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Heteroclinic Jumps for Whiskered Tori in Nearly Integrable Hamiltonian Systems

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1 Introduction

We consider three-degree-of-freedom Hamiltonian systems of the from

$$\begin{aligned}
\dot{x} &= J\mathrm{D}_x H_0(x,I) + \epsilon J \mathrm{D}_x H_1(x,I,\theta),\\
\dot{I} &= -\epsilon \mathrm{D}_\theta H_1(x,I,\theta), \qquad\qquad (x,I,\theta)\in\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{T}^2,\\
\dot{\theta} &= \mathrm{D}_I H_0(x,I) + \epsilon \mathrm{D}_I H_1(x,I,\theta),
\end{aligned} \tag{1}$$

where ϵ is a small parameter such that $0<\epsilon\ll 1$, $H=H_0(x,I)+\epsilon H_1(x,I,\theta)$ is a real analytic function and J is 2×2 symplectic matrix, i.e.,

$$J=\begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix}.$$

When $\epsilon=0$, Eq. (1) becomes

$$\dot{x}=J\mathrm{D}_x H_0(x,I),\quad \dot{I}=0,\quad \dot{\theta}=\mathrm{D}_I H_0(x,I), \tag{2}$$

which is integrable. Hence, Eq. (1) represents nearly-integrable Hamiltonian systems.

For such a class of Hamiltonian systems, since the pioneer work of Arnold [1] there has been great interest in global instability known as *Arnold diffusion* [2–4]: If there is a sequence of invariant tori such that they are connected by *heteroclinic orbits*, then there exists an open set of trajectories which visit their neighborhoods in succession and go far from the initial points. A torus having stable and unstable manifolds are said to be *whiskered* and such a sequence of whiskered tori is called *transition chain*. The fact that these tori may be subjected to resonance raised a serious problem, which had been actually unsolved for many years, for complete understanding of the mechanism for Arnold diffusion. Recently, for a special case of (1) in which particularly $\mathrm{D}_I H_0(x,I)$ is independent of x, Delshams *et al.* [4] overcame the difficulty and showed that diffusion is more intense near resonant tori.

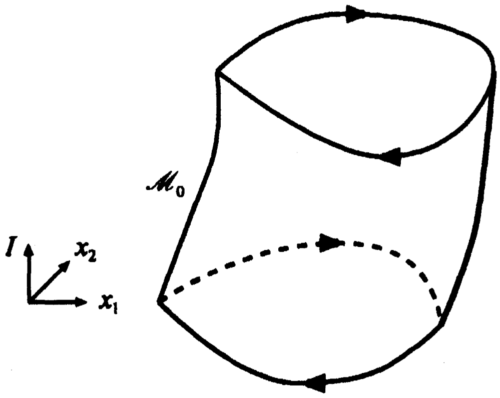

Fig. 1. Unperturbed phase space

In this paper we are interested in the size of jumps of heteroclnic orbits connecting whiskered tori in (1). We show that the jumps can be $\mathscr{O}(\sqrt{\epsilon})$ for resonant tori while they are $\mathscr{O}(\epsilon)$ for nonresonant tori. This is a contrast to the result of [4] in which the jumps of heteroclinic orbits are $\mathscr{O}(\epsilon)$ even for resonant tori. Thus, in a general case where $\mathrm{D}_I H_0(x, I)$ is not independent of x, diffusion near resonant tori can be even more intense. The proofs and technical details will be given elsewhere [5].

2 Unperturbed and perturbed phase space structures

Let $\mathscr{I}$ be a non-empty open set of $\mathbb{R}^2$ and let $\bar{\mathscr{I}} = \mathscr{I} \cup \partial\mathscr{I}$. Denote $\omega(I) = \mathrm{D}_I H_0(x_0(I), I)$. We make the following assumptions on (2).

(A1) There exists a function $x_0 : \bar{\mathscr{I}} \to \mathbb{R}^2$ such that for any $I \in \bar{\mathscr{I}}$ the point $x = x_0(I)$ is a hyperbolic saddle in the x-component of (2) and has a homoclinic orbit $x^I(t)$.

(A2) For any $I \in \bar{\mathscr{I}}$ we have

$$\det \mathrm{D}_I \omega(I) = \det \mathrm{D}_I[\mathrm{D}_I H_0(x_0(I), I)] \neq 0. \tag{3}$$

In the unperturbed system (2)

$$\mathscr{M}_0 = \{(x, I, \theta) \in \mathbb{R}^2 \times \mathscr{I} \times \mathbb{T}^2 \,|\, x = x_0(I)\}$$

is a four-dimensional, normally hyperbolic, invariant manifold whose stable and unstable manifolds, $W^{\mathrm{s}}(\mathscr{M}_0)$ and $W^{\mathrm{u}}(\mathscr{M}_0)$, coincide along a five-dimensional manifold

$$\{(x, I, \theta) \in \mathbb{R}^2 \times \mathscr{I} \times \mathbb{T}^2 \,|\, x = x^I(t), t \in \mathbb{R}\}.$$

See Fig. 1. The invariant manifold $\mathscr{M}_0$ consists of a two-parameter family of invariant tori $\mathscr{T}_0^I = \{(x_0(I), I, \theta) \mid \theta \in \mathbb{T}^2\}$ which satisfies a resonant condition

$$k \cdot \omega(I) = 0 \quad \text{for some } k \in \mathbb{Z}^2/\{0\}$$

or not, where '$\cdot$' represents the inner product. These invariant tori are *whiskered* in the meaning that they have stable and unstable manifolds.

For $\epsilon \neq 0$ sufficiently small it follows from the invariant manifold theory [6, 7] that there exists a four-dimensional, normally hyperbolic, locally invariant manifold $\mathscr{M}_\epsilon$ in an $\mathscr{O}(\epsilon)$-neighborhood of $\mathscr{M}_0$. Moreover, $\mathscr{M}_\epsilon$ has local stable and unstable manifolds $W^{\mathrm{s}}_{\mathrm{loc}}(\mathscr{M}_0)$ and $W^{\mathrm{u}}_{\mathrm{loc}}(\mathscr{M}_0)$, from which the global stable and unstable manifolds $W^{\mathrm{s}}(\mathscr{M}_0)$ and $W^{\mathrm{u}}(\mathscr{M}_0)$ are obtained, near $W^{\mathrm{s}}(\mathscr{M}_0)$ and $W^{\mathrm{u}}(\mathscr{M}_0)$. Define the Melnikov function as

$$M^I(\theta) = \int_{-\infty}^{\infty} \mathrm{D}_x H_0(x^I(t), I) \cdot J\mathrm{D}_x H_1(x^I(t), I, \theta^I(t) + \theta)\mathrm{d}t, \tag{4}$$

where

$$\theta^I(t) = \int_0^t \mathrm{D}_I H_0(x^I(t), I)\mathrm{d}t. \tag{5}$$

Using a standard argument in the Melnikov method (see, e.g., [8]), we can prove the following result.

Theorem 1. Suppose that for some point $(I, \theta) = (I_0, \theta_0)$ in $\mathbb{R}^2 \times \mathbb{T}^2$

$$M^I(\theta) = 0, \quad \mathrm{D}_\theta M^I(\theta) \neq 0.$$

Then for $\epsilon > 0$ sufficiently small the stable and unstable manifolds $W^{\mathrm{s}}(\mathscr{M}_\epsilon)$ and $W^{\mathrm{u}}(\mathscr{M}_\epsilon)$ of $\mathscr{M}_\epsilon$ intersect transversely in a four dimensional manifold.

We can show that on $\mathscr{M}_\epsilon$ there still exist many whiskered tori near the unperturbed nonresonant or resonant whiskered tori. The transverse intersection between $W^{\mathrm{s}}(\mathscr{M}_\epsilon)$ and $W^{\mathrm{u}}(\mathscr{M}_\epsilon)$ implies that the whiskered tori have homoclinic or heteroclinic orbits. The existence of such heteroclinic orbits is especially of importance since they provides a mechanism for Arnold diffusion, as stated in Section 1.

3 Heteroclinic jumps

We first treat the case of nonresonant tori. Let

$$\tilde{H}_1(x, I, \theta) = H_1(x, I, \theta) - H_1(x_0(I), I, \theta)$$

and define

$$\Delta I(I, \theta) = -\int_{-\infty}^{\infty} \mathrm{D}_\theta \tilde{H}_1(x^I(t), I, \theta^I(t) + \theta)\mathrm{d}t. \tag{6}$$

Using the averaging method [9], the KAM theorem [10] and Theorem 1, we can prove the following result.

Theorem 2. Suppose that for some $(I,\theta) = (I_0,\theta_0)$ the hypothesis of Theorem 1 and Diophantine condition

$$|k \cdot \omega(I)| \geq \gamma |k|^{-\tau}, \quad k \in \mathbb{Z}^2 \backslash \{0\}, \quad \tau > 1 \tag{7}$$

hold. Then for $\epsilon > 0$ sufficiently small there exist a pair of whiskered tori in an $\mathscr{O}(\epsilon)$-neighborhood of $I = I_0$ on $\mathscr{M}_\epsilon$ such that the distance between them is $\epsilon \Delta I(I_0,\theta_0) + \mathscr{O}(\epsilon^2)$ and they have a heteroclinic orbit.

We next consider the case of resonant tori and assume that

$$k_* \cdot \omega(I_*) = 0 \tag{8}$$

for some $I_* \in \mathscr{I}$ and $k_* \in \mathbb{Z}^2 \backslash \{0\}$. Expand $H_1(x_0(I), I, \theta)$ to a Fourier series as

$$H_1(x_0(I), I, \theta) = \sum_{k \in \mathbb{Z}^2} h_k(I) e^{ik\theta}, \quad h_k(I) = \frac{1}{(2\pi)^2} \int_{\mathbb{T}^2} H_1(x_0(I), I, \theta) e^{-ik \cdot \theta} d\theta$$

and set

$$h^*(\phi) = \sum_{j \neq 0} h_{jk_*}(I_*) e^{ij\phi}.$$

Define a function $\Delta h(\phi)$ as

$$\Delta h(\phi) = h^*(\Delta\phi_+ + \phi) - h^*(\Delta\phi_- + \phi), \quad \Delta\phi_\pm = \int_0^{\pm\infty} k_* \cdot \mathrm{D}_I H_0(x^{I_*}(t), I_*) dt. \tag{9}$$

Again, we use the averaging method [9], the KAM theorem [10] and Theorem 1 to prove the following result.

Theorem 3. Suppose that for some $\theta_* \in \mathbb{T}^2$ the hypothesis of Theorem 1 with $(I,\theta) = (I_*,\theta_*)$ and

$$\Delta h(k_* \cdot \theta_*) \neq 0. \tag{10}$$

Then for $\epsilon > 0$ sufficiently small there exist a pair of whiskered tori in an $\mathscr{O}(\sqrt{\epsilon})$-neighborhood of $I = I_*$ on $\mathscr{M}_\epsilon$ such that the distance between them is $\mathscr{O}(\sqrt{\epsilon})$ and they have a heteroclinic orbit.

4 Example

To illustrate the above theory, we consider the following example:

$$\begin{aligned} H_0(x, I) &= \frac{1}{3}(\cos x_1 + 2) I_1 + I_2 + \frac{1}{2}(x_2^2 + I_1^2 + I_2^2), \\ H_1(x, I, \theta) &= (\cos(\theta_1 - \theta_2) + \cos\theta_1 + \cos\theta_2) \cos x_1, \end{aligned} \tag{11}$$

where $I_1, I_2 > 0$. We easily see that the unperturbed Hamiltonian H_0 satisfies assumptions (A1) and (A2). In particular, in the x-component of the unperturbed system (2) the point $x_0(I) = (0,0)$ is a hyperbolic saddle and has a pair of homoclinic orbits

$$x_\pm^I(t) = \left(\pm 2\arcsin\left(\tanh\sqrt{\frac{I_1}{3}}\,t\right) + \pi, \pm 2\sqrt{\frac{I_1}{3}}\operatorname{sech}\sqrt{\frac{I_1}{3}}\,t\right).$$

Moreover, the frequency vector is given by

$$\omega(I) = (I_1 + 1, I_2 + 1) \tag{12}$$

and satisfies the nondegeneracy condition (3).

We compute (5) as

$$\theta_\pm^I(t) = \left((I_1+1)t - \frac{2}{3}\sqrt{\frac{3}{I_1}}\tanh\sqrt{\frac{I_1}{3}}\,t,\ (I_2+1)t\right)$$

and estimate the Melnikov function (4) as

$$\begin{aligned} M^I(\theta) =& 4\Bigg[A\left(\sqrt{\frac{3}{I_1}}, I_1 - I_2\right)\sin(\theta_1 - \theta_2) \\ &+ A\left(\sqrt{\frac{3}{I_1}}, I_1 + 1\right)\sin\theta_1 - A_0\left(\sqrt{\frac{3}{I_1}}(I_2+1)\right)\sin\theta_2\Bigg], \end{aligned}$$

where

$$\begin{aligned} A(a,b) =& \int_{-\infty}^{\infty} \tanh\tau\operatorname{sech}^2\tau\cos\left(\frac{2}{3}a\tanh\tau\right)\sin ab\tau\,dt \\ &- \int_{-\infty}^{\infty} \tanh\tau\operatorname{sech}^2\tau\sin\left(\frac{2}{3}a\tanh\tau\right)\cos ab\tau\,dt, \\ A_0(\nu) =& \frac{\pi\nu^2}{2}\operatorname{cosech}\left(\frac{\pi\nu}{2}\right) > 0 \quad \text{for } \nu > 0. \end{aligned}$$

Hence, the hypothesis of Theorem 1 holds for some $\theta_0 = (\theta_{10}, \theta_{20})$ so that $W^s(\mathcal{M}_\varepsilon)$ and $W^u(\mathcal{M}_\varepsilon)$ intersect transversely. In particular, if

$$A_0\left(\sqrt{\frac{3}{I_1}}(I_2+1)\right) \neq \pm A\left(\sqrt{\frac{3}{I_1}}, I_1 + 1\right), \tag{13}$$

then we can take θ_0 such that

$$\theta_{10} \neq \theta_{20}, \theta_{20} + \pi. \tag{14}$$

Note that for any $I_1 > 0$ condition (13) holds for almost all $I_2 > 0$.

Now we assume that the frequency vector (12) satisfies the Diophantine condition (7) for $I = I_0$. We have

$$\tilde{H}_1(x, I, \theta) = (\cos(\theta_1 - \theta_2) + \cos\theta_1 + \cos\theta_2)(\cos x_1 - 1)$$

and estimate (6) as

$$\Delta I_1(I,\theta) = 2\sqrt{\frac{3}{I_1}}\left[-B\left(\sqrt{\frac{3}{I_1}}, I_1 - I_2\right)\sin(\theta_1-\theta_2) - B\left(\sqrt{\frac{3}{I_1}}, I_1+1\right)\sin\theta_1\right],$$

$$\Delta I_2(I,\theta) = 2\sqrt{\frac{3}{I_1}}\left[B\left(\sqrt{\frac{3}{I_1}}, I_1 - I_2\right)\sin(\theta_1-\theta_2) - B_0\left(\sqrt{\frac{3}{I_1}}(I_2+1)\right)\sin\theta_2\right],$$

where

$$\begin{aligned} B(a,b) &= \int_{-\infty}^{\infty} \operatorname{sech}^2\tau \cos\left(\frac{2}{3}a\tanh\tau\right)\cos ab\tau\, dt \\ &\quad + \int_{-\infty}^{\infty} \operatorname{sech}^2\tau \sin\left(\frac{2}{3}a\tanh\tau\right)\sin ab\tau\, dt, \\ B_0(\nu) &= \pi\nu \operatorname{cosech}\left(\frac{\pi\nu}{2}\right) > 0 \quad \text{for } \nu > 0. \end{aligned}$$

By Theorem 2, in an $\mathscr{O}(\epsilon)$-neighborhood of $I = I_0$ on $\mathscr{M}_\epsilon$, there exist a pair of whiskered tori which are at distance of $\epsilon\Delta I(I_0,\theta_0) + \mathscr{O}(\epsilon^2)$ and connected by a heteroclinic orbit.

Finally, we consider the case in which the unperturbed tori are resonant and assume that $I_1 = I_2$ so that the resonance condition (8) holds with $k_* = (1,-1)$. The Fourier coefficients $h_k(I)$ of $H_1(x_0(I), I, \theta)$ are given by

$$h_k(I) = \begin{cases} \frac{1}{2} & \text{if } (k_1,k_2) = (\pm 1, \mp 1),\ (\pm 1, 0) \text{ or } (0, \pm 1); \\ 0 & \text{otherwise}, \end{cases}$$

so that

$$h^*(\phi) = \cos\phi.$$

We estimate

$$\Delta\phi_\pm = \mp\frac{2}{3}\sqrt{\frac{I_1}{3}}$$

to obtain

$$\Delta h(\phi) = \cos\left(\phi - \frac{2}{3}\sqrt{\frac{I_1}{3}}\right) - \cos\left(\phi + \frac{2}{3}\sqrt{\frac{I_1}{3}}\right).$$

Suppose that condition (13) holds. Then since as in (14) we can take $\theta_0 = \theta_*$ for the hypothesis of Theorem 1 to hold for $I_1 = I_2$, we have (10). Hence it follows from Theorem 3 that in an $\mathscr{O}(\sqrt{\epsilon})$-neighborhood of $I_1 = I_2$ on $\mathscr{M}_\epsilon$, there exist a pair of whiskered tori which are at distance of $\mathscr{O}(\sqrt{\epsilon})$ and connected by a heteroclinic orbit.

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