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Continuity properties for logarithmic potentials of functions in Morrey spaces of variable exponent

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1 Introduction

Let $\mathbf{R}^n$ be the n-dimensional Euclidean space and $B(x,r)$ denote the open ball centered at x with radius r.

Following Kováčik and Rákosník [1], we consider a positive continuous function $p(\cdot)$ on $\mathbf{R}^n$, which is called a variable exponent. For $0 \le \nu \le n$, a real number β and a locally integrable function f on an open set Ω in $\mathbf{R}^n$, we define the $L^{p(\cdot),\nu,\beta}$ norm by

$$\|f\|_{p(\cdot),\nu,\beta,\Omega} = \inf\left\{\lambda > 0 : \sup_{x\in\Omega, r>0} r^{-\nu}(\log(2+r^{-1}))^{\beta} \int_{\Omega\cap B(x,r)} \left|\frac{f(y)}{\lambda}\right|^{p(y)} dy \le 1\right\},$$

where $\beta \ge 0$ when $\nu = 0$ and $\beta \le 0$ when $\nu = n$. We denote by $L^{p(\cdot),\nu,\beta}(\Omega)$ the space of all measurable functions f on Ω with $\|f\|_{p(\cdot),\nu,\beta,\Omega} < \infty$. This space $L^{p(\cdot),\nu,\beta}(\Omega)$ is referred to as a generalized Morrey space of variable exponent. In particular, $L^{p(\cdot),0,0}(\Omega)$ is equal to the generalized Lebesgue space $L^{p(\cdot)}(\Omega)$.

In the second section, we consider a function $p(\cdot)$ satisfying a log-Hölder condition such that $p(0) = p_0 \ge 1$,

$$p(r) = p_0 + \frac{a\log(\log(1/r))}{\log(1/r)} + \frac{b}{\log(1/r)}$$

for $0 < r < r_0$ and $p(r) = p(r_0)$ for $r \ge r_0$, where the numbers a, b and r_0 are chosen so that $p(r)$ is nondecreasing on $[0, r_0)$. For a compact set K in a bounded open set G, we define

$$K(r) = \{x \in G : \delta_K(x) < r\},$$

where $\delta_K(x)$ denotes the distance of x from K. For $0 \le \alpha \le n$, we say that the Minkowski $(n-\alpha)$-content of K is finite if

$$|K(r)| \le Cr^{\alpha} \qquad \text{for small } r > 0,$$

where $|E|$ denotes the Lebesgue measure of a set E. Note here that if K is a singleton, then its Minkowski 0-content is finite, and if K is a spherical surface, then its Minkowski $(n-1)$-content is finite. As another examples of K, we

may consider fractal type sets like Cantor sets or Koch curves. Now we define a variable exponent $p(\cdot)$ by

$$p(x) = p(\delta_K(x))$$

for $x \in G$; set $p(x) = p_0$ on K.

In the case $\nu = 0$ and $\beta = 0$, we know the following fact (see [3, Remark 4.4]): if the Minkowski $(n-\alpha)$-content of K is finite, then there exists a constant $C > 0$ such that

$$\int_G |f(x)|^{p_0} (\log(1 + |f(x)|))^{a\alpha/p_0} \, dx \leq C$$

for all measurable functions f on G with $\|f\|_{p(\cdot),G} \leq 1$. Our first aim in this paper is to give an extension of the above fact to the generalized Morrey space of variable exponent.

In the third section, we consider the logarithmic potential of a locally integrable function f on $\mathbf{R}^n$, which is defined by

$$Lf(x) = \int (\log(1/|x-y|)) \, f(y) dy.$$

Here it is natural to assume that

$$\int (\log(2 + |y|)) |f(y)| dy < \infty, \tag{1.1}$$

which is equivalent to the condition that $-\infty < Lf \not\equiv \infty$ (see [2, Section 2.6]). If f is a locally integrable function on $\mathbf{R}^n$ satisfying (1.1) and

$$\int |f(y)| (\log(2 + |f(y)|)) dy < \infty,$$

then it is known that Lf is continuous on $\mathbf{R}^n$ (see [2, Theorem 9.1, Section 5.9]). Our second aim is to study the continuity for logarithmic potentials in Morrey spaces.

In the final section, we consider a positive continuous function $p(\cdot)$ such that $p_0 = 1$. Our final aim is to study the continuity for logarithmic potentials in Morrey spaces of variable exponent.

2 Morrey spaces of variable exponent

Throughout this paper, let C denote various constants independent of the variables in question.

We say that a positive function φ on $(0,\infty)$ is quasi-increasing if there exists a constant $C > 1$ such that

$$\varphi(s) \le C\varphi(t) \qquad \text{whenever } 0 < s \le t.$$

A positive function φ is quasi-decreasing if $\varphi(t)^{-1}$ is quasi-increasing and a positive function φ is quasi-monotone if φ is quasi-increasing or quasi-decreasing. Our typical example of φ is of the form

$$\varphi(r) = a(\log_{(1)}(1/r))^b(\log_{(2)}(1/r))^c$$

for $r > 0$, where $a > 0$ and $b, c \in \mathbf{R}$ and $\log_{(0)} t = e$, $\log_{(1)} t = \log(e+t)$ and $\log_{(m+1)} t = \log(e + \log_{(m)} t)$ for $m = 1, 2, \ldots$. From now on we assume that φ is quasi-monotone on $(0,\infty)$ and there exists a constant $C_2 > 1$ such that

$(\varphi 1)$ $$C_2^{-1}\varphi(r) \le \varphi(r^2) \le C_2\varphi(r) \qquad \text{whenever } r > 0,$$

which implies the doubling condition on φ; that is, there exists a constant $C > 1$ such that

$(\varphi 2)$ $$C^{-1}\varphi(r) \le \varphi(2r) \le C\varphi(r) \qquad \text{whenever } r > 0.$$

LEMMA 1 [2, Lemma 3.1, Section 5.3]. *If $\gamma > 0$, then $t^\gamma\varphi(t)$ is quasi-increasing on $(0,\infty)$.*

LEMMA 2 *There exists a constant $\kappa_0 > 0$ such that $(\log(2+t^{-1}))^{-\kappa_0}\varphi(t)$ is quasi-increasing on $(0,\infty)$.*

LEMMA 3 (cf. [3, Lemma 2.3]). *Suppose $0 \le \alpha \le n$ and the Minkowski $(n-\alpha)$-content of K is finite. If $\psi(t)$ is a quasi-increasig function on $(0,\infty)$ satisfying the doubling condition, then there exists a constant $C > 0$ such that*

$$\int_{G\cap B(x,r)} \psi(\delta_K(y))^{-1}dy \le C\int_0^{4r} t^\alpha\psi(t)^{-1}\frac{dt}{t}$$

for all $x \in \mathbf{R}^n$ and $r > 0$.

Consider a positive continuous nonincreasing function k on $(0,\infty)$ for which there exist $\varepsilon_0 \ge 0$ and $r_0 > 0$ such that

(k) $(\log(1/r))^{-\varepsilon_0}k(r)$ is nondecreasing on $(0, r_0)$.

Further we assume that

$$k(r_0) \geq e^{\varepsilon_0}.$$

By (k) we see that

$$C^{-1}k(r) \leq k(r^2) \leq Ck(r) \qquad \text{whenever } 0 < r < r_0, \tag{2.1}$$

which implies the doubling condition on k. Our typical example of k is of the form

$$k(r) = a(\log_{(1)}(1/r))^b(\log_{(2)}(1/r))^c$$

for $r \in (0, r_0)$, where $a > 0$ and the numbers b, c and r_0 are chosen so that $k(r)$ is nonincreasing on $(0, r_0]$.

LEMMA 4 [3, Lemma 2.1]. *There exists* $0 < r^* < r_0$ *such that* $\log k(r)/\log(1/r)$ *is nondecreasing on* $(0, r^*)$.

In this paper, consider a positive continuous function $p(\cdot)$ such that

$$p(x) = p_0 + \frac{\log k(\delta_K(x))}{\log(1/\delta_K(x))}$$

for $\delta_K(x) < r_0$ and $p(x) = p_0 + \log k(r_0)/\log(1/r_0)$ for $\delta_K(x) \geq r_0$, where $p_0 \geq 1$ and the number r_0 is chosen so that $\log k(r)/\log(1/r)$ is nondecreasing on $[0, \infty)$ (see Lemma 4).

For $0 \leq \nu \leq n$ and a locally integrable function f on G, we define the $L^{p(\cdot),\nu,\varphi}$ norm by

$$\|f\|_{p(\cdot),\nu,\varphi,G} = \inf\left\{\lambda > 0 : \sup_{x\in G, r>0} r^{-\nu}\varphi(r)\int_{G\cap B(x,r)}\left|\frac{f(y)}{\lambda}\right|^{p(y)} dy \leq 1\right\},$$

where $\varphi(r)$ is quasi-decreasing on $(0, \infty)$ when $\nu = 0$ and $\limsup_{r\to 0}\varphi(r)^{-1} > 0$ when $\nu = n$. We denote by $L^{p(\cdot),\nu,\varphi}(G)$ the space of all measurable functions f on G with $\|f\|_{p(\cdot),\nu,\varphi,G} < \infty$.

The following theorem is an extension of [3, Remark 4.4].

THEOREM 5 (cf. [3, Lemma 2.4]). *Suppose* $0 \leq \nu \leq \alpha \leq n$ *and the Minkowski* $(n-\alpha)$*-content of* K *is finite. Then there exists a constant* $C > 0$ *such that*

$$\int_{G\cap B(x,r)} |f(y)|^{p_0} k(|f(y)|^{-1})^{(\alpha-\nu)/p_0} dy \leq Cr^\nu\varphi(r)^{-1}$$

for all $x \in G$, $r > 0$ *and measurable functions* f *on* G *with* $\|f\|_{p(\cdot),\nu,\varphi,G} \leq 1$.

REMARK 6 We set $\Psi_k(t) = t^{p_0} k(t^{-1})^{(\alpha-\nu)/p_0}$ for $0 \le t < r_0$; otherwise set $\Psi_k(t) = t^{p_0} k(r_0^{-1})^{(\alpha-\nu)/p_0}$. For $0 \le \nu \le n$ and a locally integrable function f on G, we define the $L^{\Psi_k,\nu,\varphi}$ norm by

$$\|f\|_{\Psi_k,\nu,\varphi,G} = \inf\left\{\lambda > 0 : \sup_{x\in G, r>0} r^{-\nu}\varphi(r) \int_{G\cap B(x,r)} \Psi_k\left(\left|\frac{f(y)}{\lambda}\right|\right) dy \le 1\right\},$$

where $\varphi(r)$ is quasi-decreasing on $(0,\infty)$ when $\nu = 0$ and $\limsup_{r\to 0} \varphi(r)^{-1} > 0$ when $\nu = n$. We denote by $L^{\Psi_k,\nu,\varphi}(G)$ the space of all measurable functions f on G with $\|f\|_{\Psi_k,\nu,\varphi,G} < \infty$. We see that $\Psi_k(t)$ satisfies the doubling condition. This implies that $\|\cdot\|_{\Psi_k,\nu,\varphi,G}$ is a quasi-norm. Then it follows from Theorem 5 that

$$\|f\|_{\Psi_k,\nu,\varphi,G} \le C\|f\|_{p(\cdot),\nu,\varphi,G} \qquad \text{whenever } f \in L^{p(\cdot),\nu,\varphi}(G).$$

REMARK 7 For $\alpha > 0$, let K be a compact subset of G such that

$$|K(r)| \le Cr^\alpha \qquad \text{for all } 0 < r < r_0$$

and

$$C^{-1}r^\alpha \le |K(r) \cap B(x_0,t)|$$

for some $x_0 \in K$ and all $0 < r \le t < r_0$. Set $\delta(x) = \delta_K(x)$ for simplicity. Let

$$p(x) = p_0 + \frac{a\log(\log_{(m)}(1/\delta(x)))}{\log(1/\delta(x))}$$

for $a > 0$ and an integer $m \ge 0$ when $\delta(x) \le r_0$; otherwise $p(x) = p_0 + a\log(\log_{(m)}(1/r_0))/\log(1/r_0)$ and

$$\varphi(t) = (\log_{(\ell)}(1/t))^b$$

for an integer $\ell \ge 0$ and $b \in \mathbf{R}$. Then Theorem 5 is the best in the following sense: if $0 < \nu \le \alpha \le n$, then we can find $f \in L^{p(\cdot),\nu,\varphi}(G)$ satisfying

$$\int_{G\cap B(x_0,r)} |f(y)|^{p_0}(\log_{(m)}|f(y)|)^{a(\alpha-\nu)/p_0} dy \ge Cr^\nu(\log_{(\ell)}(1/r))^{-b}$$

for all $0 < r < r_0$.

For $\nu = 0$ and $b \ge 0$, we consider the function

$$f(y) = \begin{cases} \delta(y)^{-\alpha/p_0}(\log(1/\delta(y)))^{-(b-1)/p_0}(\log_{(m)}(1/\delta(y)))^{-a\alpha/p_0^2} & \text{if } \ell = 1, \\ \delta(y)^{-\alpha/p_0}(\log_{(\ell)}(1/\delta(y)))^{-(b-1)/p_0}(\log_{(m)}(1/\delta(y)))^{-a\alpha/p_0^2} & \\ \qquad \times \prod_{j=1}^{\ell-1}(\log_{(j)}(1/\delta(y)))^{-1/p_0} & \text{if } \ell \ge 2 \end{cases}$$

for $y \in G$ with $\delta(y) \le r_0$; set $f(y) = 0$ when $\delta(y) > r_0$. Then we can show that $f \in L^{p(\cdot),\nu,\varphi}(G)$ and

$$\int_{G\cap B(x_0,r)} f(y)^{p_0}(\log_{(m)} f(y))^{a\alpha/p_0}dy \ge C(\log_{(\ell)}(1/r))^{-b}$$

for all $0 < r < r_0$.

PROPOSITION 8 *Suppose $0 \le \nu \le \alpha \le n$. Then there exists a constant $C > 0$ such that*

$$\int_{G\cap B(x,r)} |f(y)|^{p_0}dy \le Cr^{\nu}\varphi(r)^{-1}k(r)^{-(\alpha-\nu)/p_0}$$

for all $x \in G$, $r > 0$ and measurable functions f on G satisfying the conclusion of Theorem 5.

REMARK 9 We set $\Phi_k(t) = \varphi(t)^{-1}k(t)^{-(\alpha-\nu)/p_0}$ for $0 \le t < r_0$; otherwise set $\Phi_k(t) = \varphi(t)^{-1}k(r_0)^{-(\alpha-\nu)/p_0}$. Proposition 8 implies that

$$\|f\|_{p_0,\nu,\Phi_k,G} \le C\|f\|_{\Psi_k,\nu,\varphi,G} \qquad \text{whenever } f \in L^{\Psi_k,\nu,\varphi}(G).$$

Moreover, Proposition 8 is seen to be sharp in the following sense: let

$$k(t) = (\log_{(m)}(1/t))^a$$

for an integer $m \ge 0$ and $a > 0$ when $t \le r_0$ and

$$\varphi(t) = (\log_{(\ell)}(1/t))^b$$

for an integer $\ell \ge 0$ and $b \in \mathbf{R}$. If $0 < \nu \le \alpha \le n$, then we can find $f \in L^{\Psi_k,\nu,\varphi}(G)$ satisfying

$$\int_{G\cap B(0,r)} |f(y)|^{p_0}dy \ge Cr^{\nu}(\log_{(\ell)}(1/r))^{-b}(\log_{(m)}(1/r))^{-a(\alpha-\nu)/p_0}$$

for all $0 < r \le r_0$.

For $\nu = 0$, $b \ge 0$ and integers $1 \le \ell \le m$, we consider the function

$$f(y) = \begin{cases} |y|^{-\alpha/p_0}(\log(1/|y|))^{-(b-1)/p_0}(\log_{(m)}(1/|y|))^{-a\alpha/p_0^2}\chi_{B(0,r_0)}(y) & \text{if } \ell = 1, \\ |y|^{-\alpha/p_0}(\log_{(\ell)}(1/|y|))^{-(b-1)/p_0}(\log_{(m)}(1/|y|))^{-a\alpha/p_0^2} & \\ \qquad \times \prod_{j=1}^{\ell-1}(\log_{(j)}(1/|y|))^{-1/p_0}\chi_{B(0,r_0)}(y) & \text{if } \ell \ge 2. \end{cases}$$

Then we can show that $f \in L^{\Psi_k,\nu,\varphi}(G)$ and

$$\int_{G\cap B(0,r)} f(y)^{p_0}dy \ge C(\log_{(\ell)}(1/r))^{-b}(\log_{(m)}(1/r))^{-a\alpha/p_0}$$

for all $0 < r \le r_0$.

3 Continuity of logarithmic potentials in Morrey spaces

In this section, we deduce the continuity of the logarithmic potential Lf. We consider a nondecresing function φ_1 on $(0,1/2]$ and a nonincresing function φ_2 on $(0,1/2]$ such that

$$\varphi_1(r)=\int_0^r \varphi(t)^{-1}\frac{dt}{t} \text{ and } \varphi_2(r)=\int_r^1 \varphi(t)^{-1}\frac{dt}{t}$$

for $0<r\le 1/2$. We set

$$\Phi(r)=\begin{cases} \varphi_1(r) & \text{if } \nu=0,\\ \varphi(r)^{-1} & \text{if } 0<\nu<1,\\ \varphi_2(r) & \text{if } \nu=1. \end{cases}$$

REMARK 10 Let $\varphi(t)=(\log(1/t))^{\beta}$ for $\beta\in\mathbf{R}$. Then

$$\varphi_1(r)=C\begin{cases} (\log(1/r))^{-\beta+1} & \text{if } \beta>1,\\ \infty & \text{if } \beta\le 1 \end{cases}$$

and

$$\varphi_2(r)=C\begin{cases} (\log(1/r))^{-\beta+1} & \text{if } \beta<1,\\ \log(\log(1/r)) & \text{if } \beta=1,\\ 1 & \text{if } \beta>1. \end{cases}$$

LEMMA 11 *Suppose $0\le\nu\le 1$ and $\varphi_1(1/2)<\infty$ when $\nu=0$. Then there exists a constant $C>0$ such that*

$$\int_{B(x,\delta)}(\log(\delta/|x-y|))\,f(y)dy\le C\begin{cases} \varphi_1(\delta) & \text{if } \nu=0,\\ \delta^{\nu}\varphi(\delta)^{-1} & \text{if } 0<\nu\le 1 \end{cases}$$

for all $x\in\mathbf{R}^n$, $0<\delta<1/2$ and nonnegative measurable functions f with $\|f\|_{1,\nu,\varphi,\mathbf{R}^n}\le 1$.

LEMMA 12 *Suppose $0\le\nu\le 1$. If f is a nonnegative measurable function satisfying (1.1) and $\|f\|_{1,\nu,\varphi,\mathbf{R}^n}\le 1$, then*

$$\int_{\mathbf{R}^n\setminus B(x,\delta)}|x-y|^{-1}f(y)dy\le C\begin{cases} \delta^{\nu-1}\varphi(\delta)^{-1} & \text{if } 0\le\nu<1,\\ \varphi_2(\delta) & \text{if } \nu=1 \end{cases}$$

for all $x\in\mathbf{R}^n$ and $0<\delta<1/2$.

Our aim in this section is to establish the following result, which deals with the continuity of logarithmic potentials in Morrey spaces.

THEOREM 13 *Assume that* $0 \le \nu \le 1$ *and* $\varphi_1(1/2) < \infty$ *when* $\nu = 0$. *If* f *is a nonnegative measurable function on* $\mathbf{R}^n$ *satisfying* (1.1) *and* $\|f\|_{1,\nu,\varphi,\mathbf{R}^n} \le 1$, *then* Lf *is continuous on* $\mathbf{R}^n$ *and satisfies*

$$|Lf(x) - Lf(z)| \le C|x-z|^{\nu}\Phi(|x-z|)$$

whenever $0 < |x-z| < 1/2$.

REMARK 14 In the case $\nu = 0$, we need the condition $\varphi_1(1/2) < \infty$ for the Hölder continuity of Lf.

For this, consider the functions

$$\varphi(t) = \begin{cases} (\log(1/t))^a & \text{if } m = 1, \\ (\log_{(m)}(1/t))^a \prod_{j=1}^{m-1}(\log_{(j)}(1/t)) & \text{if } m \ge 2 \end{cases}$$

and

$$f(y) = \begin{cases} |y|^{-n}(\log(1/|y|))^{-2}\chi_{B(0,1/2)}(y) & \text{if } m = 1, \\ |y|^{-n}(\log(1/|y|))^{-2}\prod_{j=2}^{m}(\log_{(j)}(1/|y|))^{-1}\chi_{B(0,1/2)}(y) & \text{if } m \ge 2. \end{cases}$$

If $a \le 1$, then we see that

(1) $\displaystyle\int (\log(1/|y|))f(y)dy = \infty;$

(2) $\displaystyle\int_{B(x,r)} f(y)dy \le C\prod_{j=1}^{m}(\log_{(j)}(1/r))^{-1} \le C\varphi(r)^{-1}$ for all $x \in \mathbf{R}^n$ and $0 < r < 1/2$.

This implies that Lf is not continuous at the origin.

REMARK 15 Theorem 13 is seen to be sharp in the following sense: let

$$\varphi(t) = (\log_{(m)}(1/t))^a$$

for an integer $m \ge 0$ and $a \in \mathbf{R}$. If $0 < \nu \le 1$, then we can find $f \in L^{1,\nu,\varphi}(\mathbf{R}^n)$ satisfying

$$|Lf(0) - Lf(x_i)| \ge C|x_i|^{\nu}\Phi(|x_i|)$$

for some sequence $\{x_i\}$ which tends to the origin.

Let

$$\varphi(t) = \begin{cases} (\log(1/t))^a & \text{if } m = 1, \\ (\log_{(m)}(1/t))^a \prod_{j=1}^{m-1}(\log_{(j)}(1/t)) & \text{if } m \ge 2 \end{cases}$$

for $a > 1$ and an integer $m \ge 1$. If $\nu = 0$, then we can find $f \in L^{1,\nu,\varphi}(\mathbf{R}^n)$ satisfying

$$|Lf(0) - Lf(x_i)| \ge C(\log_{(m)}(1/|x_i|))^{-a+1}$$

for some sequence $\{x_i\}$ which tends to the origin.

4 Continuity of logarithmic potentials in Morrey spaces of variable exponent

We set

$$\Phi_K(r) = \begin{cases} \int_0^r \varphi(t)^{-1} k(t)^{-\alpha} \frac{dt}{t} & \text{if } \nu = 0, \\ \varphi(r)^{-1} k(r)^{-(\alpha-\nu)} & \text{if } 0 < \nu < 1, \\ \int_r^1 \varphi(t)^{-1} k(t)^{-(\alpha-1)} \frac{dt}{t} & \text{if } \nu = 1. \end{cases}$$

By Theorems 5, 13 and Proposition 8, we have the following result, which deals with the continuity of logarithmic potentials in Morrey spaces of variable exponent.

THEOREM 16 *Assume that* $0 \le \nu \le 1$, $\nu \le \alpha \le n$ *and*

$$\int_0^{1/2} \varphi(t)^{-1} k(t)^{-\alpha} \frac{dt}{t} < \infty$$

when $\nu = 0$. *Let the Minkowski* $(n-\alpha)$*-content of* K *be finite. If* f *is a nonnegative measurable function on* $\mathbf{R}^n$ *satisfying* (1.1) *and* $\|f\|_{p(\cdot),\nu,\varphi,\mathbf{R}^n} \le 1$, *then* Lf *is continuous on* $\mathbf{R}^n$ *and satisfies*

$$|Lf(x) - Lf(z)| \le C|x-z|^{\nu} \Phi_K(|x-z|)$$

whenever $0 < |x-z| < 1/2$.

We set $A = a(n-\nu) + \beta$,

$$\Psi(r) = \begin{cases} (\log(1/r))^{-A+1} & \text{if } \nu = 0, \\ (\log(1/r))^{-A} & \text{if } 0 < \nu < 1, \\ (\log(1/r))^{-A+1} & \text{if } \nu = 1 \text{ and } A < 1, \\ \log(\log(1/r)) & \text{if } \nu = 1 \text{ and } A = 1, \\ 1 & \text{if } \nu = 1 \text{ and } A > 1 \end{cases}$$

in case $n \ge 2$ and

$$\Psi(r) = \begin{cases} (\log(1/r))^{-A+1} & \text{if } \nu = 0, \\ (\log(1/r))^{-A} & \text{if } 0 < \nu < 1, \\ (\log(1/r))^{-A+1} & \text{if } \nu = 1 \text{ and } \beta \le 0 \end{cases}$$

in case $n = 1$.

By Theorem 16 and Remark 10, we have the following result.

COROLLARY 17 *Let*

$$p(x) = 1 + \frac{a \log(\log(1/|x_0 - x|))}{\log(1/|x_0 - x|)} + \frac{b}{\log(1/|x_0 - x|)} = 1 + \omega_{a,b}(|x_0 - x|)$$

for $x \in B(x_0, r_0)$ *and* $p(x) = 1 + \omega_{a,b}(r_0)$ *for* $x \in \mathbf{R}^n \setminus B(x_0, r_0)$, *where the numbers* a, b *and* r_0 *are chosen so that* $\omega_{a,b}(r)$ *is nondecreasing on* $(0, r_0)$ *and* $p(x) \geq 1$. *Assume that* $0 \leq \nu \leq 1$ *and* $A > 1$ *when* $\nu = 0$. *If* f *is a nonnegative measurable function on* $\mathbf{R}^n$ *satisfying* (1.1) *and* $\|f\|_{p(\cdot),\nu,\beta,\mathbf{R}^n} \leq 1$, *then* Lf *is continuous on* $\mathbf{R}^n$ *and satisfies*

$$|Lf(x) - Lf(z)| \leq C|x - z|^{\nu}\Psi(|x - z|)$$

whenever $0 < |x - z| < 1/2$.

We have three remarks for Corollary 17.

REMARK 18 When $\nu = 0$ and $\beta = 0$, we showed that

$$\int_G f(y)(\log(1 + f(y)))^{an} dy < \infty$$

for nonnegative measurable functions $f \in L^{p(\cdot)}(\mathbf{R}^n)$ (see Theorem 5). It follows from [2, Theorem 9.1, Section 5.9] that if $\nu = 0, \beta = 0$ and $an = 1$, then Lf is continuous on $\mathbf{R}^n$.

REMARK 19 Set $x_0 = 0$. In case $\nu = 0$ and $a > 0$, we need the condition $-an + 1 < \beta$ for the Hölder continuity of Lf.

For this, consider the function

$$f(y) = |y|^{-n}(\log(1/|y|))^{-2}\chi_{B(0,1/2)}(y).$$

If $-an + 1 \geq \beta$ and $an \neq 1$ (see Remark 18), then we see that

(1) $\int (\log(1/|y|))f(y)dy = \infty$;

(2) $\int_{B(x,r)} f(y)^{p(y)}dy \leq C \int_{B(x,r)} |y|^{-n}(\log(1/|y|))^{an-2}dy \leq C(\log(1/r))^{an-1}$ for all $x \in \mathbf{R}^n$ and $0 < r < 1/2$.

This implies that Lf is not continuous at the origin.

Similarly, in case $\nu = 0$ and $a = 0$, we need the condition $\beta > 1$ for the Hölder continuity of Lf.

REMARK 20 Set $x_0 = 0$. Corollary 17 is seen to be sharp in the following sense: for $0 < \nu \leq 1$, we can find $f \in L^{p(\cdot),\nu,\beta}(\mathbf{R}^n)$ satisfying

$$|Lf(0) - Lf(x_i)| \geq C|x_i|^{\nu}\Psi(|x_i|)$$

for some sequence $\{x_i\}$ which tends to the origin.

Similarly, for $\nu = 0$, we can find $f \in L^{p(\cdot),\nu,\beta}(\mathbf{R}^n)$ satisfying

$$|Lf(0) - Lf(x_i)| \geq C(\log(1/|x_i|))^{-A+1}$$

for some sequence $\{x_i\}$ which tends to the origin.

By Theorem 16, we have the following remark.

REMARK 21 We consider a positive continuous function $p(\cdot)$ such that

$$p(x) = 1 + \frac{a\log(\log(1/|x_n|))}{\log(1/|x_n|)} + \frac{b}{\log(1/|x_n|)} = 1 + \omega_{a,b}(|x_n|)$$

for $x \in L(r_0) = \{x = (x_1, x_2, \cdots, x_n) \in \mathbf{R}^n : |x_n| \leq r_0\}$ and $p(x) = 1 + \omega_{a,b}(r_0)$ for $x \in \mathbf{R}^n \setminus L(r_0)$, where the numbers a, b and r_0 are chosen so that $\omega_{a,b}(r)$ is nondecreasing on $(0, r_0)$ and $p(x) \geq 1$.

We set $A_L = a(1-\nu) + \beta$,

$$\Psi_L(r) = \begin{cases} (\log(1/r))^{-A_L+1} & \text{if } \nu = 0, \\ (\log(1/r))^{-A_L} & \text{if } 0 < \nu < 1, \\ (\log(1/r))^{-A_L+1} & \text{if } \nu = 1 \text{ and } A_L < 1, \\ \log(\log(1/r)) & \text{if } \nu = 1 \text{ and } A_L = 1, \\ 1 & \text{if } \nu = 1 \text{ and } A_L > 1 \end{cases}$$

for $n \geq 2$.

Assume that $0 \leq \nu \leq 1$ and $A_L > 1$ when $\nu = 0$. If f is a nonnegative measurable function on $\mathbf{R}^n$ satisfying (1.1) and $\|f\|_{p(\cdot),\nu,\beta,\mathbf{R}^n} \leq 1$, then Lf is continuous on $\mathbf{R}^n$ and satisfies

$$|Lf(x) - Lf(z)| \leq C|x-z|^{\nu}\Psi_L(|x-z|)$$

whenever $0 < |x-z| < 1/2$.

But I conjecture that the conclusion of Corollary 17 still holds for this exponent.

References

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