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Continuity properties and vanishing exponential integrability of Riesz potentials of Orlicz functions

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1 Introduction and statement of results

For $0 < \alpha < n$ and a locally integrable function f on $\mathbf{R}^n$, we define the Riesz potential $U_\alpha f$ of order α by

$$U_\alpha f(x) = \int_{\mathbf{R}^n} |x-y|^{\alpha-n} f(y)dy.$$

Here it is natural to assume that $U_\alpha|f| \not\equiv \infty$, which is equivalent to

$$\int_{\mathbf{R}^n} (1+|y|)^{\alpha-n}|f(y)|dy < \infty. \tag{1.1}$$

In the present paper, we treat functions f satisfying an Orlicz condition :

$$\int_{\mathbf{R}^n} \Phi_{p,\varphi}(|f(y)|)dy < \infty. \tag{1.2}$$

Here $\Phi_{p,\varphi}(r)$ is a positive nondecreasing function on the interval $(0,\infty)$ of the form

$$\Phi_{p,\varphi}(r) = r^p\varphi(r),$$

where $p > 1$ and $\varphi(r)$ is a positive monotone function on $[0,\infty)$ which is of logarithmic type; that is, there exists $c_1 > 0$ such that

(φ1) $\qquad c_1^{-1}\varphi(r) \le \varphi(r^2) \le c_1\varphi(r) \qquad$ whenever $r > 0$.

We set

$$\Phi_{p,\varphi}(0) = 0,$$

because we will see in the proof of Lemma 2.1 below that

$$\lim_{r\to 0+} \Phi_{p,\varphi}(r) = 0 = \Phi_{p,\varphi}(0).$$

For an open set $G \subset \mathbf{R}^n$, we denote by $L^{\Phi_{p,\varphi}}(G)$ the family of all locally integrable functions g on G such that

$$\int_G \Phi_{p,\varphi}(|g(x)|)\, dx < \infty,$$

and define

$$\|g\|_{\Phi_{p,\varphi}} = \|g\|_{\Phi_{p,\varphi},G} = \inf\left\{\lambda > 0 : \int_G \Phi_{p,\varphi}(|g(x)|/\lambda)\, dx \le 1\right\}.$$

This is a quasi-norm in $L^{\Phi_{p,\varphi}}(G)$.

Our first aim in the present paper is to establish integral inequalities for Riesz potentials of functions in $L^{\Phi_{p,\varphi}}$. For this purpose, if $1 < p < n/\alpha$, then we set

$$\varphi_p^*(r) = \left[\int_0^r \{t^{\alpha p - n}\varphi(t)\}^{-p'/p} t^{-1} dt\right]^{1/p'} \quad \text{for } r \ge 0,$$

where $1/p + 1/p' = 1$; if $p = n/\alpha > 1$, then we set

$$\varphi_p^*(r) = \left[\int_1^r \{\varphi(t)\}^{-p'/p} t^{-1} dt\right]^{1/p'} \quad \text{for } r \ge 2,$$

and extend it to be a (strictly) increasing continuous function on $[0,\infty)$ such that $\varphi_p^*(t) = (t/2)\varphi_p^*(2)$ for $t \in [0,2)$. Following Alberico and Cianchi [3], we consider the Sobolev conjugate $\Psi_{p,\varphi}$ of $\Phi_{p,\varphi}$ defined by

$$\Psi_{p,\varphi}(r) = (\psi_n \circ (\varphi_p^*)^{-1})(r) \qquad \text{for } r \ge 0,$$

where $\psi_n(r) = r^n$ and $(\varphi_p^*)^{-1}$ is the inverse of the function φ_p^*. Note that $\Psi_{p,\varphi}(r)$ is continuous on $[0,\infty)$ and $\Psi_{p,\varphi}(0) = 0$.

As an extension of Alberico and Cianchi [3, Theorem 2.3], we state our first result in the following.

THEOREM A. *Let $\alpha p \le n$ and G be a bounded open set in $\mathbf{R}^n$. Then there exists $\varepsilon_0 > 0$ such that*

$$\int_G \Psi_{p,\varphi}(\varepsilon_0 U_\alpha |f|(x))\, dx \le 1$$

whenever f is a locally integrable function on G such that $\|f\|_{\Phi_{p,\varphi}} \le 1$.

Cianchi [2, Theorem 2] gave a necessary and sufficient condition that the operator $f \longrightarrow U_\alpha f$ is bounded from one Orlicz space L^Φ to another Orlicz space L^Ψ; but our statement is straightforward and simple. Further Edmunds and Evans [4, Theorems 3.6.10, 3.6.16] discussed the boundedness of Bessel potentials in Lorenz-Karamata space setting.

Since our function $\Phi_{p,\varphi}$ may not be convex, for the reader's convenience, we give a proof of Theorem A different from Cianchi [2] in the next section.

REMARK 1.1 Theorem A implies that

$$\|U_\alpha f\|_{\Psi_{p,\varphi}} \le \varepsilon_0^{-1}\|f\|_{\Phi_{p,\varphi}} \qquad \text{whenever } f \in L^{\Phi_{p,\varphi}}(G),$$

where the quasi-norm $\|\cdot\|_{\Psi_{p,\varphi}}$ is defined in the same way as $\|\cdot\|_{\Phi_{p,\varphi}}$.

EXAMPLE 1.2 Consider $\Phi_{p,q}(r) = r^p(\log r)^q$ for large $r > 0$, where $p = n/\alpha > 1$ and $q \le p-1$. If $q < p-1$, then

$$\Psi_{p,q}(r) \ge C \exp(nr^{p/(p-1-q)})$$

and if $q = p-1$, then

$$\Psi_{p,q}(r) \ge C \exp(n \exp(r^{p'}))$$

for $r \ge 1$. Hence we have the exponential integrability obtained by Edmunds, Gurka and Opic [5, Theorem 4.6], [6, Theorems 3.1 and 3.2] and the authors [12, Theorems A and B].

COROLLARY 1.3 *Let $\alpha p = n$ and G be a bounded open set in $\mathbf{R}^n$. Let $\Phi_{p,q}(r) = r^p(\log r)^q$ for large $r > 0$.*

(1) If $q < p-1$, then there exists $\varepsilon_0 > 0$ such that

$$\int_G \{\exp(\varepsilon_0 U_\alpha |f|(x)^\beta) - 1\} dx \le 1$$

whenever f is a locally integrable function on G such that $\|f\|_{\Phi_{p,q}} \le 1$, where $\beta = p/(p-1-q)$.

(2) If $q = p-1$, then there exists $\varepsilon_0 > 0$ such that

$$\int_G \{\exp(\exp(\varepsilon_0 U_\alpha |f|(x)^\beta) - e\} dx \le 1$$

whenever f is a locally integrable function on G such that $\|f\|_{\Phi_{p,q}} \le 1$, where $\beta = p/(p-1)$.

In the case $q > p-1$, $U_\alpha f$ is shown to be continuous in G; see Remark 1.5. Denote by $p^\sharp$ the Sobolev conjugate of p which is defined by

$$\frac{1}{p^\sharp} = \frac{1}{p} - \frac{\alpha}{n} > 0.$$

We also obtain Sobolev's type inequality for Riesz potentials in the following:

COROLLARY 1.4 *Let $\alpha p < n$. Then*

$$\int_{\mathbf{R}^n} \{U_\alpha |f|(x) \varphi(U_\alpha |f|(x))^{1/p}\}^{p^\sharp} \, dx \le C$$

whenever f is a locally integrable function on $\mathbf{R}^n$ such that $\|f\|_{\Phi_{p,\varphi}} \le 1$, where C is a positive constant independent of f.

For a measurable function u on $\mathbf{R}^n$, we define the integral mean over a measurable set $E \subset \mathbf{R}^n$ of positive measure by

$$\fint_E u(x)\,dx = \frac{1}{|E|}\int_E u(x)\,dx.$$

As an application of Theorem A, we discuss continuity properties for Riesz potentials of functions in $L^{\Phi_{p,\varphi}}(\mathbf{R}^n)$, as an extension of Adams and Hurri-Syrjänen [1, Theorem 1.6] and the authors [14, Theorems A and B].

Our main result is now stated as follows:

THEOREM B. *Let f be a locally integrable function on $\mathbf{R}^n$ satisfying* (1.1) *and* (1.2). *Set*

$$\begin{aligned} E_\infty &= \{x \in \mathbf{R}^n : \int_{\mathbf{R}^n} |x-y|^{\alpha-n}|f(y)|\,dy = \infty\}, \\ E_* &= \{x \in \mathbf{R}^n : \limsup_{r\to 0} r^{\alpha p-n}\varphi(r^{-1})^{-1}\int_{B(x,r)} \Phi_{p,\varphi}(|f(y)|)\,dy > 0\}, \\ E^* &= \{x \in \mathbf{R}^n : \limsup_{r\to 0} r^{\alpha p-n}\int_{B(x,r)} \Phi_{p,\varphi}(|f(y)|)\,dy > 0\}. \end{aligned}$$

If $x_0 \in \mathbf{R}^n \setminus (E_\infty \cup E_* \cup E^*)$, then

$$\lim_{r\to 0+}\fint_{B(x_0,r)} \Psi_{p,\varphi}(A|U_\alpha f(x) - U_\alpha f(x_0)|)\,dx = 0 \tag{1.3}$$

holds for all $A > 0$.

We discuss the size of the exceptional sets after proving this theorem, in the final section.

REMARK 1.5 Suppose

$$\int_1^\infty \{t^{\alpha p-n}\varphi(t)\}^{-p'/p}t^{-1}dt < \infty \tag{1.4}$$

and set

$$\varphi_p(r) = \left(\int_r^\infty \{t^{\alpha p-n}\varphi(t)\}^{-p'/p}t^{-1}dt\right)^{1/p'}.$$

Then it is known (see [9, Theorem 1] and [10, Corollary 3.1]) that $U_\alpha f$ is continuous on $\mathbf{R}^n$ and

$$|U_\alpha f(x) - U_\alpha f(x_0)| = o(\varphi_p(1/|x-x_0|)) \qquad \text{as } x \to x_0$$

for all $x_0 \in \mathbf{R}^n$, whenever f satisfies (1.1) and (1.2). On the contrary, if (1.4) does not hold, then we can find an f satisfying (1.1) and (1.2) such that $U_\alpha f$ is not continuous (see [13, Remark 3.3]).

2 Proof of Theorem A

In spite of the fact that $\Phi_{p,\varphi}$ may not be convex, Theorem A must be a consequence of Cianchi [2] in spirit. But we here give a proof of Theorem A, because our method is straightforward and several materials are also needed for a proof of our main Theorem B. In fact, our proof is based on the boundedness of maximal functions, by use of the methods in the paper by Hedberg [7].

Throughout this paper, let C, C_1, C_2, ... denote various constants independent of the variables in question.

First we collect properties which follow from condition (φ1) (see [11] and [13]).

(φ2) φ satisfies the doubling condition, that is, there exists $c > 1$ such that

$$c^{-1}\varphi(r) \le \varphi(2r) \le c\varphi(r) \qquad \text{whenever } r > 0.$$

(φ3) For each $\gamma > 0$, there exists $c = c(\gamma) \ge 1$ such that

$$c^{-1}\varphi(r) \le \varphi(r^\gamma) \le c\varphi(r) \qquad \text{whenever } r > 0.$$

(φ4) If $\gamma > 0$, then there exists $c = c(\gamma) \ge 1$ such that

$$s^\gamma\varphi(s) \le ct^\gamma\varphi(t) \qquad \text{whenever } 0 < s < t.$$

(φ5) If $\gamma > 0$, then there exists $c = c(\gamma) \ge 1$ such that

$$t^{-\gamma}\varphi(t) \le cs^{-\gamma}\varphi(s) \qquad \text{whenever } 0 < s < t.$$

LEMMA 2.1 *Let $1 < p_1 < p < p_2$. Then there exists $C > 1$ such that*

$$C^{-1}A^{p_1}\Phi_{p,\varphi}(r) \le \Phi_{p,\varphi}(Ar) \le CA^{p_2}\Phi_{p,\varphi}(r)$$

whenever $r > 0$ and $A > 1$.

COROLLARY 2.2 *Let $\alpha p \le n$ and $1 < p_1 < p < p_2$. Let G be a bounded open set in $\mathbf{R}^n$. Then there exists a positive constant C such that*

$$C^{-1}\{\|f\|_{\Phi_{p,\varphi}}\}^{p_2} \le \int_G \Phi_{p,\varphi}(|f(y)|)dy \le C\{\|f\|_{\Phi_{p,\varphi}}\}^{p_1}$$

whenever f is a locally integrable function on G such that $\|f\|_{\Phi_{p,\varphi}} \le 1$.

LEMMA 2.3 (cf. [13, Lemma 2.5]) *Let G be a bounded open set in $\mathbf{R}^n$ and $\varepsilon > 0$. Let p_0 be given so that $p_0 = p$ if φ is nondecreasing, and $1 < p_0 < p$ if φ is nonincreasing. If $x \in G$, $\delta > 0$ and f is a nonnegative measurable function on G, then*

$$\int_{G-B(x,\delta)} |x-y|^{\alpha-n} f(y)dy \le C\varphi_p^*(\delta^{-1}) \left\{ \varepsilon + c(\varepsilon) \left(\int_G \Phi_{p,\varphi}(f(y))dy \right)^{1/p_0} \right\},$$

where C and $c(\varepsilon)$ are positive constants such that C is independent of ε but $c(\varepsilon)$ may depend on ε. In case $\alpha p < n$,

$$\int_{\mathbf{R}^n - B(x,\delta)} |x-y|^{\alpha-n} f(y)dy \le C\varphi_p^*(\delta^{-1}) \left\{ 1 + \left(\int_{\mathbf{R}^n} \Phi_{p,\varphi}(f(y))dy \right)^{1/p_0} \right\},$$

for all $x \in \mathbf{R}^n$ and nonnegative measurable functions f on $\mathbf{R}^n$.

For a locally integrable function f on $\mathbf{R}^n$, define the maximal function by

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{G \cap B(x,r)} |f(y)|dy,$$

where $|B(x,r)|$ denotes the n-dimensional Lebesgue measure of the ball $B(x,r)$ centered at x of radius $r > 0$.

We denote by $c(\varepsilon)$ various constants which may depend on ε.

LEMMA 2.4 *Let $\alpha p = n$ and G be a bounded open set in $\mathbf{R}^n$. Then, for each $\eta > 0$, there exist $\varepsilon_0 > 0$ and $c(\varepsilon_0) > 0$ such that*

$$(\varphi_p^*)^{-1}(U_\alpha f(x)) \le c(\varepsilon_0)\{\Phi_{p,\varphi}(Mf(x))\}^{1/n} + \eta$$

for all nonnegative measurable functions f on G satisfying $\int_G \Phi_{p,\varphi}(f(y))dy \le \varepsilon_0$.

In case $\alpha p < n$, we find from (φ4) and (φ5) that

$$C^{-1} r^{(n-\alpha p)/p} \{\varphi(r)\}^{-1/p} \le \varphi_p^*(r) \le C r^{(n-\alpha p)/p} \{\varphi(r)\}^{-1/p}, \tag{2.1}$$

so that

$$C^{-1} r^{p/(n-\alpha p)} \{\varphi(r)\}^{1/(n-\alpha p)} \le (\varphi_p^*)^{-1}(r) \le C r^{p/(n-\alpha p)} \{\varphi(r)\}^{1/(n-\alpha p)} \tag{2.2}$$

for $r > 0$.

LEMMA 2.5 *Let $\alpha p < n$. Then*

$$(\varphi_p^*)^{-1}(U_\alpha f(x)) \le C\{\Phi_{p,\varphi}(Mf(x))\}^{1/n}$$

for all nonnegative measurable functions f on $\mathbf{R}^n$ satisfying $\int_{\mathbf{R}^n} \Phi_{p,\varphi}(f(y))dy \le 1$.

Note that

$$C^{-1}\frac{\Phi_{p,\varphi}(t)}{t} \le \int_0^t s^{-1} d\Phi_{p,\varphi}(s) \le C\frac{\Phi_{p,\varphi}(t)}{t} \tag{2.3}$$

for all $t > 0$ by (φ4) and (φ5).

The next lemma is an extension of Stein [15, Chapter 1], whose proof will be done along the same lines as in Stein [15, Chapter 1].

LEMMA 2.6 *For a locally integrable function f on $\mathbf{R}^n$,*

$$\int \Phi_{p,\varphi}(Mf(x))\, dx \le C \int \Phi_{p,\varphi}(|f(x)|)\, dx.$$

PROOF OF THEOREM A. We give a proof of Theorem A only in case $\alpha p = n$. With the aid of Lemma 2.4, for $\eta > 0$ we find $\varepsilon_1 > 0$ such that

$$(\varphi_p^*)^{-1}(U_\alpha f(x)) \le C(\varepsilon_1)\{\Phi_{p,\varphi}(Mf(x))\}^{1/n} + \eta$$

for all nonnegative measurable functions f on G satisfying $\int_G \Phi_{p,\varphi}(f(y))dy \le \varepsilon_1$. Hence, in view of Lemma 2.6, we obtain

$$\begin{aligned}\int_G \Psi_{p,\varphi}(U_\alpha f(x))dx &\le C(\varepsilon_1)\int_G \Phi_{p,\varphi}(Mf(x))dx + C\eta^n|G| \\ &\le C(\varepsilon_1)\int_G \Phi_{p,\varphi}(f(y))dy + C\eta^n|G|\end{aligned}$$

for all nonnegative measurable functions f on G satisfying $\int_G \Phi_{p,\varphi}(f(y))dy \le \varepsilon_1$. Now, letting $C\eta^n|G| \le 1/2$ and using Corollary 2.2, we find $0 < \varepsilon_0 < \varepsilon_1$ such that

$$\int_G \Psi_{p,\varphi}(U_\alpha f(x))dx \le 1$$

for all nonnegative measurable functions f on G satisfying $\|f\|_{\Phi_{p,\varphi}} \le \varepsilon_0$. This implies that

$$\int_G \Psi_{p,\varphi}(\varepsilon_0 U_\alpha f(x))dx \le 1$$

for all nonnegative measurable functions f on G satisfying $\|f\|_{\Phi_{p,\varphi}} \le 1$. Now the proof is completed. □

3 Proof of Theorem B

For a proof of Theorem B, we prepare a series of lemmas.

LEMMA 3.1 *Let $\alpha p \le n$. Then there exist $\beta > 1$ and $C > 0$ such that*

$$\varphi_p^*(Ar) \le CA^\beta \varphi_p^*(r)$$

for all $r > 0$ and $A > 1$.

With the aid of Lemma 3.1, we establish the following result.

LEMMA 3.2 *There exist $C > 1$ and $0 < \varepsilon_0 < 1$ such that*

$$\int_G \Psi_{p,\varphi}(U_\alpha |f|(y))dy \le C\{\|f\|_{\Phi_{p,\varphi}}\}^{n/\beta}$$

whenever f is a locally integrable function on G such that $\|f\|_{\Phi_{p,\varphi}} \le \varepsilon_0$, where β is given in Lemma 3.1.

We further need the following result.

LEMMA 3.3 *Let $\alpha p \le n$. For a nonnegative measurable function f on $\mathbf{R}^n$ satisfying (1.2), set*

$$E_* = \left\{x \in \mathbf{R}^n : \limsup_{r\to 0+} r^{\alpha p - n}\varphi(r^{-1})^{-1} \int_{B(x,r)} \Phi_{p,\varphi}(f(y))dy > 0\right\}$$

and

$$E^* = \left\{x \in \mathbf{R}^n : \limsup_{r\to 0+} r^{\alpha p - n} \int_{B(x,r)} \Phi_{p,\varphi}(f(y))dy > 0\right\}.$$

If $x_0 \in \mathbf{R}^n \setminus (E_* \cup E^*)$, then

$$\lim_{r\to 0} r^{-n} \int_{B(x_0,r)} \Phi_{p,\varphi}(r^\alpha f(y))\, dy = 0.$$

For $x_0 \in \mathbf{R}^n$ and $r > 0$, set $f_{x_0,r}(w) = r^\alpha f(x_0 + rw)\chi_{B(0,1)}$, where χ_E denotes the characteristic function of E. Then note that

$$\begin{aligned} \int_{B(x_0,r)} |x-y|^{\alpha-n} f(y)dy &= \int_{B(0,1)} |z-w|^{\alpha-n}\left(r^\alpha f(x_0+rw)\right)dw \\ &= U_\alpha f_{x_0,r}(z) \end{aligned} \tag{3.1}$$

for $x = x_0 + rz$.

We are now ready to prove our main Theorem B.

PROOF OF THEOREM B. For a nonnegative measurable function f on $\mathbf{R}^n$ satisfying (1.1) and (1.2), it suffices to show that (1.3) holds for $x_0 \in \mathbf{R}^n \setminus (E_\infty \cup E_* \cup E^*)$. Write

$$\begin{aligned} U_\alpha f(x) - U_\alpha f(x_0) &= \int_{B(x_0,2|x-x_0|)} |x-y|^{\alpha-n} f(y)dy \\ &\quad + \int_{\mathbf{R}^n\setminus B(x_0,2|x-x_0|)} |x-y|^{\alpha-n} f(y)dy - U_\alpha f(x_0) \\ &= U_1(x) + U_2(x). \end{aligned}$$

If $y \in \mathbf{R}^n - B(x_0, 2|x-x_0|)$, then $|x_0 - y| \le 2|x-y|$, so that, since $U_\alpha f(x_0) < \infty$, we can apply Lebesgue's dominated convergence theorem to obtain

$$\lim_{x \to x_0} U_2(x) = 0. \tag{3.2}$$

Since $(\varphi_p^*)^{-1}$ is nondecreasing, we have

$$\begin{aligned}(\varphi_p^*)^{-1}(A|U_\alpha f(x) - U_\alpha f(x_0)|) &\le (\varphi_p^*)^{-1}(AU_1(x) + A|U_2(x)|) \\ &\le (\varphi_p^*)^{-1}(2AU_1(x)) + (\varphi_p^*)^{-1}(2A|U_2(x)|),\end{aligned}$$

so that

$$\begin{aligned}\Psi_{p,\varphi}(A|U_\alpha f(x) - U_\alpha f(x_0)|) &\le C\psi_n((\varphi_p^*)^{-1}(2AU_1(x))) + C\psi_n((\varphi_p^*)^{-1}(2A|U_2(x)|)) \\ &= C\Psi_{p,\varphi}(2AU_1(x)) + C\Psi_{p,\varphi}(2A|U_2(x)|).\end{aligned}$$

In view of (3.2), we have

$$\lim_{x \to x_0} \Psi_{p,\varphi}(2A|U_2(x)|) = 0.$$

Note that

$$U_1(x) \le \int_{B(x_0,r)} |x-y|^{\alpha-n} f(y)\, dy = U_\alpha f_r(x)$$

for $x \in B(x_0, r/2)$, where $f_r = f\chi_{B(x_0,r)}$. Hence, we have only to show that

$$\lim_{r \to 0+} ⨍_{B(x_0,r)} \Psi_{p,\varphi}(2AU_\alpha f_r(x))dx = 0.$$

Note that $U_\alpha(f_r)(x) = U_\alpha(f_r)_{x_0,r}(z)$ for $x = x_0 + rz$ and

$$\int_{B(0,1)} \Phi_{p,\varphi}((f_r)_{x_0,r}(w))\, dw = r^{-n} \int_{B(x_0,r)} \Phi_{p,\varphi}(r^\alpha f(y))\, dy$$

which tends to zero as $r \to +0$ by Lemma 3.3. Hence we have by Lemma 3.2 and Corollary 2.2

$$\begin{aligned}⨍_{B(x_0,r)} \Psi_{p,\varphi}(2AU_1(x))dx &\le ⨍_{B(0,1)} \Psi_{p,\varphi}(U_\alpha(2A(f_r)_{x_0,r})(z))dz \\ &\le C\{\|2A(f_r)_{x_0,r}\|_{\Phi_{p,\varphi}}\}^{n/\beta} \\ &\le C(2A)^{n/\beta}\left(\int_{B(0,1)} \Phi_{p,\varphi}((f_r)_{x_0,r}(z))dz\right)^{n/(p_2\beta)} \\ &\le C(2A)^{n/\beta}\left(r^{-n}\int_{B(x_0,r)} \Phi_{p,\varphi}(r^\alpha f(y))dy\right)^{n/(p_2\beta)}.\end{aligned}$$

Consequently it follows from Lemma 3.3 that the left hand side tends to zero as $r \to 0+$. Thus the proof is completed. □

4 Size of exceptional sets

To evaluate the size of exceptional sets in Theorem B, we introduce the notion of capacity. For a set $E \subset \mathbf{R}^n$ and an open set $G \subset \mathbf{R}^n$, we define

$$C_{\alpha,\Phi_{p,\varphi}}(E;G) = \inf_f \int_G \Phi_{p,\varphi}(f(y))dy,$$

where the infimum is taken over all nonnegative measurable functions f on $\mathbf{R}^n$ such that f vanishes outside G and $U_\alpha f(x) \geq 1$ for every $x \in E$ (cf. Meyers [8] and the first author [11]). When $\varphi \equiv 1$, we write $C_{\alpha,p}$ for $C_{\alpha,\Phi_{p,\varphi}}$. We say that E is of $C_{\alpha,\Phi_{p,\varphi}}$-capacity zero, written as $C_{\alpha,\Phi_{p,\varphi}}(E) = 0$, if

$$C_{\alpha,\Phi_{p,\varphi}}(E \cap G; G) = 0 \qquad \text{for every bounded open set } G.$$

The following can be obtained readily from the definition of $C_{\alpha,\Phi_{p,\varphi}}$; see [11, Theorem 1.1, Chapter 2].

LEMMA 4.1 *For a nonnegative measurable function f on $\mathbf{R}^n$ satisfying* (1.1) *and* (1.2), *set*

$$E_\infty = \{x \in \mathbf{R}^n : \int |x-y|^{\alpha-n} f(y)\, dy = \infty\}.$$

Then

$$C_{\alpha,\Phi_{p,\varphi}}(E_\infty) = 0.$$

As in the proof of Lemma 7.3 and Corollary 7.2 in [10], we can prove the following results.

LEMMA 4.2 *Let $\alpha p \leq n$. For a nonnegative measurable function f on $\mathbf{R}^n$ satisfying* (1.2), *set*

$$E_* = \{x \in \mathbf{R}^n : \limsup_{r\to 0} r^{\alpha p-n}\varphi(r^{-1})^{-1} \int_{B(x,r)} \Phi_{p,\varphi}(f(y))\, dy > 0\}.$$

Then $C_{\alpha,\Phi_{p,\varphi}}(E_*) = 0$.

LEMMA 4.3 *For a nonnegative measurable function f in $L^p(\mathbf{R}^n)$, set*

$$E^* = \{x \in \mathbf{R}^n : \limsup_{r\to 0} r^{\alpha p-n} \int_{B(x,r)} f(y)^p\, dy > 0\}.$$

If $\alpha p < n$, then $C_{\alpha,p}(E^*) = 0$; and if $\alpha p = n$, then E^* is empty.

Finally, in view of Theorem B and Lemmas 4.1 - 4.3, we establish the following result.

COROLLARY 4.4 *Let $\alpha p \le n$. If f is a locally integrable function on $\mathbf{R}^n$ satisfying* (1.1) *and* (1.2), *then*

$$\lim_{r\to 0+} \fint_{B(x_0,r)} \Psi_{p,\varphi}(A|U_\alpha f(x) - U_\alpha f(x_0)|)\, dx = 0$$

holds for all $A > 0$ and all $x_0 \in \mathbf{R}^n \setminus E$, where $C_{\alpha,\Phi_{p,\varphi}}(E) = 0$ when $\alpha p = n$ or φ is nonincreasing and $C_{\alpha,p}(E) = 0$ when $\alpha p < n$ and φ is nondecreasing.

COROLLARY 4.5 *Let $\alpha p = n$ and $\varphi(r)$ be of the form $(\log r)^{q_1}(\log\log r)^{q_2}$ for large $r > 0$, where q_1 and q_2 are real numbers. Set $\Phi(r) = \Phi_{p,\varphi}(r) = r^p\varphi(r)$. Suppose f is a locally integrable function on $\mathbf{R}^n$ satisfying* (1.1) *and* (1.2).

(1) If $q_1 < p-1$, then

$$\lim_{r\to 0+} \fint_{B(x_0,r)} \{\exp(A|U_\alpha f(x)-U_\alpha f(x_0)|^{\beta_1}(\log(1+|U_\alpha f(x)-U_\alpha f(x_0)|))^{\beta_2})-1\}dx = 0$$

for every $A > 0$ and every $x_0 \in \mathbf{R}^n$ except in a set of $C_{\alpha,\Phi_{p,\varphi}}$-capacity zero, where $\beta_1 = p/(p-1-q_1)$ and $\beta_2 = q_2/(p-1-q_1)$.

(2) If $q_1 > p-1$, then $U_\alpha f$ is continuous on $\mathbf{R}^n$ and

$$|U_\alpha f(x)-U_\alpha f(x_0)| = o((\log(1/|x-x_0|))^{1/\beta_1}(\log\log(1/|x-x_0|))^{-q_2/p}) \qquad \text{as } x \to x_0$$

for every $x_0 \in \mathbf{R}^n$.

For the continuity of $U_\alpha f$ (case (2)), see Remark 1.5. The case $q_1 = p-1$ is treated as follows:

COROLLARY 4.6 *Let $\alpha p = n$, $\varphi(r) = \varphi_{p-1,q}(r) = (\log r)^{p-1}(\log\log r)^q$ for large $r > 0$ and $\Phi_{p,\varphi}(r) = r^p\varphi(r)$. Suppose f is a locally integrable function on $\mathbf{R}^n$ satisfying* (1.1) *and* (1.2).

(1) If $q < p-1$, then

$$\lim_{r\to 0+} \fint_{B(x_0,r)} \{\exp(\exp(A|U_\alpha f(x) - U_\alpha f(x_0)|^{\beta})) - e\}dx = 0$$

for every $A > 0$ and every $x_0 \in \mathbf{R}^n$ except in a set of $C_{\alpha,\Phi_{p,\varphi}}$-capacity zero, where $\beta = p/(p-1-q)$.

(2) If $q = p-1$, then

$$\lim_{r\to 0+} \fint_{B(x_0,r)} \{\exp(\exp(\exp(A|U_\alpha f(x) - U_\alpha f(x_0)|^{\beta}))) - e^e\}dx = 0$$

for every $A > 0$ and every $x_0 \in \mathbf{R}^n$ except in a set of $C_{\alpha,\Phi_{p,\varphi}}$-capacity zero, where $\beta = p/(p-1)$.

(3) If $q > p-1$, then $U_\alpha f$ is continuous on $\mathbf{R}^n$ and

$$|U_\alpha f(x) - U_\alpha f(x_0)| = o((\log(\log(1/|x-x_0|)))^{(p-1-q)/p}) \qquad \text{as } x \to x_0$$

for every $x_0 \in \mathbf{R}^n$.

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