

## CAN PHYSARUM SOLVE THE SHORTEST PATH PROBLEM MATHEMATICALLY RIGOROUSLY ?

真性粘菌は最短経路問題を数学的に厳密に解きうるか？

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**1. Introduction.** The plasmodium of true slime mold *Physarum polycephalum* is a large amoeba-like organism. Its body contains a tube network by means of which nutrients and signals circulate through the body in effective manner. When food sources were presented to a starved plasmodium that was spread over the entire agar surface, it concentrated at every food source, respectively. Almost the entire plasmodium accumulated at the food sources and covered each of them in order to absorb nutrients[8]. Only a few tube remained connecting the quasi-separated components of the plasmodium through the short path. Nakagaki *et al.* showed that this simple organism had the ability to find the minimum-length solution of a maze[9, 10]. The connecting tube traces the shortest path even in a complicated maze. This adaptation process of the tube network is based on an underlying physiological mechanism, that is, a tube becomes thicker as a flux in the tube is larger. This insight might be based on the research on the rhythmic oscillation of *Physarum polycephalum*[11]. Tero *et al.* made a mathematical model in consideration of the qualitative mechanisms clarified by experiments[12]. They considered the tube network of *Physarum polycephalum* on a maze to be a plane graph, set some variables on vertices and edges of the graph, and described the process of growth and degeneracy of the tube. The model consists of two parts, equations for flux in tubes and nonlinear ODEs for adaptation of tubes. In a special case of nonlinear terms, the model is called *Physarum solver*. According to numerical simulation results, the minimum-length solution of a maze can be obtained as an asymptotic steady state of *Physarum solver*[12, 13]. We have already obtained partial results in some simple cases[4]. Recently, we have gotten a further general result. This report is the first announcement of our result, that is, for two specified vertices  $s, t$  on the same face of any planar graph *Physarum solver* can find the shortest  $s-t$  path. This means that the equilibrium point corresponding to the shortest path is globally asymptotically stable for *Physarum solver*. See the forthcoming paper [5] in details.

## 2. Preliminaries.

**2.1. Graphs.** *Physarum solver* is defined on a finite graph. First, we briefly introduce some notations. We will assume that the reader is familiar with basic terms and results from graph theory. See, for details, [2, 3, 7].

A graph  $G = (V, E)$  is a pair of sets,  $V$  and  $E$ , where  $V$  is a nonempty and  $E$  is a set of 2-element subsets of  $V$ . Throughout this paper we assume that  $V$  is finite. The elements of  $V$  are called vertices of  $G$ , the elements of  $E$  are the edges of  $G$ . Let  $|G|$  denote the number of vertices.  $|G|$  is called the order of the graph  $G$ . The degree of a vertex  $v$ , denoted  $d(v)$ , is the number of edges incident with  $v$ .

Let  $G_1, \dots, G_k$  be subgraphs of the graph  $G$ . The union  $G_1 \cup \dots \cup G_k$  is the graph  $H \subseteq G$  with  $V(H) = V(G_1) \cup \dots \cup V(G_k)$  and  $E(H) = E(G_1) \cup \dots \cup E(G_k)$ .

For  $U \subset V$  we denote by  $G - U$  the subgraph of  $G$  obtained by deleting from  $G$  the vertices in  $U$  and all edges incident with them. If  $F \subset E$ , then  $G - F$  is the subgraph of  $G$  obtained by removing from  $G$  the edges in the set  $F$ .

A path is a nonempty graph  $P = (V, E)$  such that

$$V = \{x_0, x_1, \dots, x_k\} \quad E = \{x_0x_1, x_1x_2, \dots, x_{k-1}x_k\}.$$

We denote  $P = x_0x_1 \dots x_k$  and call it the path from  $x_0$  to  $x_k$ , or  $x_0$ - $x_k$  path.

A graph is planar if it can be drawn in the plane in such a way that no edges intersect, except at a common end-vertex. A plane graph is a graph drawn in such a way. For any graph  $G$ , a set  $\mathbb{R}^2 \setminus G$  is an open subset. Its region is called a face of  $G$ . Since  $G$  is bounded, just one of the faces is unbounded. The unbounded face of  $G$  is called outer face, and the remainder is called inner face.

**2.2. Physarum solver.** Now, let us briefly introduce *Physarum solver*. The formulation and physiological backgrounds are detailed in [12](See also [13]).

Let  $G = (V, E)$  be a graph. We consider a set  $N = (G, s, t, L)$ , where  $s, t \in V$  are two distinguished vertices, and  $L$  is a function from  $E$  to  $\mathbb{R}_+$ . Assume that  $G$  is a connected graph and  $|G| \geq 2$ . Let  $V = \{v_0, v_1, \dots, v_n\}$ , where  $n \geq 1$ . Let  $e_{ij}$  denote the edge joining  $v_i$  and  $v_j$  if it exists. If  $G$  has multiple edges joining  $v_i$  and  $v_j$ , we denote them by  $v_{ij}^1, v_{ij}^2, \dots$ . Let  $L(e_{ij}) = L_{ij} > 0$  be a length (or weight) of the edge  $e_{ij}$ . Assume that  $L(e_{ij}) = L(e_{ji})$ .  $G$  is considered as a flow network flowing out from  $v_0$  and sinking into  $v_n$ . To distinguish those two vertices, let us call  $s = v_0$  a source and  $t = v_n$  a sink (or a target). Assume that there exist exactly one source and one target. For a path  $P = v_{\beta_0} \dots v_{\beta_k}$  in  $G$ , define the length of the path  $P$  to be

$$L(P) = \sum_{i=0}^{k-1} L(e_{\beta_i, \beta_{i+1}}).$$

Throughout this paper, the length of a path does not mean the number of edges which compose the path.

Let  $\tau$  denote the time variable. For each  $i$ , the variable  $p_i(\tau)$  is a pressure at the vertex  $v_i$ . For each edge  $e_{ij}$ ,  $D_{ij}(\tau)$  and  $Q_{ij}(\tau)$  are its conductivity (corresponding to a thickness of tube) and flux, respectively. In addition, as described above,  $e_{ij}$  has its length  $L_{ij}$ . For each edge,  $D_{ij}$  should be nonnegative and  $D_{ij} = D_{ji}$ . Let  $\mathbf{D}(\tau) = (D_{ij}(\tau))_{i,j}$  be the set of all  $D_{ij}(\tau)$ 's. Note that  $p_i, D_{ij}$  and  $Q_{ij}$  are variables depending on time  $\tau$ , and  $L_{ij}$  is a positive constant. We want to obtain the  $s$ - $t$  path such that its length is smaller than that of any other  $s$ - $t$  paths.

First, we give two rules for flux. The flux  $Q_{ij}$  is given by

$$(2.1) \quad Q_{ij} = \frac{D_{ij}}{L_{ij}}(p_i - p_j) = g_{ij}(p_i - p_j),$$

where  $g_{ij} = D_{ij}/L_{ij}$  is the conductance of the edge  $e_{ij}$ . (2.1) is an analogy of *Ohm's law* for electric circuits. It is clear that  $Q_{ij} = -Q_{ji}$ . Moreover, we assume the *Kirchhoff's law* at each node:

$$(2.2) \quad \sum_{j \neq i} Q_{ij} = \begin{cases} I_0 & \text{if } i = 0, \\ 0 & \text{if } 0 < i < n, \\ -I_0 & \text{if } i = n, \end{cases}$$

where  $I_0$  is the flux from the source vertex. In this model, it is assumed that  $I_0$  is a positive constant.

Next, we give an adaptation rule of conductivity:

$$(2.3) \quad \dot{D}_{ij} = |Q_{ij}| - D_{ij},$$

where we use  $\dot{x}$  to represent the derivative  $dx/d\tau$ .

By setting  $p_n = 0$  as a basic pressure level, all  $p_i$ 's are determined by (2.2). Then  $Q_{ij}$ 's are determined by (2.1), and the evolution of  $D_{ij}$ 's is described by (2.3). We call the system (2.1),(2.2),(2.3) *Physarum solver* for  $N = (G, s, t, L)$ .

We sometimes consider an orientation of graph  $G$  by means of the direction of flux. If  $Q_{ij} > 0$ , then we suppose that  $e_{ij}$  is oriented from  $v_i$  to  $v_j$ . Naturally, this orientation can change as time passes.

**3. Mathematical analysis.** In this section, we state important lemmas and propositions, and our main result without proof. People who would like to see the details may refer to the forthcoming paper [5].

**3.1. Kirchhoff's Law.** Here we discuss a system of linear equations derived from Kirchhoff's law (2.2). Solving this system, we obtain the values of pressure at each vertex.

Assume that  $|G| = n + 1$  for an integer  $n \geq 1$ . For simplicity, we interpret that  $g_{ij} > 0$  if  $e_{ij} \in G$ , otherwise  $g_{ij} = 0$ . Substitute (2.1) for (2.2), then the equation to be solved is

$$(3.1) \quad \sum_{j \neq i} g_{ij}(p_i - p_j) = \begin{cases} I_0 & \text{if } i = 0, \\ 0 & \text{if } 1 \leq i \leq n - 1, \end{cases}$$

where  $p_n = 0$  and  $g_{ij} > 0$ . (3.1) is written in matrix form

$$(3.2) \quad \mathbf{A}\mathbf{p} = \mathbf{b},$$

where

$$(3.3) \quad \mathbf{p} = {}^t(p_0, p_1, \dots, p_{n-1}), \quad \mathbf{b} = {}^t(I_0, 0, \dots, 0),$$

and  $\mathbf{A} = (A_{ij})$  is a square matrix of order  $n$  given by

$$(3.4) \quad A_{ij} = \begin{cases} \sum_{l \neq i} g_{il} & \text{if } i = j, \\ -g_{ij} & \text{otherwise,} \end{cases}$$

where  $i, j = 0, \dots, n - 1$ . We first study the solution of (3.2).

**PROPOSITION 3.1.** *The coefficient matrix  $\mathbf{A}$  is a symmetric nonsingular  $M$ -matrix.*

**PROPOSITION 3.2.** *The system (3.2) has a unique non-negative solution  $\mathbf{p}$ .*

The next proposition guarantees that the vertex  $s = v_0$  actually works as a source and  $t = v_n$  works as a sink.

**PROPOSITION 3.3.** *For any  $v_j \in G$ ,  $p_0 \geq p_j$ .*

It follows that all  $Q_{ij}$ 's are bounded.

**PROPOSITION 3.4.** *For any  $e_{ij} \in G$ ,  $|Q_{ij}| \leq I_0$ .*

**PROPOSITION 3.5.** *Let  $\beta = \{\beta_i\}_{i=0}^k \subset \{0, 1, \dots, n\}$  and  $\beta_0 = 0, \beta_l = n$ , and  $l < n$ . Suppose that  $D_{\beta_i, \beta_{i+1}} > 0$  for  $0 \leq i \leq l - 1$ . When  $D_{r,s} \rightarrow 0$  for every  $r, s$  such that  $(r, s) \neq (\beta_i, \beta_{i+1})$  for any  $0 \leq i \leq l - 1$ , all  $p_i$ 's remain finite.*

Now we consider the orientation of  $G$  by the direction of flow as described in the previous section. If  $p_i > p_j$  and  $e_{ij} \in G$  for  $v_i, v_j \in G$ , let  $e_{ij}$  be oriented from  $v_i$  to  $v_j$ . Let  $\vec{E}$  be the set of orientable edges in such a way and  $\vec{G} = (V, \vec{E})$ , then  $\vec{G}$  is a directed graph with one source  $s = v_0$  and one target  $t = v_n$ . Note that  $E = \vec{E}$  does not always hold and  $\vec{E}$  can change as time passes. The next is, however, a universal property which holds through the adaptation process.

**PROPOSITION 3.6.**  $\vec{G}$  is acyclic, that is,  $\vec{G}$  has no directed cycle.

**3.2. Equilibria and  $s$ - $t$  paths.** Here we discuss the relation between equilibria of an adaptation equation and  $s$ - $t$  paths in a graph  $G$ . In an equilibrium state, i.e.  $\dot{D}_{ij} = 0$ , it holds that

$$(3.5) \quad D_{ij} (|p_i - p_j| - L_{ij}) = 0$$

$$(3.6) \quad \sum_j \frac{D_{ij}}{L_{ij}} (p_i - p_j) = \begin{cases} I_0 & \text{if } i = 0, \\ 0 & \text{if } i \neq 0, n, \\ -I_0 & \text{if } i = n. \end{cases}$$

According to (3.6), there always exists at least one  $s$ - $t$  path such that  $D_{ij} > 0$  for all  $e_{ij}$  in the path.

We assume that the length of  $s$ - $t$  path is different each other. For given any  $s$ - $t$  path  $P = v_{\beta_0} \cdots v_{\beta_k}$ , let  $\bar{D}_P = (D_{ij})_{i,j}$  be the conductivities with

$$(3.7) \quad \begin{cases} D_{ij} > 0 \text{ and } |p_i - p_j| = L_{ij} & \text{if } e_{ij} \in P, \\ D_{ij} = 0 & \text{otherwise,} \end{cases}$$

where  $v_{\beta_0} = s$  and  $v_{\beta_k} = t$ . Then  $\bar{D}_P$  satisfies (3.5). Let us call  $\bar{D}_P$  the equilibrium point corresponding to the path  $P$ .

It is easy to characterize  $\bar{D}_P$ . The next proposition immediately follows from (3.5)-(3.7).

**PROPOSITION 3.7.** The following two hold.

1. The pressures at vertex  $v_{\beta_i} \in P$  is given by

$$(3.8) \quad p_i = \begin{cases} \sum_{j=i}^{k-1} L_{\beta_j \beta_{j+1}} & \text{if } i = 0, \dots, k-1, \\ 0 & \text{if } i = k. \end{cases}$$

Especially,  $p_0$  is equal to the length of  $P$ .

2. The flux and the conductivity at edge  $e_{\beta_i \beta_{i+1}} \in P$  are

$$Q_{\beta_i \beta_{i+1}} = D_{\beta_i \beta_{i+1}} = I_0.$$

Therefore, the equilibrium point  $\bar{D}_P$  is given by

$$(3.9) \quad D_{ij} = \begin{cases} I_0 & \text{if } e_{ij} \in P, \\ 0 & \text{otherwise.} \end{cases}$$

Our goal is to prove that the equilibrium point  $\bar{D}_{P_*}$  corresponding to the shortest  $s$ - $t$  path  $P_*$  is globally asymptotically stable for Physarum solver.

*Remark.* If  $G$  has  $h$  edges such that  $L(P_1) = \cdots = L(P_h)$  ( $h > 2$ ), the number of equilibria corresponding to  $P_1, \dots, P_h$  is uncountably infinite. Let  $G' = P_1 \cup P_2 \cup \cdots \cup P_h$ ,  $V(G') = \{v_{\gamma_0}, \dots, v_{\gamma_k}\}$ , and  $v_{\gamma_0} = s, v_{\gamma_k} = t$ . The set of equilibria corresponding to the paths consists of all the point such that

$$\begin{cases} \sum_{j=1}^k D_{\gamma_0 \gamma_j} = \sum_{j=0}^{k-1} D_{\gamma_j \gamma_k} = I_0, \\ \sum_{j>i} D_{\gamma_i \gamma_j} = \sum_{j<i} D_{\gamma_i \gamma_j} & \text{if } i \neq 0, k. \end{cases}$$

**3.3. Main Theorem.** Here we prove the main theorem of this paper:

**THEOREM 3.8.** Assume that  $N = (G, s, t, L)$  satisfies the following properties:

- (i)  $G$  is a connected planar graph with  $|G| \geq 2$ .
- (ii) The source  $s$  and the target  $t$  are on the same face of  $G$ .
- (iii)  $G$  has exactly one shortest  $s$ - $t$  path  $P_* = v_{\alpha_0} \cdots v_{\alpha_k}$ , where  $v_{\alpha_0} = s, v_{\alpha_k} = t$ .

Then the equilibrium point  $\bar{\mathbf{D}}_{P_*}$  corresponding to the path  $P_*$  is globally asymptotically stable for Physarum solver (2.1)-(2.9).

First, the results in the previous section allow us to restrict the phase space of Physarum solver.

**LEMMA 3.9.** *The hypercube*

$$(3.10) \quad H = \{\mathbf{D} \mid 0 \leq D_{ij} \leq I_0 \text{ for all } i, j\}$$

is attracting and invariant for Physarum solver.

Therefore, we can always restrict the phase space into  $H$ . Hereafter we suppose that  $\mathbf{D}(0) \in H$ . The lower boundary of  $H$  is invariant for Physarum solver.

**LEMMA 3.10.** *For any edge  $e_{ij}$ , the set  $\{\mathbf{D} \mid D_{ij} = 0\}$  is invariant for Physarum solver. More generally, for a  $s$ - $t$  path  $P$  the set*

$$(3.11) \quad \{\mathbf{D} \mid D_{ij} = 0 \text{ for } e_{ij} \notin P\}$$

is an invariant subset.

**LEMMA 3.11.** *Let  $G$  be a connected graph, then the following three hold.*

1. *If  $G$  has a vertex  $v (\neq s, t)$  with  $d(v) = 1$ , then the conductivity of the incident edge tends to zero as  $\tau \rightarrow \infty$ .*

2. *If  $G$  is  $s$ - $t$  path, i.e.  $G = P_*$ , then  $\bar{\mathbf{D}}_{P_*}$  is globally asymptotically stable.*

3. *If  $G$  is a tree, then  $\bar{\mathbf{D}}_{P_*}$  is globally asymptotically stable*

**LEMMA 3.12.** *Assume that  $G$  is not a path and satisfies the assumption (iii). Then there is no inner equilibrium point, that is, the interior of  $H$  contains no equilibrium point.*

Now, we introduce the main tool.

**DEFINITION 3.13.** *For an edge  $e_{ij} \in G$ , we define a function  $F_{ij}(\mathbf{D})$  as*

$$(3.12) \quad F_{ij}(\mathbf{D}) = L_{ij} \log D_{ij}.$$

For a path  $P = v_{\beta_0} \dots v_{\beta_k}$  in  $G$ , we define a function  $F_P(\mathbf{D})$  as

$$(3.13) \quad F_P(\mathbf{D}) = \sum_{i=0}^{k-1} F_{\beta_i \beta_{i+1}}.$$

**LEMMA 3.14.** *The derivative of  $F_{ij}$  is calculated as*

$$(3.14) \quad \dot{F}_{ij} = |p_i - p_j| - L_{ij}.$$

Therefore we obtain

$$(3.15) \quad \dot{F}_P = \sum_{i=0}^{k-1} |p_{\beta_i} - p_{\beta_{i+1}}| - L(P).$$

LEMMA 3.15. Let  $P = v_{\beta_0} \cdots v_{\beta_i} \subset G$  be a  $s$ - $t$  path such that  $P \neq P_*$ . Assume that  $Q_{\beta_i, \beta_{i+1}}(\tau) > 0$  for all  $i$  and  $\tau \geq 0$ . Then there exists at least one edge  $e_{ij} \in P \setminus P_*$  such that  $D_{ij} \rightarrow 0$  as  $\tau \rightarrow \infty$ . Therefore any orbit is attracted into an invariant subset

$$(3.16) \quad \{\mathbf{D} \in H \mid D_{ij} = D_{ji} = 0\}.$$

According to Lemma 3.15, we can restrict the system into (3.16) to know the  $\omega$ -limit set of Physarum solver for  $N$ . The reduced system corresponds to the Physarum solver on the graph  $G - e_{ij}$ .

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