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On complementary spaces of the Lizorkin spaces

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§1. Introduction

Let R^n be the n-dimensional Euclidean space. For a multi-index $\alpha=(\alpha_1,\cdots,\alpha_n)$ and $x=(x_1,\cdots,x_n)\in R^n$, we let

$$x^\alpha=x_1^{\alpha_1}\cdots x_n^{\alpha_n},\quad D^\alpha=\frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}}.$$

The Schwartz space $\mathcal{S}(R^n)$ is defined to be the class of all C^∞-functions φ on R^n such that

$$p_{\alpha,\beta}(\varphi)=\sup_{x\in R^n}|x^\alpha D^\beta\varphi(x)|<\infty$$

for all multi-indices α and β. We introduce two kinds of the Lizorkin spaces $\Phi_1(R^n)$ and $\Phi_2(R^n)$. The Lizorkin space $\Phi_1(R^n)$ of the first kind is defined to be the class of all functions $\varphi\in\mathcal{S}(R^n)$ which satisfy

$$\int_{R^n}\varphi(x)x^\alpha dx=0$$

for any multi-index α. The Lizorkin space $\Phi_2(R^n)$ of the second kind is defined to be the class of all functions $\varphi\in\mathcal{S}(R^n)$ which satify

$$\int_{-\infty}^{\infty}\varphi(x_1,\cdots,x_j,\cdots,x_n)x_j^\ell dx_j=0$$

for $j=1,\cdots,n$ and $\ell=0,1,2,\cdots$. Clealy $\Phi_1(R^n)\supset\Phi_2(R^n)$. An example of a function belonging to $\Phi_1(R^n)$ (resp. $\Phi_2(R^n)$) is $\mathcal{F}(e^{-|y|^2-(1/|y|^2)})(x)$ (resp. $\mathcal{F}(e^{-|y|^2-\sum_{j=1}^n 1/y_j^2})(x)$) where $\mathcal{F}\varphi$ is the Fourier transform of φ:

$$\mathcal{F}\varphi(x)=\int_{R^n}e^{-ixy}dy.$$

The Lizorkin spaces appeared in the theory of fractional derivatives, hypersingular integrals and Riesz potentials ([Sa2] and [SKM]). The properties of the Lizorkin spaces have studied by several authors. The denseness of the Lizorkin spaces in the Lebegue spaces was proved in O.I.Lizorkin [Li2] and S.G.Samko [Sa1]. Moreover P.I.Lizorkin [Li3] showed that the space $\Phi_1(R^n)$ is dense in the Sobolev spaces and T.Kurokawa [Ku] deals with the denseness of the space $\Phi_1(R^n)$ in the spaces of Beppo Levi type. The invariance of the space $\Phi_1(R^n)$ relative to Riesz potentials was noted by V.I.Semyanistyi [Se], P.I.Lizorkin [Li3] and S.Helgason [He]. T.Kurokawa [Ku] establish the invariance of the space $\Phi_1(R^n)$ relative to more general operators. In this note we are concerned with comlementary spaces of $\Phi_1(R^n)$ and $\Phi_2(R^n)$ in $\mathcal{S}(R^n)$. For a subspace $V \subset \mathcal{S}(R^n)$, if a subspace $W \subset \mathcal{S}(R^n)$ satisfies the condition

$$\mathcal{S}(R^n) = V \oplus W,$$

then we call W a complementary space of V in $\mathcal{S}(R^n)$ where the symbol $\oplus$ indicates a direct sum. In section 2 as a preparation we introduce dual functions of polynomials and tensor product functions, and study their properties. In section 3 we sketch our plan to give comlementary spaces of $\Phi_1(R^n)$ and $\Phi_2(R^n)$ in $\mathcal{S}(R^n)$.

§2. Dual functions of polynomials and tensor product functions

Let $h \in C^\infty(R^1)$ be a function which satisfies the conditions $0 \leq h(t) \leq 1, h(-t) = h(t)$ and

$$h(t) = \begin{cases} 1, & \text{for } |t| \leq 1/2 \\ 0, & \text{for } |t| \geq 1. \end{cases}$$

We fix the function $h(t)$. We denote by $\mathcal{A}$ the set of sequences $a = \{a_j\}_{j=0,1,\cdots}$ which satify $0 < a_j \leq 1$ and $a_j \geq a_{j+1}$. For $a =$

$\{a_j\}_{j=0,1,\cdots} \in \mathcal{A}$ we put

$$\eta_j^a(t) = \frac{t^j}{j!} h(\frac{t}{a_j}), \quad j = 0, 1, 2, \cdots$$

and

$$\theta_j^a(t) = \frac{i^j}{2\pi} \mathcal{F}\eta_j^a(t), \quad j = 0, 1, 2, \cdots.$$

Then $\theta_j^a \in \mathcal{S}(R^1)$ and

$$\text{(2.1)} \qquad \int_{-\infty}^{\infty} \theta_j^a(t) t^k dt = \begin{cases} 1, & k = j \\ 0, & k \neq j, \end{cases} \quad k, j = 0, 1, 2, \cdots.$$

Since $\{\theta_j^a\}_{j=0,1,\cdots}$ satisfy (2.1), we call $\{\theta_j^a\}_{j=0,1,\cdots}$ dual functions of polynomials associated with a sequence $a \in \mathcal{A}$. For $1 \le p \le n$ we denote by M_p the set of subsets of $\{1, 2, \cdots, n\}$ which have p elements. For $\{i_1, i_2, \cdots, i_p\} \in M_p$ we always assume that $i_1 < i_2 < \cdots < i_p$. For multi-index $\alpha = (\alpha_1, \cdots, \alpha_n)$ and $\{i_1, \cdots, i_p\} \in M_p$ the notation $(\{\alpha_{i_1}, \cdots, \alpha_{i_p}\}^c)$ stands for

$$(\{\alpha_{i_1}, \cdots, \alpha_{i_p}\}^c) = (\alpha_{k_1}, \cdots, \alpha_{k_{n-p}})$$

where $\{k_1, \cdots, k_{n-p}\} = \{1, \cdots, n\} - \{i_1, \cdots, i_p\}$. Similarly, for $x = (x_1, \cdots, x_n)$ we denote

$$(\{x_{i_1}, \cdots, x_{i_p}\}^c) = (x_{k_1}, \cdots, x_{k_{n-p}}).$$

Moreover we denote

$$(\{x_{i_1}, \cdots, x_{i_p}\}^c)^{(\{\alpha_{i_1}, \cdots, \alpha_{i_p}\}^c)} = x_{k_1}^{\alpha_{k_1}} \cdots x_{k_{n-p}}^{\alpha_{k_{n-p}}},$$

$$(\{D_{i_1}, \cdots, D_{i_p}\}^c)^{(\{\alpha_{i_1}, \cdots, \alpha_{i_p}\}^c)} = D_{k_1}^{\alpha_{k_1}} \cdots D_{k_{n-p}}^{\alpha_{k_{n-p}}}.$$

Let α, β be multi-indices and $\{i_1, \cdots, i_p\} \in M_p$. For a function $\varphi(\{x_{i_1}, \cdots, x_{i_p}\}^c) \in \mathcal{S}(R^{n-p})$ we define

$$\begin{aligned} & p_{(\{\alpha_{i_1}, \cdots, \alpha_{i_p}\}^c), (\{\beta_{i_1}, \cdots, \beta_{i_p}\}^c)}(\varphi) \\ & = \sup_{(\{x_{i_1}, \cdots, x_{i_p}\}^c) \in R^{n-p}} |(\{x_{i_1}, \cdots, x_{i_p}\}^c)^{(\{\alpha_{i_1}, \cdots, \alpha_{i_p}\}^c)} \\ & \quad (\{D_{i_1}, \cdots, D_{i_p}\}^c)^{(\{\beta_{i_1}, \cdots, \beta_{i_p}\}^c)} \varphi(\{x_{i_1}, \cdots, x_{i_p}\}^c)|. \end{aligned}$$

For $\{i_1, \cdots, i_p\} \in M_p$ we denote by $C^a_{i_1,\cdots,i_p}$ the set of p-multiple sequences of functions $\{\varphi_{s_1\cdots s_p}(\{x_{i_1}, \cdots, x_{i_p}\}^c)\}_{s_1,\cdots,s_p=0,1,\cdots} \subset \mathcal{S}(R^{n-p})$ which satisfy

$$\sum_{s_1,\cdots,s_p=0}^{\infty} p_{(\{\alpha_{i_1},\cdots,\alpha_{i_p}\}^c),(\{\beta_{i_1},\cdots,\beta_{i_p}\}^c)}(\varphi_{s_1\cdots s_p}) a_{s_1} \cdots a_{s_p} < \infty$$

for all multi-idices α and β. We note that the sequence $\{\varphi_{s_1\cdots s_n}(\{x_1, \cdots, x_n\}^c)\}_{s_1,\cdots,s_n=0,1,\cdots}$ is a n-multiple seqence of numbers $\{b_{s_1\cdots s_n}\}_{s_1,\cdots,s_n=0,1,\cdots}$ and

$$p_{(\{\alpha_1,\cdots,\alpha_n\}^c),(\{\beta_1,\cdots,\beta_n\}^c)}(b_{s_1\cdots s_n}) = |b_{s_1\cdots s_n}|.$$

Therefore

$$C^a_{1,\cdots,n} = \{\{b_{s_1\cdots s_n}\}_{s_1,\cdots,s_n=0,1,\cdots} : \sum_{s_1,\cdots,s_n=0}^{\infty} |b_{s_1\cdots s_n}| a_{s_1} \cdots a_{s_n} < \infty\}.$$

The basic fact is

LEMMA 1. *Let $\{i_1, \cdots, i_p\} \in M_p$. If a p-multiple sequence of functions $\{\varphi_{s_1\cdots s_p}(\{x_{i_1}, \cdots, x_{i_p}\}^c)\}_{s_1,\cdots,s_p=0,1,\cdots}$ belongs to $C^a_{i_1,\cdots,i_p}$, then the p-multiple series*

$$\sum_{s_1,\cdots,s_p=0}^{\infty} \varphi_{s_1\cdots s_p}(\{x_{i_1}, \cdots, x_{i_p}\}^c) \theta^a_{s_1}(x_{i_1}) \cdots \theta^a_{s_p}(x_{i_p})$$

converges in $\mathcal{S}(R^n)$.

We introduce two kinds of tensor product functions associated with $\{\theta^a_j\}$. If a function f has the following form

$$f(x) = \sum_{s_1,\cdots,s_n=0}^{\infty} b_{s_1\cdots s_n} \theta^a_{s_1}(x_1) \cdots \theta^a_{s_n}(x_n) \tag{2.2}$$

where $\{b_{s_1\cdots s_n}\} \in C^a_{1,\cdots,n}$, then f is called a tensor product function of the first kind associated with $\{\theta^a_j\}$. If a function f which has the

form
(2.3)

$$f(x) = \sum_{p=1}^{n}(-1)^p \sum_{\{i_1,\cdots,i_p\}\in M_p} \sum_{s_1,\cdots,s_p=0}^{\infty} \lambda_{i_1,\cdots,i_p;s_1,\cdots,s_p}(\{x_{i_1},\cdots,x_{i_p}\}^c)$$

$$\theta^a_{s_1}(x_{i_1})\cdots\theta^a_{s_p}(x_{i_p})$$

satisfies the conditions

(i) $\{\lambda_{i_1,\cdots,i_p;s_1,\cdots,s_p}\}_{s_1,\cdots,s_p=0,1,\cdots} \in \mathcal{C}^a_{i_1,\cdots,i_p}$,

(ii) for $2 \le p \le n, \{i_1,\cdots,i_p\} \in M_p$ and $s_1,\cdots,s_p \ge 0$,

$$\lambda_{i_1,\cdots,i_p;s_1,\cdots,s_p}(\{x_{i_1},\cdots,x_{i_p}\}^c)$$

$$= \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \lambda_{i_\ell;s_\ell}(\{x_{i_\ell}\}^c) x_{i_1}^{s_1}\cdots\widehat{x_{i_\ell}^{s_\ell}}\cdots x_{i_p}^{s_p} dx_{i_1}\cdots\widehat{dx_{i_\ell}}\cdots dx_{i_p}$$

where $\ell = 1,\cdots,p$, then we call f a tensor product function of the second kind associated with $\{\theta^a_j\}$ where the symbol $\widehat{\ }$ indicates that the variable underneath is deleted. We denote by $\mathcal{T}^a_1(R^n)$ (resp. $\mathcal{T}^a_2(R^n)$) the class of all tensor product functions of the first kind (resp. the second kind) associated woth $\{\theta^a_j\}$. By Lemma 1, we see that $\mathcal{T}^a_1(R^n), \mathcal{T}^a_2(R^n) \subset \mathcal{S}(R^n)$. A fundamental property of the tensor product functions is the following.

LEMMA 2. (i) *Let f be a tensor product function of the first kind with the form (2.2). Then*

$$\int_{R^n} f(x_1,\cdots,x_n) x_1^{t_1}\cdots x_n^{t_n} dx_1\cdots dx_n = b_{t_1\cdots t_n}$$

for $t_1,\cdots,t_n \ge 0$.

(ii) *Let f be a tensor product function of the second kind with the form (2.3). Then*

$$\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} f(x_1,\cdots,x_n) x_{k_1}^{t_1}\cdots x_{k_q}^{t_q} dx_{k_1}\cdots dx_{k_q}$$

$$= \lambda_{k_1,\cdots,k_q;t_1,\cdots,t_q}(\{x_{k_1},\cdots x_{k_q}\}^c)$$

for $1 \leq q \leq n, \{k_1, \cdots, k_q\} \in M_q$ *and* $t_1, \cdots, t_q \geq 0$.

§3. Complementary spaces of the Lizorkin spaces

For $\{i_1, \cdots, i_p\} \in M_p, s_1, \cdots, s_p \geq 0$ and $\varphi \in \mathcal{S}(R^n)$, we define

$$\mu_{i_1,\cdots,i_p;s_1,\cdots,s_p}(\varphi)(\{x_{i_1}, \cdots, x_{i_p}\}^c)$$
$$= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \varphi(x_1, \cdots, x_n) x_{i_1}^{s_1} \cdots x_{i_p}^{s_p} dx_{i_1} \cdots dx_{i_p}.$$

Moreover, for $a \in \mathcal{A}$ and $\{i_1, \cdots, i_p\} \in M_p$ we set

$$\mathcal{S}^a_{i_1,\cdots,i_p}$$
$$= \{\varphi \in \mathcal{S}(R^n) : \{\mu_{i_1,\cdots,i_p;s_1,\cdots,s_p}(\varphi)(\{x_{i_1}, \cdots, x_{i_p}\}^c)\}_{s_1,\cdots,s_p=0,1,\cdots}$$
$$\in \mathcal{C}^a_{i_1,\cdots,i_p}\}$$

and

$$\mathcal{S}^a(R^n) = \cap_{p=1}^n \cap_{\{i_1,\cdots,i_p\}\in M_p} \mathcal{S}^a_{i_1,\cdots,i_p}(R^n).$$

If $\varphi \in \Phi_1(R^n)$, then $\mu_{1,\cdots,n;s_1,\cdots,s_n}(\varphi) = 0$ for $s_1, \cdots, s_n \geq 0$. Hence $\Phi_1(R^n) \subset \mathcal{S}^a_{1,\cdots,n}$ for any $a \in \mathcal{A}$. If $\varphi \in \Phi_2(R^n)$, then $\mu_{i_1,\cdots,i_p;s_1,\cdots,s_p}(\varphi) = 0$ for $1 \leq p \leq n, \{i_1, \cdots, i_p\} \in M_p$ and $s_1, \cdots, s_p \geq 0$. Hence $\Phi_2(R^n) \subset \mathcal{S}^a(R^n)$ for any $a \in \mathcal{A}$. Moreover, By Lemma 2 (i), (ii) and the definitions of $\mathcal{T}^a_1, \mathcal{T}^a_2$ we see that $\mathcal{T}^a_1(R^n) \subset \mathcal{S}^a_{1,\cdots,n}(R^n)$ and $\mathcal{T}^a_2(R^n) \subset \mathcal{S}^a(R^n)$. We introduce some operators. For $\varphi \in \mathcal{S}^a_{1,\cdots,n}(R^n)$, we define

$$T^a_{1,\cdots,n}\varphi(x) = \sum_{s_1,\cdots,s_n=0}^{\infty} \mu_{1,\cdots,n;s_1,\cdots,s_n}(\varphi)\theta^a_{s_1}(x_1) \cdots \theta^a_{s_n}(x_n)$$

and

$$U^a_{1,\cdots,n}\varphi = \varphi - T^a_{1,\cdots,n}\varphi.$$

Further, for $\varphi \in \mathcal{S}^a(R^n)$ we define

$$T^a_j\varphi(x) = \sum_{s=0}^{\infty} \mu_{j;s}(\varphi)(\{x_j\}^c)\theta^a_s(x_j), \qquad j = 1, \cdots, n$$

and

$$U_j^a\varphi = \varphi - T_j^a\varphi.$$

Moreover

$$U^a\varphi = U_1^a \cdots U_n^a\varphi.$$

We see that

$$U^a\varphi = \varphi - \sum_{p=1}^{n}(-1)^{p+1} \sum_{\{i_1,\cdots,i_p\}\in M_p} T^a_{i_1,\cdots,i_p}\varphi$$

where

$$T^a_{i_1,\cdots,i_p}\varphi(x)$$
$$= \sum_{s_1,\cdots,s_p=0}^{\infty} \mu_{i_1,\cdots,i_p;s_1,\cdots,s_p}(\varphi)(\{x_{i_1},\cdots,x_{i_p}\}^c)\theta^a_{s_1}(x_{i_1})\cdots\theta^a_{s_p}(x_{i_p}).$$

We put

$$T^a = \sum_{p=1}^{n}(-1)^{p+1} \sum_{\{i_1,\cdots,i_p\}\in M_p} T^a_{i_1,\cdots,i_p}.$$

We establish properties of these operators which are necessary for decompositions of $\mathcal{S}^a_{1,\cdots,n}(R^n)$ and $\mathcal{S}^a(R^n)$. About ranges of these operators we have

LEMMA 3. (i) *If* $\varphi \in \mathcal{S}^a_{1,\cdots,n}(R^n)$, *then* $T^a_{1,\cdots,n}\varphi, U^a_{1,\cdots,n}\varphi \in \mathcal{S}^a_{1,\cdots,n}(R^n)$.
(ii) *If* $\varphi \in \mathcal{S}^a(R^n)$, *then* $T^a\varphi, U^a\varphi \in \mathcal{S}^a(R^n)$.

LEMMA 4. (i) $\varphi \in \mathcal{S}^a_{1,\cdots,n}(R^n)$, *then* $U^a_{1,\cdots,n}\varphi \in \Phi_1(R^n)$.
(ii) *If* $\varphi \in \mathcal{S}^a(R^n)$, *then* $U^a\varphi \in \Phi_2(R^n)$.

LEMMA 5. (i) $\varphi \in \mathcal{S}^a_{1,\cdots,n}(R^n)$, *then* $T^a_{1,\cdots,n}\varphi \in \mathcal{T}^a_1(R^n)$.
(ii) *If* $\varphi \in \mathcal{S}^a(R^n)$, *then* $T^a\varphi \in \mathcal{T}^a_2(R^n)$.

These operators become the identity operators on each proper subspace. In fact we have

LEMMA 6. (i) $\varphi \in \Phi_1(R^n)$, *then* $U^a_{1,\cdots,n}\varphi = \varphi$.

(ii) *If* $\varphi \in \Phi_2(R^n)$, *then* $U^a\varphi = \varphi$

LEMMA 7. $\varphi \in \mathcal{T}_1^a(R^n)$, *then* $T^a_{1,\cdots,n}\varphi = \varphi$.
(ii) *If* $\varphi \in \mathcal{T}_2^a(R^n)$, *then* $T^a\varphi = \varphi$.

Now we give decompositions of $\mathcal{S}^a_{1,\cdots,n}(R^n)$ and $\mathcal{S}^a(R^n)$.

THEOREM 8. (i) $\mathcal{S}^a_{1,\cdots,n}(R^n) = \Phi_1(R^n) \oplus \mathcal{T}_1^a(R^n)$.
(ii) $\mathcal{S}^a(R^n) = \Phi_2(R^n) \oplus \mathcal{T}_2^a(R^n)$.

In order to give a decomposition of $\mathcal{S}(R^n)$, we need a relation between $\mathcal{S}(R^n)$ and $\mathcal{S}^a(R^n)$ (or $\mathcal{S}^a_{1,\cdots,n}(R^n)$). We have

LEMMA 9. $\mathcal{S}(R^n) = \cup_{a\in\mathcal{A}}\mathcal{S}^a(R^n)$, $\mathcal{S}(R^n) = \cup_{a\in\mathcal{A}}\mathcal{S}^a_{1,\cdots,n}(R^n)$.

Taking Lemma 9 into account we put

$$\mathcal{T}_1(R^n) = \cup_{a\in\mathcal{A}}\mathcal{T}_1^a(R^n), \quad \mathcal{T}_2(R^n) = \cup_{a\in\mathcal{A}}\mathcal{T}_2^a(R^n).$$

Then we have

TEOREM 10. (i) $\mathcal{S}(R^n) = \Phi_1(R^n) \oplus \mathcal{T}_1(R^n)$.
(ii) $\mathcal{S}(R^n) = \Phi_2(R^n) \oplus \mathcal{T}_2(R^n)$.

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