $\mathcal{K}$ be a simplical complex and X a contravariant fuctor from $\mathcal{K}$ to the category of Hopf algebroids. We use the same notations as in the last section. For a Hopf algebroid $\mathcal{U}$ on X and $x, y \in X$, we define a linear homomorphism *ten* of degree -1 by

$$\begin{array}{rcc} ten : K^\bullet(\mathcal{U})_{x,y} \otimes K^\bullet(\mathcal{U})_{x,y} & \to & K^\bullet(\mathcal{U})_{x,y} \\ (a_0 \otimes \cdots \otimes a_i) \otimes (b_0 \otimes \cdots \otimes b_j) & \mapsto & a_0 \otimes \cdots \otimes a_i \otimes b_0 \otimes \cdots \otimes b_j. \end{array}$$

Note that this homogeneous linear map is not a homomorphism of complexes.

Let $n \geq 2$ be a natural number and $\sigma < \tau$ be simplices in $\mathcal{K}$ of codimension n. Then $\tau - \sigma$ is a totally ordered set S with $\#S = n$. There is a one to one correspondece between the cube $\square^S$ and the set of simplices γ contained in τ cotaining σ, i.e. $\sigma < \gamma < \tau$. By this correspondence, the set $\square^S$ is regarded as a subset of simplices in $\mathcal{K}$.

The pair of simplices $(\alpha < \beta)$ of codimension one in $\square^S$ is denoted as $Mor(\sigma, \tau)$. We put

$$Chain^\bullet_{\sigma,\tau} := \otimes_{(\alpha<\beta)\in Mor(\sigma,\tau)} K^\bullet_{\alpha,\beta}$$

Let $\kappa \in \Sigma(S)$ be a minimal path in $\square^S$. This corresponds to a sequence of simplices $\kappa_0 < \cdots < \kappa_n$ in $\square^S$. We define a homomorphism $c(\kappa) : Chain^\bullet_{\sigma,\tau} \to K^\bullet_{\sigma,\tau}$ of complexes by

$$\begin{array}{ccc} \otimes_{(\alpha,\beta)\in Mor(\sigma,\tau)} K^\bullet_{\alpha,\beta} \simeq & & \\ \otimes_{(\alpha,\beta)\notin\kappa} K^\bullet_{\alpha,\beta} \otimes \otimes_{(\alpha,\beta)\in\kappa} K^\bullet_{\alpha,\beta} & \ni & (a_{\alpha,\beta}, b_{\kappa_i,\kappa_{i+1}}) \\ \downarrow & & \\ \otimes_{(\alpha,\beta)\in\kappa} K^\bullet_{\alpha,\beta} & \ni & \prod \epsilon(a_{\alpha,\beta}) \cdot (b_{\kappa_i,\kappa_{i+1}}) \\ \downarrow & & \\ K^\bullet_{\sigma,\tau} & \ni & \prod \epsilon(a_{\alpha,\beta}) \cdot b_{\kappa_{n-1},\kappa_n} \cdot \ldots \cdot b_{\kappa_0,\kappa_1} \end{array}$$

Here we used the dot product defined in the last section.

We define a homogeneous linear map (not a homomorphism of complex in general)

$$h(S) : Chain^\bullet_{\sigma,\tau} \to K^\bullet_{\sigma,\tau}$$

of degree $-n+1$ by the induction of n as follows.

1. If $n = 2$, we may assume that $S = \{1, 2\}$.

$$h(S) = ten \circ (c(12) \otimes c(21)) \circ \Delta :$$
$$Chain^\bullet_{\sigma,\tau} \to Chain^\bullet_{\sigma,\tau} \otimes Chain^\bullet_{\sigma,\tau} \to K^\bullet_{\sigma,\tau} \otimes K^\bullet_{\sigma,\tau} \to K^\bullet_{\sigma,\tau}$$

where

$$\Delta : Chain^\bullet_{\sigma,\tau} \to Chain^\bullet_{\sigma,\tau} \otimes Chain^\bullet_{\sigma,\tau}$$

is the coproduct homomorphism obtained from that of $K^\bullet_{\alpha,\beta}$. (See Definition 3.3. 2.)

2. For a simplex γ such that $\sigma < \gamma < \tau$, we define

$$split_\gamma : Chain^\bullet_{\sigma,\tau} \to Chain^\bullet_{\gamma,\tau} \otimes Chain^\bullet_{\sigma,\gamma}$$

by operating the augmentation homomorphism ϵ for a component (α, β) satisfying neither $\sigma < \alpha < \beta < \gamma$ nor $\gamma < \alpha < \beta < \tau$. The subset of S corresponding to the simplex γ is denoted as T. For $T \subset S$, we define a linear homomorphism $h(S-T) \cdot h(T) : Chain^\bullet_{\sigma,\tau} \to K^\bullet_{\sigma,\tau}$ by the composite

$$h(S-T) \cdot h(T) : Chain^\bullet_{\sigma,\tau} \underset{split_\gamma}{\longrightarrow} Chain^\bullet_{\gamma,\tau} \otimes Chain^\bullet_{\sigma,\gamma}$$
$$\underset{h(S-T)\otimes h(T)}{\longrightarrow} K^\bullet_{\gamma,\tau} \otimes K^\bullet_{\sigma,\gamma} \underset{\text{dot product}}{\longrightarrow} K^\bullet_{\sigma,\tau}.$$

3. We consider a homogeneous linear map

$$ten \circ \Big(c(tot(S)) \otimes (h(S-T) \cdot h(T))\Big) \circ \Delta :$$
$$Chain^\bullet_{\sigma,\tau} \underset{\Delta}{\to} Chain^\bullet_{\sigma,\tau} \otimes Chain^\bullet_{\sigma,\tau}$$
$$\underset{c(tot(S))\otimes(h(S-T)\cdot h(T))}{\longrightarrow} K^\bullet_{\sigma,\tau} \otimes K^\bullet_{\sigma,\tau} \underset{ten}{\to} K^\bullet_{\sigma,\tau}.$$

of degree $-n+1$. We define $h(S) : Chain^\bullet_{\sigma,\tau} \to K^\bullet_{\sigma,\tau}$ by

$$h(S) = \sum_{\emptyset \neq T \subsetneq S} (-1)^{\#(S-T)} sign(S, S-T) \cdot$$
$$ten \circ (c(tot(S)) \otimes (h(S-T) \cdot h(T))) \circ \Delta$$

The following proposition is a direct consequene of Proposition 4.3.

Proposition 4.6. 1. *Under the notation as above, we have*

$$dh(S) - h(S)d = \sum_{\emptyset \neq T \subsetneq S} (-1)^{\#(S-T)} sign(S, S-T) h(S-T) \cdot h(T). \tag{4.1}$$

2. *The righthand side of (4.1) is a homomorphism of complexes.*

4.4. **Patching differential graded algebra.** Let $\sigma < \tau$ be simplices of $\mathcal{K}$ of codimension $n \geq 2$. We define a homogeneous linear map $h(\sigma, \tau) : \tilde{\Omega}^\bullet(\sigma) \to \tilde{\Omega}^\bullet(\tau)$ of degree $-n+1$ by

$$\begin{aligned}
\widetilde{\Omega}^\bullet(\sigma) &= \bigotimes_{(\alpha,\beta)\in Mor(\sigma)} K^\bullet_{\alpha,\beta}\otimes \Omega^\bullet(\mathcal{U}(\sigma))_{b_\sigma} \\
&\simeq \bigotimes_{(\alpha,\beta)\in Mor(\sigma)-Mor(\sigma,\tau)} K^\bullet_{\alpha,\beta}\otimes Chain^\bullet_{\sigma,\tau}\otimes \Omega^\bullet(\mathcal{U}(\sigma))_{b_\sigma} \\
&\xrightarrow[1\otimes h(S)]{} \bigotimes_{(\alpha,\beta)\in Mor(\sigma)-Mor(\sigma,\tau)} K^\bullet_{\alpha,\beta}\otimes K^\bullet_{\sigma,\tau}\otimes \Omega^\bullet(\mathcal{U}(\sigma))_{b_\sigma} \\
&\longrightarrow \bigotimes_{(\alpha,\beta)\in Mor(\sigma)-Mor(\sigma,\tau)} K^\bullet_{\alpha,\beta}\otimes \Omega^\bullet(\mathcal{U}(\tau))_{b_\tau} \\
&\simeq \bigotimes_{(\alpha,\beta)\in Mor(\sigma)-Mor(\sigma,\tau)-Mor(\tau)} K^\bullet_{\alpha,\beta}\otimes \bigotimes_{(\alpha,\beta)\in Mor(\tau)} K^\bullet_{\alpha,\beta}\otimes \Omega^\bullet(\mathcal{U}(\tau))_{b_\tau} \\
&\xrightarrow[\epsilon\otimes 1\otimes 1]{} \bigotimes_{(\alpha,\beta)\in Mor(\tau)} K^\bullet_{\alpha,\beta}\otimes \Omega^\bullet(\mathcal{U}(\tau))_{b_\tau} = \widetilde{\Omega}^\bullet(\tau)
\end{aligned}$$

Using $\delta_{\sigma,\tau}$ and $h(\sigma,\tau)$, we define a complex $\Omega^\bullet(X)$ and a homogeneous linear map $d_X : \Omega^\bullet(X)\to\Omega^\bullet(X)$ as follows:

$$\Omega^i(X)\simeq \oplus_{\sigma\in\mathcal{K}}\widetilde{\Omega}^{i-\dim(\sigma)}(\sigma),$$

$$\begin{aligned}
d_X = (-1)^{\deg}\cdot\Big(&\sum_{\sigma<\tau \text{ is codimension one}} sign(\sigma)\delta_{\sigma,\tau} \\
&+ \sum_{\text{codimension of } (\sigma<\tau)\geq 2} sign(\sigma)h(\sigma,\tau)\Big),
\end{aligned}$$

where $sing(\sigma) = \binom{tot(K)}{tot(\sigma), tot(K-\sigma)}$. Now we can state the main theorem

Theorem 4.7. *We have*

$$d_X\circ d_X = 0.$$

This is a direct consequence of Proposition 4.6.

4.5. *GT*-adimissible varieties.

Definition 4.8. *Let $\mathcal{K}$ be a simplical complex. A contravariant functor Y from $\mathcal{K}$ to the category (Fund) is called a GT-admissible variety.*

Let $\# = B$ or DR. By attaching fundamental algebroid, we defined a functor $\mathcal{U}^\#$ from the category $(Fund)$ to the category $(Hopf_{\mathbf{Q}})$ of Hopf algebroid space over $\mathbf{Q}$. Then the composite $\mathcal{U}^\#\circ Y$ is functor from $\mathcal{K}$ to the category of Hopf algebroids. We aplly the construction of the last section to the functor $X = \mathcal{U}^\#\circ Y$ and get a complex $\Omega^\bullet(X)$.

Definition 4.9. *The cohomology $H^i_\#(Y)$ of Y is defined by the cohomology $H^i(\Omega^\bullet(\mathcal{U}^\#\circ Y))$ of $\Omega^\bullet(\mathcal{U}^\#\circ Y)$.*

Theorem 4.10. 1. *For a GT varieyt Y, the cohomology $H^i_\#(Y)$ is a $GT^\#$-module.*

2. *Let Φ be an associator. Then there exists an object*
$$H^i(Y) = (H^i_B(Y), H^i_{DR}(Y), comp) \in Vec_{\mathbf{Q}} \times_{Vec_{\mathbf{C}}} Vec_{\mathbf{Q}}$$
such that the first and the second factors are functorially isomorphic to $H^i_B(Y)$ and $H^i_{DR}(Y)$, respectively.
3. *Moreover if the associator Φ is Drinfeld associator, the third factor in 2 is equal to the comparison map.*

Remark 4.11. *We fix an inclusion $\overline{\mathbf{Q}} \to \mathbf{C}$. If a GT-admissible variety Y comes form a covering of an algebraic variety $\mathcal{Y}$, $H^i_{DR}(Y)$ (resp. $H^i_B(Y)$, $H^i_B(Y)\otimes\mathbf{Q}_l$) is canonically isomorphic to the classical cohomology $H^i_{DR}(\mathcal{Y}, \mathbf{Q})$ (resp. $H^i_B(\mathcal{Y}/\mathbf{Q})$, $H^i_{et}(\mathcal{Y}, \mathbf{Q}_l)$). Moreover the action of $Gal(\overline{\mathbf{Q}}/\mathbf{Q})$ factors through the natural homomorphism $Gal(\overline{\mathbf{Q}}/\mathbf{Q}) \to GT^B(\mathbf{Q}_l)$.*