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On the boundary growth of superharmonic functions satisfying nonlinear inequalities

Kentaro Hirata (Hokkaido University)

平田 賢太郎（北海道大学）

1 Introduction

The study on the boundary behavior of harmonic functions and superharmonic functions has a long history. In 1906, Fatou [8] showed that every bounded harmonic function in the unit disc $\mathbb{D}$ has a finite nontangential limit almost everywhere on the boundary $\partial\mathbb{D}$. The extension of this result to higher dimensions was due to Bray and Evans [6] in 1927. Also, it was proved by Littlewood [17] in 1927 that nontangential approach regions in the Fatou theorem are best possible in the following sense: there exists a bounded harmonic function on $\mathbb{D}$ which admits no limits along a tangential curve a.e. on $\partial\mathbb{D}$. A simple proof by the construction of a Blaschke product was given by Zygmund [25] in 1949. Also, Aikawa [1, 2] presented a stronger version of the Littlewood theorem in 1990. In their study of maximal functions, Nagel and Stein [19] was established that there exist tangential sequences along which every bounded harmonic function on $\mathbb{D}$ converges a.e. on $\partial\mathbb{D}$. These results were generalized in several directions by many authors. For example, see [11, 12, 15, 22] for invariant harmonic functions in the unit ball of $\mathbb{C}^n$, and see [4, 10, 13, 16] for positive solutions of the Helmholtz equation in $\mathbb{R}^n$, and so on.

The decomposition theorem by Riesz states that every nonnegative superharmonic function can be represented as the sum of a nonnegative harmonic function and a Green potential. In 1928, Littlewood showed that every Green potential on $\mathbb{D}$ has radial limit zero a.e. on $\partial\mathbb{D}$. Privalov [21] extened this result to higher dimensions. However, Green potentials do not necessarily have nontangential limits. Thus we need to impose some conditions on the densities of Green potentials if we want to show the existence of nontangential limits. In 1967, Arsove and Huber [3] proved that if a nonnegative measurable function f on the unit ball B satisfies the weighted integrability condition

$$\int_B (1-|x|)^{2p-1} f(x)^p dx < \infty \tag{1.1}$$

for $p = 1$ and some $p > n/2$, then the Green potential of the density f has nontangential limit zero a.e. on ∂B. In 1979, Wu [23] gave a sufficient condition similar to (1.1) for the density functions to guarantee the existence of tangential limits of Green potentials. Other generalizations are found in [7, 18].

Let Δ denote the Laplace operator on $\mathbb{R}^n$. Given a suitable nonnegative function f on B, solutions of the Poisson equation $-\Delta u = f$ in B may be decomposed into a harmonic function and the Green potential of the density f, and hence the above classical results are applicable. We are now interested in the boundary behavior of positive solutions of nonlinear elliptic equations. For example, the Lane-Emden equation given by $-\Delta u = u^p$ is one of important equations appearing in astrophysics. It goes without saying that the classical results are not applicable to positive solutions of such equations. This is one of motivations to study the boundary behavior of superharmonic functions satisfying nonlinear inequalities as mentioned in our title. It is well known that if u is superharmonic on a bounded domain Ω in $\mathbb{R}^n$ then there exists a unique Radon measure μ_u on Ω such that

$$\int_\Omega \phi d\mu_u = -\int_\Omega u\Delta\phi dx \quad \text{for all } \phi \in C_0^\infty(\Omega).$$

If μ_u is absolutely continuous with respect to Lebesgue measure, then its density function is written as $-\Delta u$ (the usage of $-\Delta u$ means implicitly that μ_u is absolutely continuous with respect to Lebesgue measure). In this note, we study positive superharmonic functions u on Ω satisfying

$$0 \le -\Delta u \le u^p \quad \text{a.e. on } \Omega, \tag{1.2}$$

where $p > 0$ is a constant. Clearly, all positive solutions of the Lane-Emden equation satisfy (1.2). Observe that if u is a positive harmonic function on a smooth domain Ω then

$$\frac{1}{A}\delta_\Omega(x) \le u(x) \le A\delta_\Omega(x)^{1-n} \quad \text{for } x \in \Omega,$$

where $A > 1$ is a constant depending only on u, n and Ω, and $\delta_\Omega(x)$ denotes the distance from x to $\partial\Omega$. Actually, the lower estimate is extendable to all positive superharmonic functions. However, the upper estimate does not necessarily hold for superharmonic functions. The main purpose of this note is to establish the boundary growth estimate for positive superharmonic functions satisfying (1.2).

In what follows, we suppose that $n \ge 3$ and that Ω is a bounded $C^{1,1}$-domain in $\mathbb{R}^n$. The open ball of center x and radius r is denoted by $B(x, r)$. The symbol A stands for an absolute positive constant whose value is unimportant and may change from line to line. For two positive functions f and g, we write $f \approx g$ if there exists a constant $A > 1$ such that $f/A \le g \le Af$. The constant A will be called the constant of comparison.

2 Boundary growth estimate

Theorem 1. *If $0 < p \le (n+1)/(n-1)$, then every positive superharmonic function u on Ω satisfying* (1.2) *enjoys*

$$u(x) \le A\delta_\Omega(x)^{1-n} \quad \text{for } x \in \Omega, \tag{2.1}$$

where A is a constant depending only on u, p, n and Ω.

Proof. We give a sketch of the proof. Since u is a positive superharmonic function on Ω satisfying (1.2), it follows from the Riesz decomposition theorem that

$$u(x) = h(x) + \int_{\Omega} G(x,y)(-\Delta u(y))dy, \tag{2.2}$$

where $h \geq 0$ is the greatest harmonic minorant of u on Ω and $G(x,y)$ is the Green function for Ω. It is known that

$$G(x,y) \approx \min\left\{1, \frac{\delta_\Omega(x)\delta_\Omega(y)}{|x-y|^2}\right\} |x-y|^{2-n}, \tag{2.3}$$

where the constant of comparison depends only on n and Ω. See [5, 24]. Also, easy computation gives the following estimate for the Martin kernel $K(\cdot,\xi)$ of Ω with pole at $\xi \in \partial\Omega$:

$$K(x,\xi) \approx \frac{\delta_\Omega(x)}{|x-\xi|^n},$$

where the constant of comparison depends only on n and Ω. This and the Martin representation theorem shows that

$$h(x) \leq A\delta_\Omega(x)^{1-n} \quad \text{for } x \in \Omega,$$

where A depends only on h, n and Ω. Therefore, it is easy to see from (2.2) that for each $j \in \mathbb{N}$, there is a constant $c_j > 0$ depending only on j, u, n and Ω such that

$$u(x) \leq c_j\delta_\Omega(z)^{1-n} + \int_{B(z,\delta_\Omega(z)/2^j)} \frac{-\Delta u(y)}{|x-y|^{n-2}}dy \tag{2.4}$$

for $z \in \Omega$ and $x \in B(z, \delta_\Omega(z)/2^{j+1})$. Also, (2.2) and (2.3) give that for $z \in \Omega$,

$$\delta_\Omega(z) \int_{B(z,\delta_\Omega(z)/2)} (-\Delta u(y))dy \leq A, \tag{2.5}$$

where A depends only on u, n and Ω.

For simplicity, we write $B(r) = B(0,r)$. Let $z \in \Omega$ and $j \in \mathbb{N}$. Put $r = \delta_\Omega(z)$ and $\psi_z(\zeta) = r^{n+1}(-\Delta u(z+r\zeta))$. Making the change of variables $x = z + r\eta$ and $y = z + r\zeta$, we obtain from (2.4) and (2.5) that

$$\int_{B(1/2)} \psi_z(\zeta)d\zeta \le A, \tag{2.6}$$

$$r^{n-1}u(z+r\eta) \le c_j + \int_{B(2^{-j})} \frac{\psi_z(\zeta)}{|\eta-\zeta|^{n-2}}d\zeta \quad \text{for } \eta \in B(2^{-(j+1)}). \tag{2.7}$$

Suppose that $0 < p \le (n+1)/(n-1)$ and let

$$\frac{n+1}{n-1} < q < \frac{n}{n-2}, \quad \ell = \left[\frac{\log(q/(q-1))}{\log(q/p)}\right] + 1, \quad c_0 = \max_{1\le j\le \ell+1}\{c_j\}.$$

Define

$$\Psi_{z,j}(\eta) = c_0 + \int_{B(2^{-j})} \frac{\psi_z(\zeta)}{|\eta-\zeta|^{n-2}}d\zeta.$$

It follows from (2.7) and the Hölder inequality that

$$\delta_\Omega(z)^{n-1}u(z) = r^{n-1}u(z) \le \Psi_{z,\ell+1}(0) \le c_0 + A\|\psi_z\|_{L^{q/(q-1)}(B(2^{-(\ell+1)}))}.$$

To prove (2.1), it suffices to show that

$$\|\psi_z\|_{L^{q/(q-1)}(B(2^{-(\ell+1)}))} \le A, \tag{2.8}$$

where A is independent of z. Let $s = q/p$. Then $s > 1$. We claim that for each $\kappa \ge 1$, there is a constant A depending only on κ, c_0, p, q, n and Ω such that

$$\int_{B(2^{-(j+1)})} \psi_z(\zeta)^{\kappa s}d\zeta \le A + A\left(\int_{B(2^{-j})} \psi_z(\zeta)^\kappa d\zeta\right)^q. \tag{2.9}$$

Indeed, by the Jensen inequality,

$$\left(\int_{B(2^{-j})} \frac{\psi_z(\zeta)}{|\eta-\zeta|^{n-2}}d\zeta\right)^\kappa \le 2^{\kappa-1}\int_{B(2^{-j})} \frac{\psi_z(\zeta)^\kappa}{|\eta-\zeta|^{n-2}}d\zeta \quad \text{for } \eta \in B(1).$$

Therefore, by the Minkowski inequality,

$$\begin{aligned}\int_{B(2^{-j})} \Psi_{z,j}(\eta)^{\kappa q}d\eta &\le A + A\int_{B(2^{-j})}\left(\int_{B(2^{-j})} \frac{\psi_z(\zeta)^\kappa}{|\eta-\zeta|^{n-2}}d\zeta\right)^q d\eta \\ &\le A + A\left(\int_{B(2^{-j})} \psi_z(\zeta)^\kappa d\zeta\right)^q.\end{aligned}$$

By the way, it follows from (1.2) and $p \le (n+1)/(n-1)$ that

$$\psi_z(\eta) = r^{n+1}(-\Delta u(z + r\eta)) \le r^{n+1} u(z + r\eta)^p \le A\Psi_{z,j}(\eta)^p$$

for a.e. $\eta \in B(2^{-(j+1)})$. Hence (2.9) holds.

Let us apply (2.9) ℓ times. Since $s^\ell \ge q/(q-1)$, it follows from the Hölder inequality that

$$\begin{aligned}
\|\psi_z\|_{L^{q/(q-1)}(B(2^{-(\ell+1)}))} &\le A\left(\int_{B(2^{-(\ell+1)})} \psi_z(\zeta)^{s^\ell} d\zeta\right)^{1/s^\ell} \\
&\le A + A\left(\int_{B(2^{-\ell})} \psi_z(\zeta)^{s^{\ell-1}} d\zeta\right)^{q/s^\ell} \\
&\le \cdots \\
&\le A + A\left(\int_{B(1/2)} \psi_z(\zeta) d\zeta\right)^{q^\ell/s^\ell}.
\end{aligned}$$

In view of (2.6) we obtain (2.8). This completes the proof of Theorem 1. $\square$

3 Nontangential limits of Green potentials

As an application of Theorem 1, we can obtain the following result.

Theorem 2. *Let $0 < p \le (n+1)/(n-1)$ and let u be as in Theorem 1. If, in addition, the greatest harmonic minorant of u on Ω is the zero function, then u has nontangential limit zero almost everywhere on $\partial\Omega$.*

Proof. We start by recalling the Harnack inequality for nonnegative weak solutions $v \in W^{1,2}(D)$ of the stationary Schrödinger equation $\Delta v + \rho v = 0$ in D. If ρ is a measurable function on D such that $|\rho|$ is bounded by a constant ν^2, then there exists a constant A depending only on n such that

$$\sup_{B(x,r)} v \le A^{\sqrt{n}+\nu r} \inf_{B(x,r)} v,$$

whenever $B(x, 4r) \subset D$. See [9].

Let $B(x, 8r) \subset \Omega$ and let us apply this Harnack inequality to $D = B(x, 4r)$ and $\rho = -\Delta u/u$. Since $-\Delta u \in L^\infty_{loc}(\Omega)$ by (1.2) and Theorem 1, it follows that $u \in C^1(\Omega) \subset W^{1,2}(D)$ and that u is a weak solution of $\Delta u + \rho u = 0$ in D. Also, if $1 \le p \le (n+1)/(n-1)$, then Theorem 1 yields that for $y \in D$,

$$0 \le \rho(y) \le u(y)^{p-1} \le A\delta_\Omega(y)^{(p-1)(1-n)} \le Ar^{(p-1)(1-n)} \le Ar^{-2}.$$

If $0 < p < 1$, then we use the decay estimate $u(y) \ge A\delta_\Omega(y)$ in Ω to obtain $0 \le \rho(y) \le Ar^{-2}$ for $y \in D$. In any cases, we observe that there exists a constant A depending only on u, p, n and Ω such that

$$\sup_{B(x,r)} u \le A \inf_{B(x,r)} u,$$

whenever $B(x, 8r) \subset \Omega$.

Finally, we apply this Harnack inequality to show the existence of nontangential limits. By our assumption, u is the Green potential of the density $-\Delta u$. Therefore it follows from [20] that u has minimal fine limit zero at $\xi \in \partial\Omega \setminus E$, where the surface measure of E is zero. Let $\{x_j\}$ be arbitrary sequence converging to ξ within a nontangential cone at ξ. Since the bubble set $\bigcup_j B(x_j, \delta_\Omega(x_j)/8)$ is not minimally thin at ξ, we find a sequence $y_j \in B(x_j, \delta_\Omega(x_j)/8)$ converging to ξ such that $u(y_j) \to 0$ as $j \to \infty$. Hence $0 \le u(x_j) \le Au(y_j) \to 0$. This completes the proof of Theorem 2. □

4 Remarks

In this final section, we give some remarks on Theorem 1.

- If $p > (n+1)/(n-1)$, then we can construct a C^2-function u on Ω satisfying (1.2) such that (2.1) fails to hold. Hence the upper bound $p \le (n+1)/(n-1)$ is sharp in Theorem 1.

- If $0 < p < (n+1)/(n-1)$, then

$$-\Delta u = u^p \quad \text{in } \Omega$$

has a positive solution $u \in C^2(\Omega)$ satisfying

$$u(x) \approx \frac{\delta_\Omega(x)}{|x-\xi|^n} \quad \text{for } x \in \Omega.$$

Hence the growth rate $1-n$ is sharp in Theorem 1.

The proofs of these remarks can be found in [14] where we actually studied positive superharmonic functions satisfying more general inequality $0 \le -\Delta u(x) \le c\delta_\Omega(x)^{-\alpha} u(x)^p$ and positive solutions of $-\Delta u(x) = c\delta_\Omega(x)^{-\alpha} u(x)^p$.

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Kentaro Hirata
DEPARTMENT OF MATHEMATICS
HOKKAIDO UNIVERSITY
email: hirata@math.sci.hokudai.ac.jp